INTRODUCTION TO CALCULUS

1: What is calculus?

1.1 Calculus deals with two themes: **taking differences** and **summing** things up.

Differences measure how data **change**, sums quantify how quantities **accumulate**.

The process of taking differences measures a **rate of change**.

A limiting process gives the **derivative**.

The process of **summation** produces the **integral**.

The two operations are related by the fundamental theorem of calculus.

In this first part, we look at functions which are evaluated on the set integers and where there is no need for limits.

It allows us to illustrate a major benefit of calculus: it gives us the ability to predict the future by analyzing the past.

1.2. Can you figure the next entry of the sequence of numbers

When solving such a riddle, we already use already a basic idea of calculus.

$$0, 3, 8, 15, 24, 35, 48, \ldots ?$$

You might see that the differences

$$3, 5, 7, 9, 11, 13, \ldots$$

show a pattern.

Taking differences again gives

$$2, 2, 2, 2, 2, 2, \ldots$$

Now, we can go back to the previous sequence and see that that the next term is 15.

Looking at the original sequence gives 48 + 15 = 63.

Seeing such a difference pattern allows to get the future entries of the process.

This observation is important.

1.3. Let us rewrite what we just did using the concept of a **function**.

A function f takes an input x and gives an output called $y=f(x)$.

The sequence we have just seen is then the function $f(1) = 0, f(2) = 3, f(3) = 8, f(4) = 15, f(5) = 24, \ldots$

Define now a new function Df by $Df(x) = f(x+1) - f(x)$.

It is a rate of change which we also call a “derivative”.

Write also $f(x)$ instead of $f(x)$.

We have $f(1) = 3 - 0 = 3, f(2) = 8 - 3 = 5, f(3) = 15 - 8 = 7, \ldots.$

Now, we can take the derivative again and define $f(n) = f(n + 1) - f(n)$.

The function f is the function where the derivative has been applied twice.

We have seen $f(1) = 2, f(2) = 2, f(3) = 2, \ldots.$

The second derivative is constant.

We have dealt with data which measure a constant acceleration.

1.4. Functions can be visualized graphically in the form of a **graphs** $y = f(x)$.

To do so, we draw two perpendicular axes, the x-axis and the y-axis and mark down every pair $(x, f(x))$ in this **Euclidean plane**.

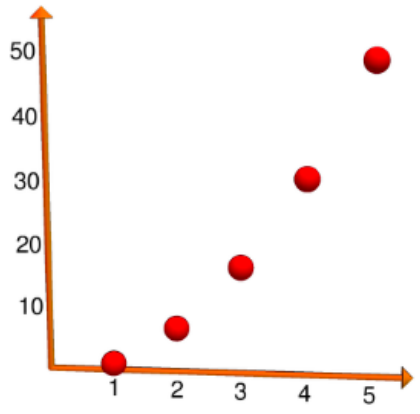


Figure 1. When plotting the sequence of numbers in the coordinate plane, the function is visualized as a graph.

1.5. When the first mathematicians were recording numbers, they marked them into tally sticks.

An artifact from tens of thousands of years ago is the Ishango bone.

We can look at the marks as a constant function

$$1, 1, 1, 1, ...$$

Over the next thousands of years, humans figured out to represent numbers as symbols like

$$1, 2, 3, 4, \ldots \; .$$

We see that $1 = 1$, $2 = 1 + 1$, $3 = 1 + 1 + 1$ etc.

If we look at this counting function $f(x) = x$, it satisfies $f'(x) = 1$, the counting function and $f''(x) = 0$.

1.6 We can now ask which function g has the property that $g' = f$.

The function g represents the summation of the terms.

For example $f(5) = 0 + 1 + 2 + 3 + 4$ and $f(3) = 0 + 1 + 2 + 3$, then $f(4 + 1) - f(4) = 4$.

We see that if we define g= S*f* as

$$Sf(x) = f(0) + f(1) + f(2) + \cdots + f(x-1)$$

then $g(x + 1) - g(x) = f(x)$.

Can we get a formula for the function g?

1.7. The new function g satisfies $g(1) = 1$, $g(2) = 3$, $g(3) = 6$, etc.

These numbers are called **triangular numbers**.

From the function g we can get *f* back by taking difference:

$$Dg(n) = g(n+1) - g(n) = f(n) \; .$$

For example $Dg(5) = g(6) - g(5) = 15 - 10 = 5$.

And indeed this is $f(5)$.

Finding a formula for the sum $Sf(n)$ is not so easy if you have not seen it yet.

We have to find the n’th term in the sequence which starts with

$$1, 3, 6, 10, 15, 21, \ldots$$

1.8. Legend tells that when Karl-Friedrich Gauss was a 9 year old school kid, his teacher, Mr. Buttner gave him the task to sum up the first 100 positive integers $1 + 2 + \cdots + 100$.

Gauss did not want to do this tedious work and looked for a better way to do it.

He discovered that pairing the numbers up would simplify the summation.

He would write the sum as $(1 + 100) + (2 + 99) + \cdots + (50 + 51)$ so that the answer is $g(x) = x(x - 1)/2 = 5050$.

We have now an explicit expression for the sum function.

Lets apply the difference function again: $Dg(x) = x(x + 1)/2 - x(x - 1)/2 = x = f(x)$.

1.9. Let us add up the new sequence again and compute $h = Sg$.

We get the sequence 0, 1, 4, 10, 20, 35,... called **tetrahedral numbers**.

The reason is that one can use $h(n)$ balls to build a tetrahedron of side length n.

For example, we need $h(4) = 20$ golf balls to build a tetrahedron of side length 4.

The formula which holds for h is $h(x) = x(x-1)(x-2)/6$.

You can easily check that summing the differences gives the function back.

1.10. The general relation

$$\boxed{SDf(x) = f(x) - f(0), \qquad DSf(x) = f(n)}$$

already is a version of the **fundamental theorem of calculus**.

It will lead to the **integral** $\boxed{\int_0^x f(x)\,dx}$, **derivative** $\boxed{\frac{d}{dx} f(x)}$ **and the fundamental theorem of calculus.**

$$\boxed{\int_0^x \frac{d}{dt} f(t)\,dt = f(x) - f(0), \qquad \frac{d}{dx}\int_0^x f(t)\,dt = f(x)}$$

1.11. This is a fantastic result.

The goal of this class is to understand this theorem, and to apply it.

Note that if we define $[n]^0 = 1$, $[n]^1 = n$, $[n]^2 = n(n-1)/2$, $[n]^3 = n(n-1)(n-2)/6$ then $D[n] = [1]$, $D[n]^2 = 2[n]$, $D[n]^3 = 3[n]^2$ and in general

$$\boxed{\frac{d}{dx}[x]^n = n[x]^{n-1}}$$

Examples

1.12. The **Fibonnacci sequence** $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ satisfies $f(x) = f(x-1) + f(x-2)$. It defines a function on the positive integers. For example, $f(6) = 8$. What is the function $g = Df$? We can assume $f(0) = 0$. **Solution:** Taking difference produces a shifted version $\boxed{Df(x) = f(x-1)}$.

1.13. Take the same function f given by the sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ but now compute the function $h(n) = Sf(n)$ obtained by summing the first n numbers up. It gives the sequence $1, 2, 4, 7, 12, 20, 33, \ldots$. What sequence is that?
Solution: Because $Df(x) = f(x-1)$ we have $f(x) - f(0) = SDf(x) = Sf(x-1)$ so that $Sf(x) = f(x+1) - f(1)$. Summing the Fibonnacci sequence produces the Fibonnacci sequence shifted to the left with $f(2) = 1$ subtracted.

1.14. Find the next term in the sequence $2 \quad 6 \quad 12 \quad 20 \quad 30 \quad 42 \quad 56 \quad 72 \quad 90 \quad 110 \quad 132$.
Solution: Take differences

$$\begin{array}{ccccccccccc} 2 & 6 & 12 & 20 & 30 & 42 & 56 & 72 & 90 & 110 & 132 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{array} .$$

Now we can add an additional number, starting from the bottom and working us up.

$$\begin{array}{cccccccccccc} 2 & 6 & 12 & 20 & 30 & 42 & 56 & 72 & 90 & 110 & 132 & \boxed{156} \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & \boxed{24} \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & \boxed{2} \end{array}$$

1.15. Look at the function $f(n)$ which gives the n'th prime number. Lets look at the derivatives $D^k f$ but take the absolute value $|D^k(f)|$. In other words, we study $T(f)(n) = |f(n+1) - f(n)|$. Let's see

n=	1	2	3	4	5	6	7	8	9	...
$f(n) =$	2	3	5	7	11	13	17	23	29	...
$Tf(n) =$	$\boxed{1}$	2	2	4	2	4	2	4	6	...
$T^2 f(n) =$	$\boxed{1}$	0	2	2	2	2	2	2	4	...
$T^3 f(n) =$	$\boxed{1}$	2	0	0	0	0	0	2	0	...

2: Functions

2.1. A **function** is a rule which assigns to a real number a new real number.

The function $f(x) = x^3 - 2x$ for example assigns to the number $x = 2$ the value $2^3 - 4 = 4$.

A function is assigned a **domain** A, the points where *f* is defined and a **codomain** B a set of numbers in which *f* is mapped to.

The **range** is $f(A)$.

2.2. Many functions like $f(x) = x^2 - 2x$ are defined everywhere.

In general, we assume that the domain is the place where the function is defined and the codomain is the set of real numbers and the range the set of numbers which are reached by *f*.

Functions can also be defined on domains which are discrete.

The prime function p(n) which gives the n'th prime is also a function.

The domain is $\mathbb{N} = \{1, 2, 3, \dots\}$ of natural numbers, the range B is the set of primes.

2.3. A function g(x) = 1/x for example cannot be evaluated at 0 so that the domain must exclude the point 0.

Its range is also $\mathbb{R} \setminus \{0\}$, the set of real numbers without 0.

The inverse of a function *f* is a function g such that g(*f*(x)) = x.

The function g(x) = $\sqrt{x}$ for example is the inverse of the function *f*(x) = x^2 on its domain $\mathbb{R}^+ = [0, \infty)$.

The function *f*(x) = 1/x is its own inverse.

2.4. Here are a few examples.

We will look at many of them in more detail during the class.

Very important are polynomials, trigonometric functions, the exponential and the logarithmic function.

Below we see some functions.

The compound interest function can also be interpreted as an exponential.

It will for h → 0 go over to the exponential function.

The logarithmic function as the inverse of the exponential function is only defined on the positive real axes.

constant	1	power	2^x
identity	x	exponential	$e^x = \exp(x)$
linear	$3x+1$	logarithm	$\log(x) = \ln(x)$
quadratic	x^2	absolute value	$\|x\|$
cosine	$\cos(x)$	devil comb	$\sin(1/x)$
sine	$\sin(x)$	bell function	e^{-x^2}
compound interest	$\exp_h(x) = (1+h)^{x/h}$	Agnesi	$\frac{1}{1+x^2}$
logarithms	$\log(x) = \ln(x)$	sinc	$\sin(x)/x$

We can build new functions by:

addition	$f(x) + g(x)$
multiplication	$f(x) * g(x)$
division	$f(x)/g(x)$
scaling	$2f(x)$
translation	$f(x+1)$
composing	$f(g(x))$
inverting	$f^{-1}(x)$

Important functions:

polynomials	$x^2 + 3x + 5$
rational functions	$(x+1)/(x^4+1)$
exponential	e^x
logarithm	$\log(x)$
trig functions	$\sin(x)$, $\tan(x)$
inverse trig functions	$\arcsin(x)$, $\arctan(x)$.
roots	$\sqrt{x}, x^{1/3}$

2.5. We will look at these functions a lot during this class.

The logarithm, exponential and trigonometric functions are especially important.

For some functions, we need to restrict the domain, where the function is defined.

For the square root function $\sqrt{x}$ or the logarithm $\log(x)$ for example, we have to assume that the number x on which we evaluate the function is positive.

We write that the domain is $(0, \infty) = \mathbf{R}^+$.

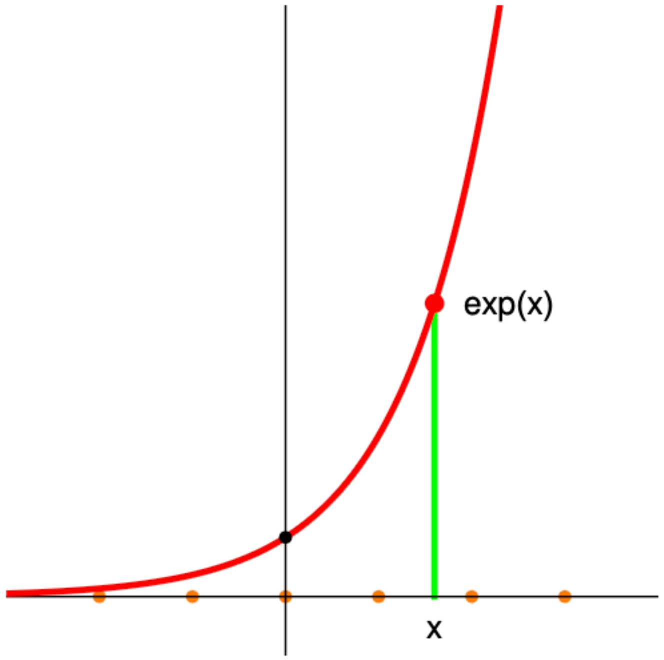
For the function $f(x) = 1/x$, we have to assume that x is different from zero.

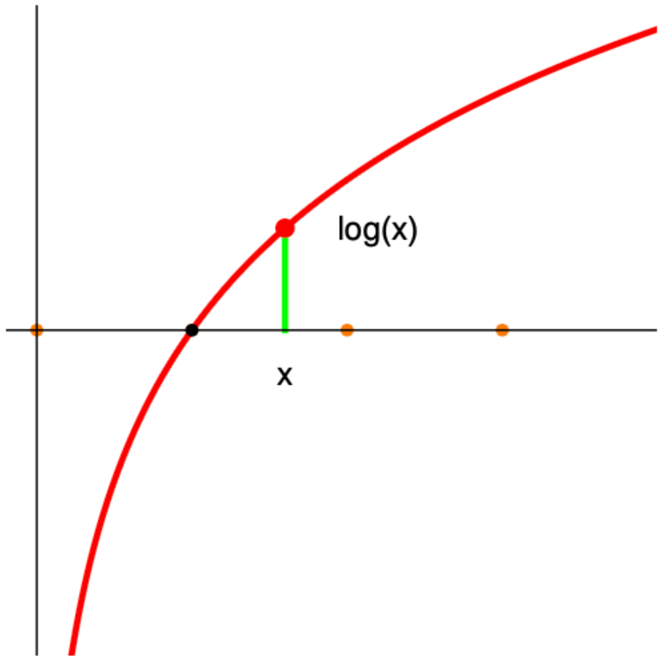
Keep these three examples in mind.

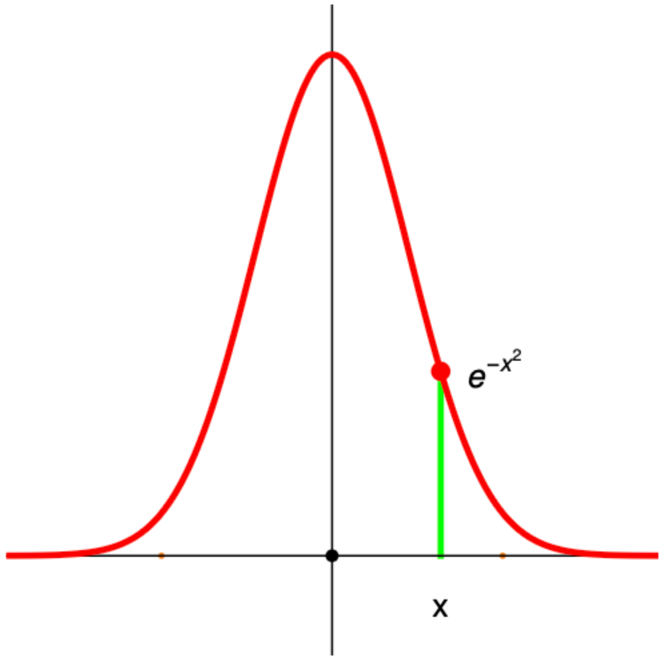
2.6. The graph of a function is the set of points $\{(x, y) = (x, f(x))\}$ in the plane, where x runs over the domain A of *f*.

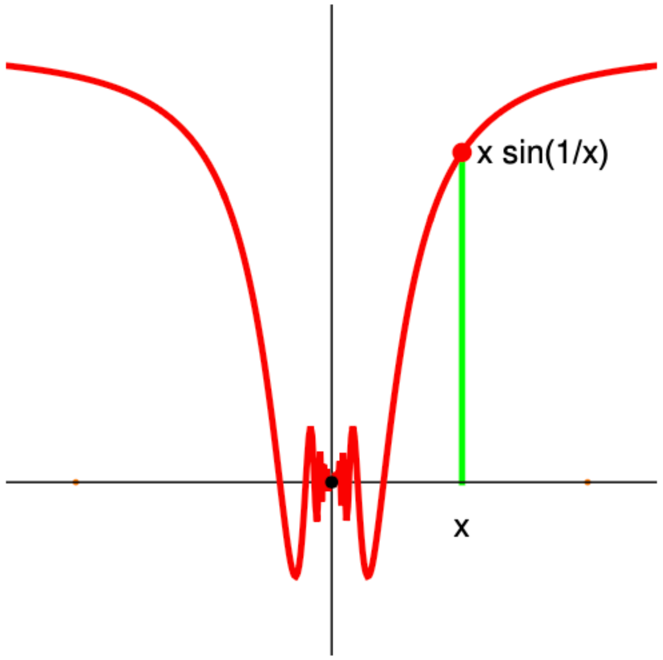
Graphs allow us to visualize functions.

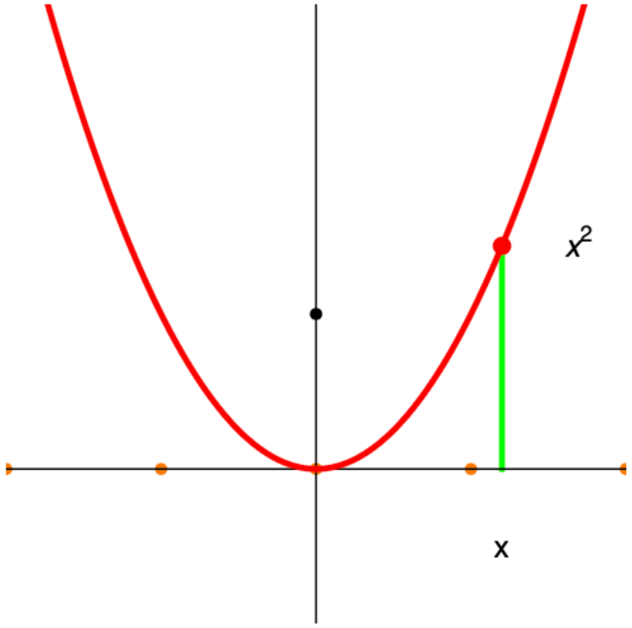
We can "see a function", when we draw the graph.

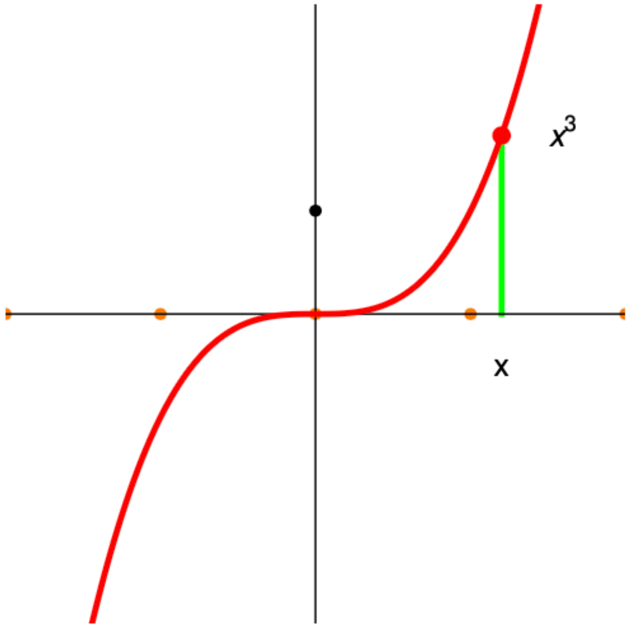


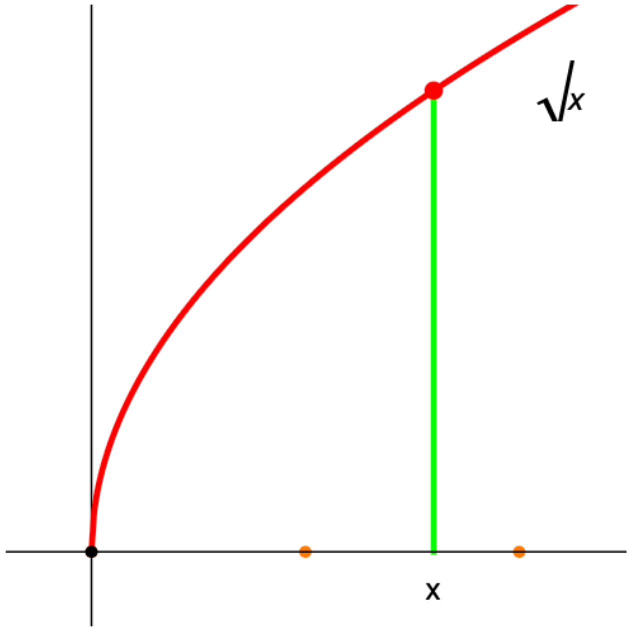


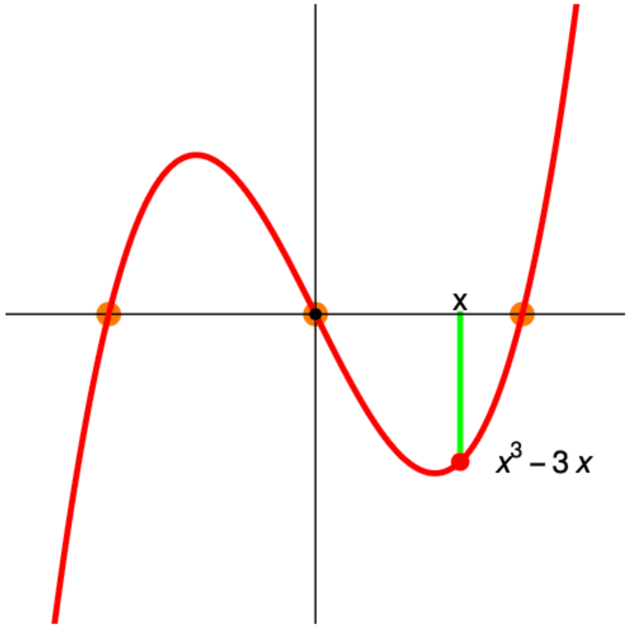


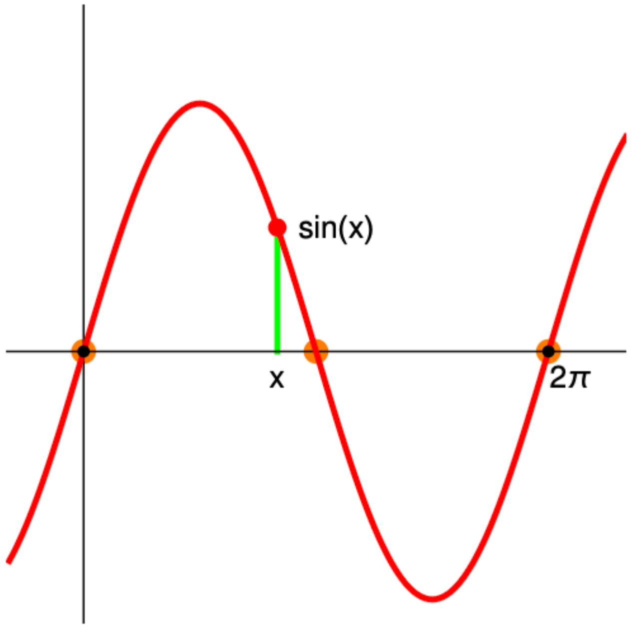


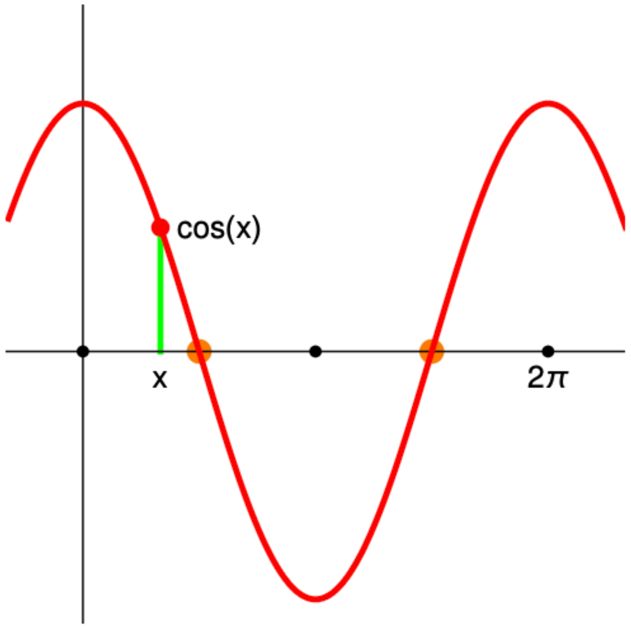

x^2
x


x^3
x


$\sqrt{x}$
x


x
$x^3 - 3x$


sin(x)
x
2π


cos(x)
x
2π

2.7. Definition: A function $f : A \to B$ is **invertible** if there is an other function g such that $g(f(x)) = x$ for all x in A and $f(g(y)) = y$ for all $y \in B$.

The function g is the **inverse** of f.

Example: $g(x) = \sqrt{x}$ is the inverse of $f(x) = x^2$ as a function from $A = [0, \infty)$ to $B = [0, \infty)$.

You can check with the horizontal line test whether an inverse exists: draw the box with base A and side B, then every horizontal line should intersect the graph exactly once.

3: Limits

3.1. The function 1/x is not defined everywhere.

It blows up at $x = 0$ where we divide by zero.

Sometimes however, a function can be **healed** at a point where it is not defined.

A silly example is $f(x) = x^2/x$ which is initially not defined at $x = 0$ because we divide by x.

The function can be “saved” by noticing that $f(x) = x$ for all x different from 0.

Functions often can be continued to “forbidden” places if we write the function differently.

This can involves dividing out a common factor.

Here are some examples:

3.2. Example. The function $f(x) = (x^3 - 1)/(x - 1)$ is at first not defined at $x = 1$.

But for x close to 1, nothing really bad happens.

We can evaluate the function at points closer and closer to 1 and get closer and closer to 3.

We say $\lim_{x \to 1} f(x) = 3$.

Indeed, you might have noticed already that $f(x) = x^2 + x + 1$ by factoring out $(x - 1)$.

While initially not defined at $x = 1$, there is a natural value $b = 3$ we can assign for $f(1) = 3$ so that the graph continues nicely through that point.

3.3. Definition. We write $x \to a$ to indicate that x **approaches** a.

This approach can be from either side, **from the left** $x \to a^-$ or **from the right** $x \to a^+$.

A function $f(x)$ has a **limit** at a point a if there exists a unique b such that $f(x) \to b$ for $x \to a$.

We write $\lim_{x \to a} f(x) = b$ if the limit exists and if it is the same value b, when approaching from either side.

3.4. Example. The **sinc function** $f(x) = \sin(x)/x$ is called sinc(x).

It is not defined at $x = 0$ at first.

It appears naturally in geometry as a quotient between the length of a side of a right angle triangle and an arc length of a sector which contains it.

We will look at this function a lot also later on and show that the limit of $f(x)$ exists for $x \to 0$.

This fact is important.

Fundamental theorem of trigonometry. $\lim_{x\to 0} \dfrac{\sin x}{x} = 1.$

3.5. Example. The function $f(x) = x/|x|$ is 1 if $x > 0$ and -1 if $x < 0$.

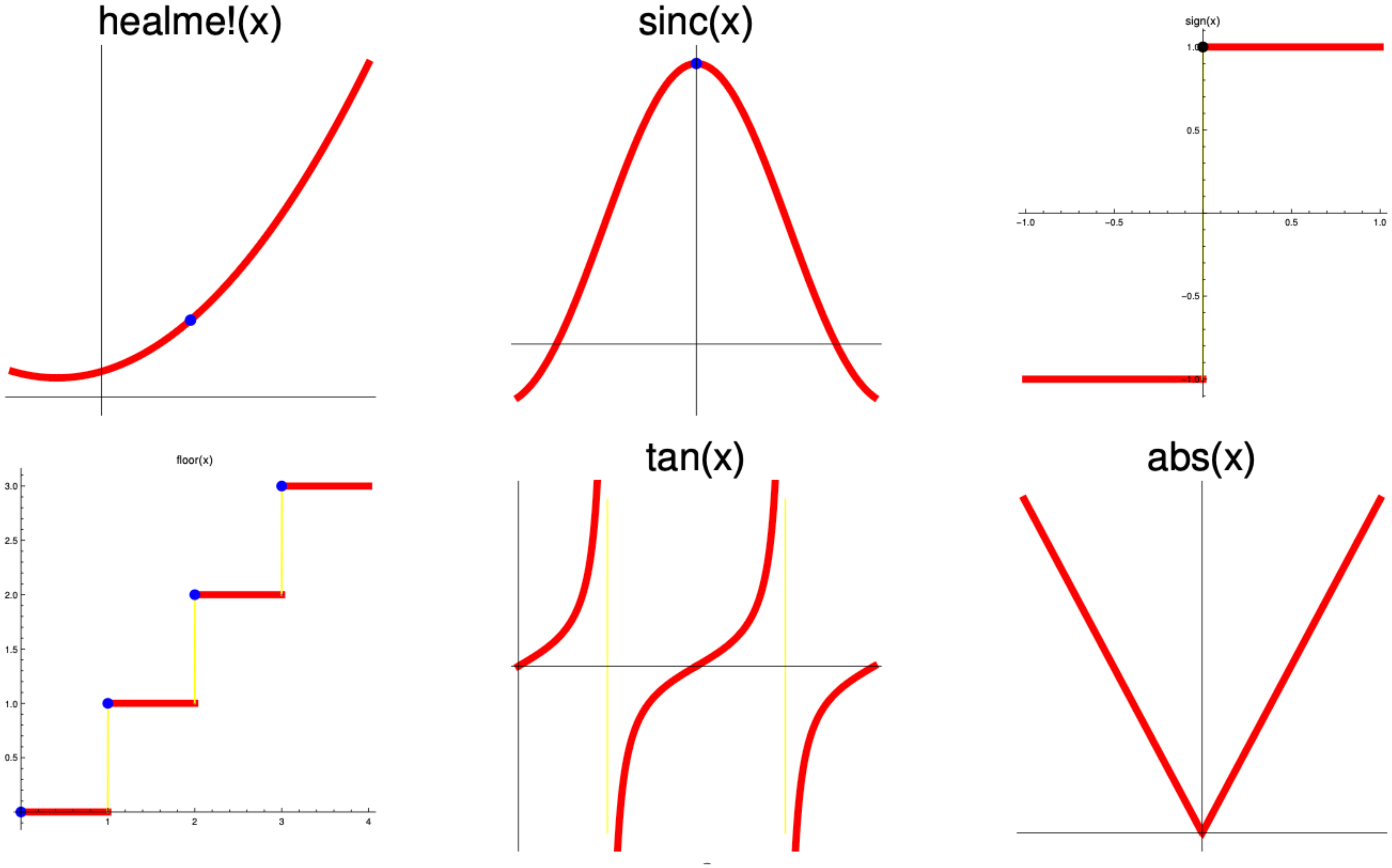
It is not defined at $x = 0$ and there is no way to assign a value b at $x = 0$ in such a way that $\lim_{x\to 0} f(x) = b$.

One could define $f(0) = 0$ and call the function the *sign* **function**.

It is defined everywhere but it is not continuous at 0 as it jumps.

We look at continuity in the next lecture.

Now to illustrate some important functions via graphs.


healme!(x)

sinc(x)

sign(x)

floor(x)

tan(x)

abs(x)

Figure: We see the graphs of f (x) = (x3– 1)/(x– 1), the sinc function sinc(x) = sin(x)/x, the sign function sign(x) = x/|x|, the floor function floor(x) giving the largest integer smaller or equal to x, the tan function and the absolute value function abs(x) = |x|.

3.6. Example. The function $f(x) = \cos(x^2)/(x^4 + 1)$ has the property that $f(x)$ approaches 1 if x approaches 0.

To evaluate functions at 0, there was no need to take a limit because $x^4 + 1$ is never zero.

The function is everywhere defined.

Actually, most functions are nice in the sense that we do not have have to worry about limits at most points.

In the overwhelming cases of real applications we only have to worry about limits when the function involves division by 0.

For example $f(x) = (x4+x2+1)/x$ needs to be investigated more carefully at $x = 0$.

You see for example that for $x = 1/1000$, the function is slightly larger than 1000.

We can simplify it to $x^3 + x + 1/x$ for $x \neq 0$.

There is no limit $\lim_{x\to 0} f(x)$ because $1/x$ has no limit.

3.7. Example. Also, for sin and cos, the limit $\lim_{x\to a} f(x) = f(a)$ is defined.

This extends to trigonometric polynomials like $\sin(3x) + \cos(5x)$.

The function $\tan(x)$ however has no limit at $x = \pi/2$.

No finite value b can be found so that $\tan(\pi/2+h) \to b$for $h \to 0$.

This is due to the fact that $\cos(x)$ is zero at $\pi/2$.

3.8. Example. The cube root function $f(x) = x^{1/3}$ is defined for all x and even $x = 0$.

For the square root function $f(x) = \sqrt{x}$ we have to be aware that for $x < -0$, it is not defined.

The domain of the is function is the positive real axis

Why do we worry about limits?

One of the main reasons will is that we will soon define the derivative and integral using limits.

A second reason is that limits of polynomials lead to function like the exponential function or logarithm function.

An other reason is that one can use limits to define numbers like $\pi = 3.1415926 \dots$

In the next lecture, we also look at the important concept of continuity which refers to limits.

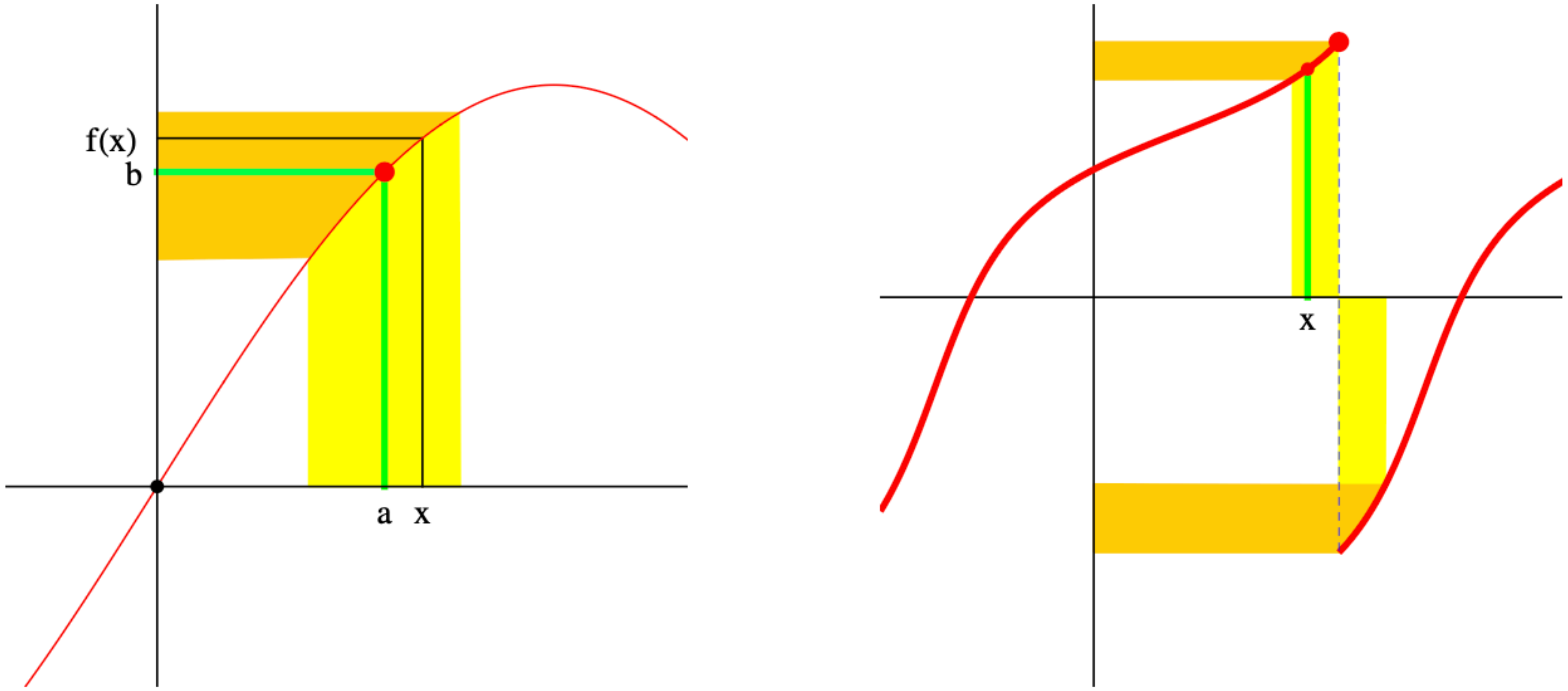


Figure: To the left we see a case, where the limit exists at $x = a$. If x approaches a then $f(x)$ approaches b. To the right we see the function $f(x) = \arctan(\tan(x) + 1)$, where arctan is the inverse of tan. The limit does not exist for $a = \pi/2$. If we approach a from the right, we get the limit $-\pi/2$. From the left, we get the limit $f(\pi/2) = \pi/2$. Note that f is not defined at $x = \pi/2$ because $\tan(x)$ becomes infinite there.

3.9. The following properties hold for limits:

$$\lim_{x\to a} f(x) = b \text{ and } \lim_{x\to a} g(x) = c \text{ implies } \lim_{x\to a} f(x) + g(x) = b + c.$$
$$\lim_{x\to a} f(x) = b \text{ and } \lim_{x\to a} g(x) = c \text{ implies } \lim_{x\to a} f(x)\cdot g(x) = b \cdot c.$$
$$\lim_{x\to a} f(x) = b \text{ and } \lim_{x\to a} g(x) = c \neq 0 \text{ implies } \lim_{x\to a} f(x)/g(x) = b/c.$$

3.10. This implies we can sum up and multiply or divide functions which have limits:

Examples:

Polynomials like $x^5 - 2x + 6$ or trig polynomials like $\sin(3x) + \cos(5x)$ have limits everywhere.

Rational functions like $(x^2 - 1)/(x^2 + 1)$ have limits everywhere if the denominator has no roots.

Functions like $\cos^2(x) \tan(x)/\sin(x)$ can be healed by simplification.

Prototype functions like $\sin(x)/x$ have limits everywhere.

4: Continuity

4.1.

Definition: A function f is continuous at a point x_0 if a value $f(x_0)$ can be found such that $f(x) \to f(x_0)$ for $x \to x_0$.

A function f is continuous on [a, b] if it is **continuous** for every point x in the interval [a, b].

4.2. In the interior (a, b), the limit needs to exist both from the right and from the left.

Intuitively, a function is continuous if one can **draw the graph of the function without lifting the pencil**.

Continuity means that small changes in x results in small changes of $f(x)$.

Some functions like $(x^2 - 1)/(x - 1)$ or $\sin(x)/x$ need to have function values filled in to become continuous.

4.3. Example. Any polynomial like x^3 or trig functions like cos(x), sin(x), exp(x) for example are continuous.

Also the **sum and products** of continuous functions is continuous.

For example, $x^5 + \sin(x^3 + e^x)$ is continuous everywhere.

We can also compose continuous functions like exp(sin(x)) and still get a continuous function.

4.4. The function $f(x) = 1/x$ is continuous except at $x = 0$.

It is a prototype with a pole discontinuity at $x = 0$.

One can draw a **vertical asymptote**.

The **division by zero** kills continuity.

Remember however that this can be salvaged in some cases like $f(x) = \sin(x)/x$ which is continuous everywhere.

The function can be healed at 0 even so it was at first not defined at $x = 0$.

4.5. The logarithm function $f(x) = \log |x|$ is continuous for all $x \neq 0$.

It is not continuous at 0 because $f(x) \to -\infty$ for $|x| \to 0$.

It might surprise you that $f(x) = (1- x2)/ \log |x|$ can be extended to a continuous function.

It is not defined at first at $x = 0$ as $\log |0| = -\infty$.

It is also not defined at $x = 1$ or $x =-1$ at first because $\log(1) = 0$.

But in both cases, we can heal it and see $f(1) = f(-1) = 0$.

The value $f(0) = 0$ is easier to see, but filling in the value $f(1) = f(-1) = -2$ is less obvious.

We will learn later to heal the function at these two points.

It will need hospitalization.

4.6. The co-secant function $\csc(x) = 1/\sin(x)$ is not continuous at $x = 0$, $x= \pi$, $x = 2\pi$ and more generally for any multiple of π.

It has poles there because $\sin(x)$ is zero at those points and because we divide by zero at such points.

The function $\cot(x) = \cos(x)/\sin(x)$ shares the same discontinuity points as $\csc(x)$.

4.7. The function $f(x) = \sin(\pi/x)$ is continuous everywhere except at $x = 0$.

It is a prototype of a function which is not continuous due to oscillation.

We can approach $x = 0$ in ways that $f(x_n) = 1$ and such that $f(z_n) = -1$.

Just pick $x_n = 2/(4k + 1)$ or $z_n = 2/(4k - 1)$.

4.8. The **signum function**

$$f(x) = \text{sign}(x) = \left[\begin{array}{cc} 1 & x > 0 \\ -1 & x < 0 \\ 0 & x = 0 \end{array}\right.$$

is not continuous at 0.

It is a prototype of a function with a jump discontinuity at 0.

There is no way we can make this continuous at 0.

Rules:

a) If f and g are continuous, then $f + g$ is continuous.

b) If f and g are continuous, then $f * g$ is continuous.

c) If f and g are continuous and if $g \neq 0$ everywhere, then f/g is continuous.

d) If f and g are continuous, then $f \circ g(x) = f(g(x))$ is continuous.

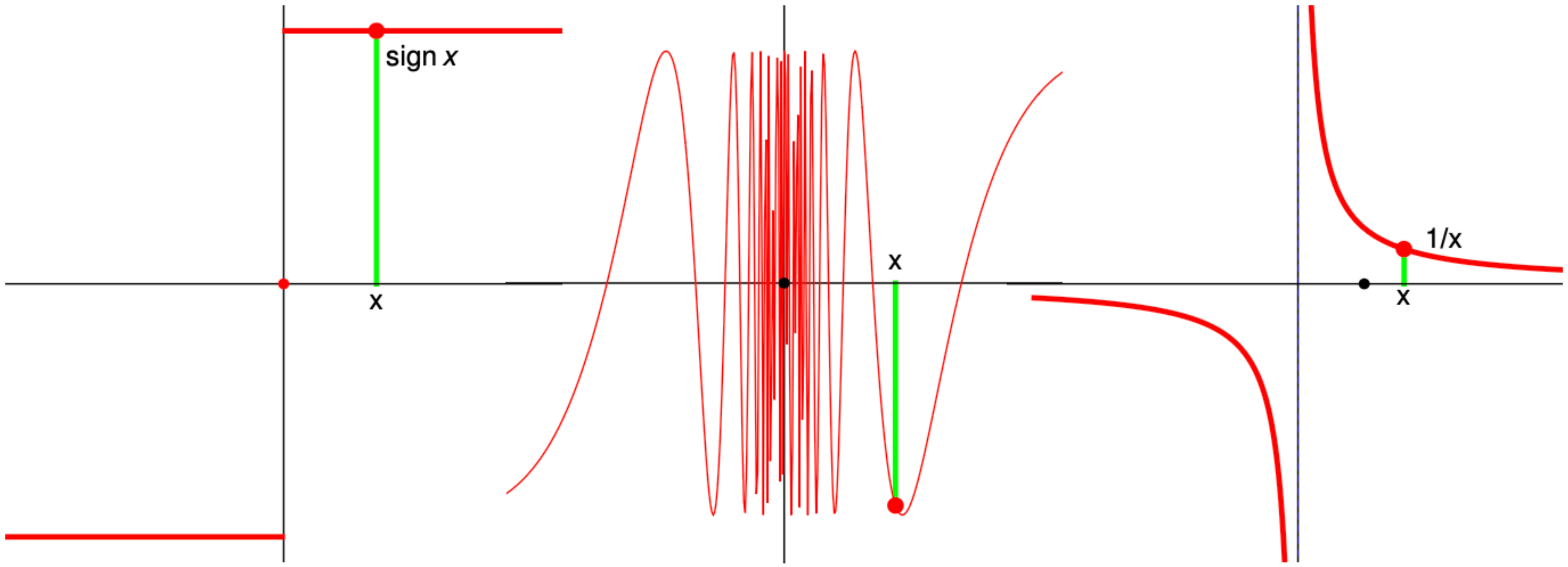
Examples. a) $f(x) = \sqrt{x^2+1}$ is continuous everywhere on the real line. b) $f(x) = \cos(x) + \sin(x)$ is continuous everywhere. c) $f(x) = \log(|x|)$ is continuous everywhere except at 0. d) $f(x) = \sin(\pi x)/\log|x^4|$ is continuous at $x = 0$. Is it continuous everywhere?

Example: The function $f(x) = [\sin(x + h) - \sin(x)]/h$ is continuous for every parameter $h > 0$.

We will see soon what happens when h becomes smaller and smaller and that the continuity will never deteriorate but indeed, we will see $f(x)$ will for smaller and smaller h get to the cos function.

4.9. There are three major reasons, why a function can be not continuous at a point: it can jump, oscillate or escape to infinity.

Here are the prototype examples we will look at more during the lecture.

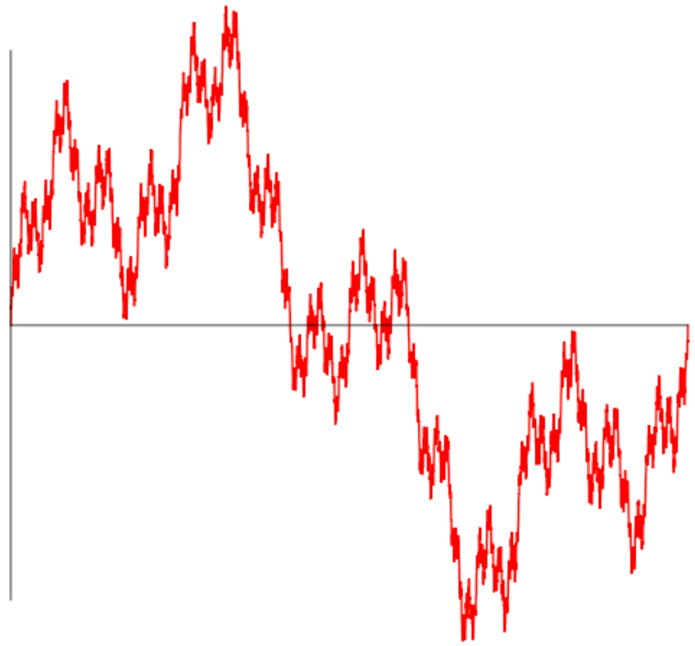


4.10. Why do we like continuity?

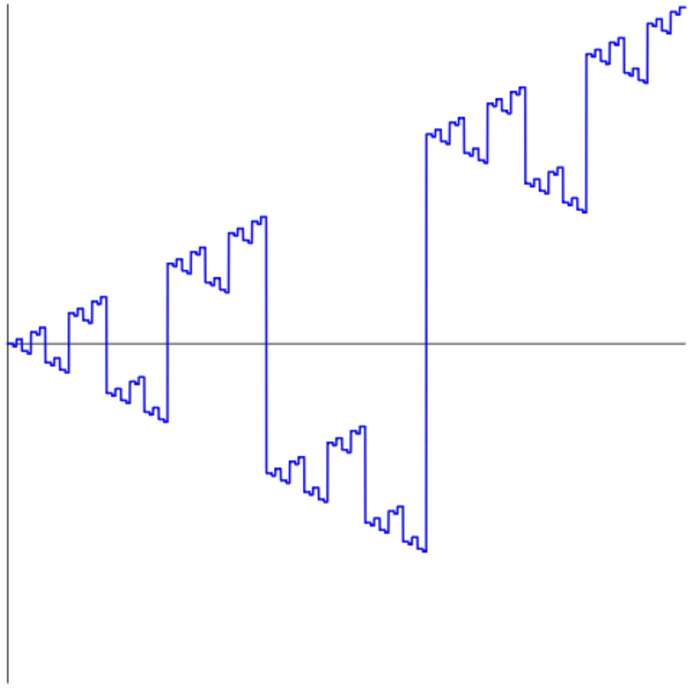
One reason important in applications is that "continuity provides stability and some sort of predictability."

Discontinuities are usually associated to catastrophes.

Discontinuities happen typically, if something breaks.



This Weierstrass function is believed to be a fractal an object of dimension between 1 and 2. But it is continuous.



This function is discontinuous at every point. The vertical connection lines put for clarity are not part of the graph.

4.11. Continuity will be useful when finding maxima and minima.

A continuous function on an interval [a, b] has a maximum and minimum.

We will see shortly that if a continuous function is negative at some place and positive at an other, there is a point between, where it is zero.

Being able to find solutions to equations $f(x) = 0$ is important and much more difficult, if f not continuous.

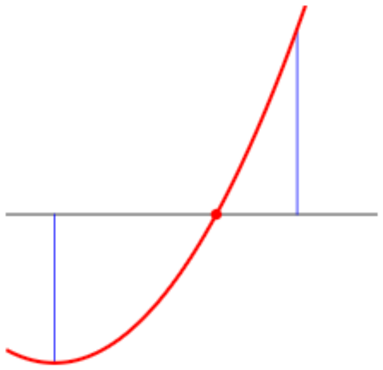
5: Intermediate value theorem

5.1. Finding solutions to $g(x) = h(x)$ is equivalent to find solutions $f(x) = g(x) - h(x) = 0$.

Definition: If $f(a) = 0$, then a is called a **root** of f .

For $f(x) = \sin(x)$ for example, there are roots at $x = 0$, $x = \pi$.

5.2. Intermediate value theorem of Bolzano.



If f is continuous on the interval $[a, b]$ and $f(a), f(b)$ have different signs, then there is a root of f in (a, b).

5.3. The proof is constructive: we can assume $f(a) < 0$ and $f(b) > 0$.

The other case is similar.

Look at $c = (a + b)/2$.

If $f(c) < 0$, then take $[c, b]$ as the new interval, otherwise, take $[a, c]$.

We get a new root problem on a smaller interval.

Repeat the procedure.

After n steps, the search is narrowed to an interval $[u_n, v_n]$ of length $2n(b - a)$.

Continuity assures that $f(u_n) - f(v_n) \to 0$ and that $f(u_n)$, $f(v_n)$ have different signs.

Both point sequences u_n, v_n converge to a root of f.

Example: Verify that the function $f(x) = x^{17} - x^3 + x^5 + 5x^7 + \sin(x)$ has a root.\

Solution. The function goes to $+\infty$ for $x \to \infty$ and to $-\infty$ for $x \to -\infty$.

We have for example $f(10000) > 0$ and $f(-1000000) < 0$.

Then use the theorem.

Definition: derivative(next discussion).

$Df(x) = (f(x + h) - f(x))/h$ is the h-derivative of f

Definition: We call a point p, where $Df(x) = 0$ an **h-critical point**.

We call a point a a **local maximum** if $f(a) \geq f(x)$ in an open interval containing a.

A **local minimum** is a point a , where $f(a) \leq f(x)$.

Example: We can check that the function $f(x) = x(x - h)(x - 2h)$ has the derivative $Df(x) = 3x(x - h)$.

With the notation $[x]^3 = x(x - h)(x - 2h)$ and $[x]^2 = x(x - h)$, we have $D[x]^3 = 3[x]^2$.

Since $[x]^2$ has exactly two roots 0, h, the function $[x]^3$ has 2 critical points.

Example: For $[x]^{n+1} = x(x - h)(x - 2h) \cdots (x - nh)$ we can compute $D[x]^{n+1} = (n + 1)D[x]^n$.

Because $[x]^n$ has exactly n roots, the function $[x]^{n+1}$ has exactly n critical points.

Keep $D[x]^n = n[x]^{n-1}$ in mind!

For the usual derivative it will be true $dx^n/dx = nx^{n-1}$.

Fermat's maximum theorem If f is continuous and has $f(a) = f(b) = f(a + h)$, then f has either a local maximum or local minimum inside the open interval (a, b).

6: Fundamental theorem

Now that we have mentioned a few new ideas and definitions, we can begin calculus.

6.1. Calculus is a theory of differentiation and integration.

We explore here this concept again in a simple setup and practice differentiation and integration without taking limits.

We fix a positive constant h and take differences and sums.

The fundamental theorem of calculus for h = 1 generalizes.

We can then differentiate and integrate polynomials, exponentials and trigonometric functions.

Later, we will do the same with actual derivatives and integrals.

But now, we can work with arbitrary continuous functions.

The constant *h* never pops up.

You can think of it as something fixed, like the Planck constant 1.6×10^{-35}m.

In the standard calculus of Newton and Leibniz the limit $h \to 0$ is taken.

Definition: Given $f(x)$, define the difference quotient

$$Df(x) = \frac{f(x+h) - f(x)}{h}$$

6.2. If f is continuous then Df is a continuous.

For simplicity, we call it “derivative”.

We keep the positive constant h fixed.

As an example, let us take the **constant function** $f(x) = 5$.

We get $Df(x) = (f(x + h) - (f(x))/h = (5 - 5)/h = 0$ everywhere.

You see that in general, if f is a constant function, then $Df(x) = 0$.

6.3. $f(x) = 3x$.

We have $Df(x) = (f(x + h) - f(x))/h = (3(x + h) - 3x)/h$ which is 3 .

You see in general that if $f(x) = mx + b$, then $Df(x) = m$.

For $f(x) = c$ we have $Df(x) = 0$.

For $f(x) = mx + b$, we have $Df(x) = m$.

6.4. For $f(x) = x^2$ we compute $Df(x) = ((x + h)^2 - x^2)/h = (2hx + h^2)/h = 2x + h$

6.5. For $f(x) = \sqrt{x}$ we compute $Df(x) = (\sqrt{x+h} - \sqrt{x})/h$

$$= \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$$

which is $\frac{1}{\sqrt{x+h} + \sqrt{x}}$

For h → 0, we get $\frac{1}{2\sqrt{x}}$.

6.6. Given a function f, define a new function $Sf(x)$ by summing up all values of $f(jh)$, where $0 \leq jh < x$ with $x = nh$.

Definition: Given $f(x)$ define the Riemann sum

$$Sf(x) = h[f(0) + f(h) + f(2h) + \cdots + f((n-1)h)]$$

In short hand, we call Sf also the ”integral” or ”anti-derivative” of f.

It will become the integral in the limit h → 0 later in the class.

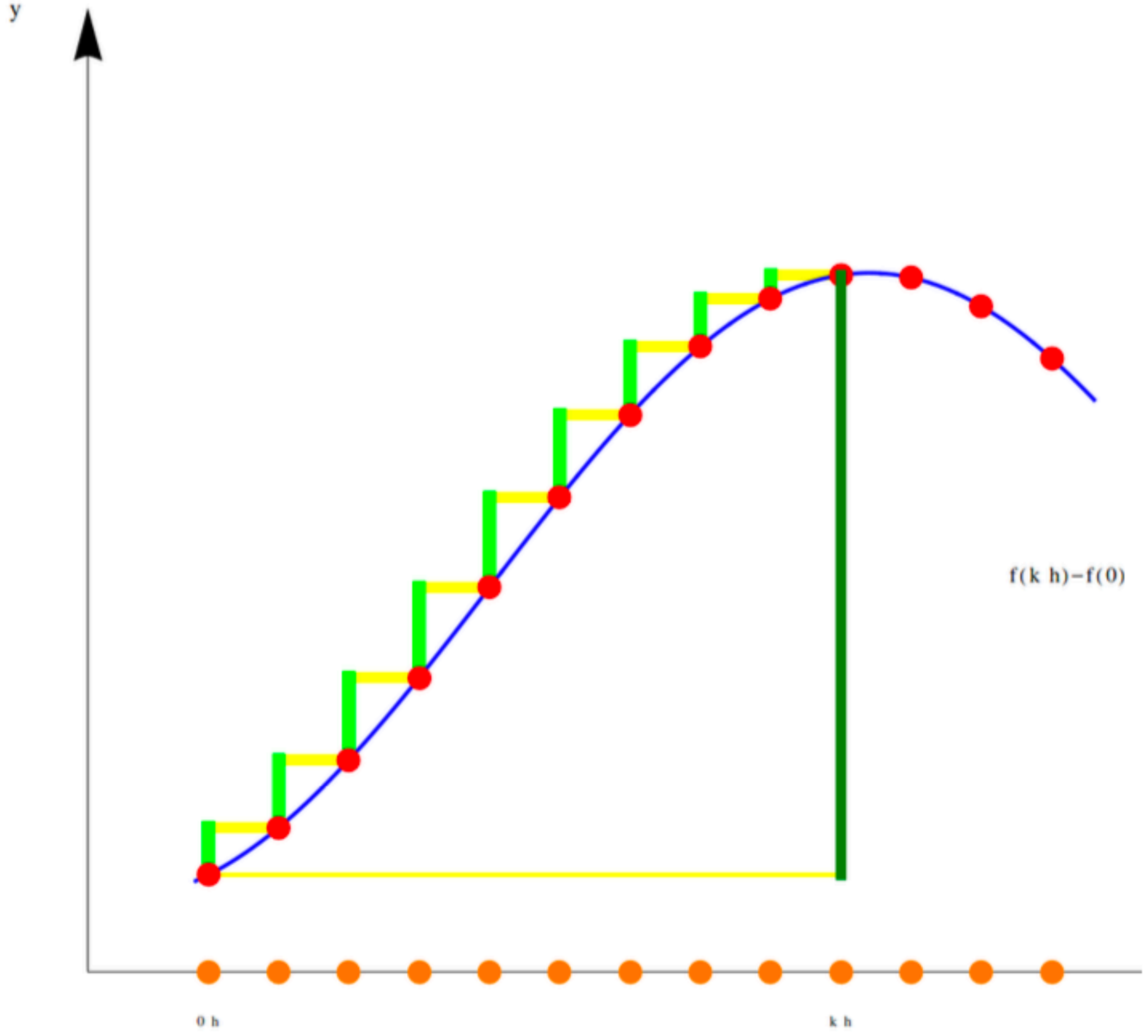
6.7. Example. Compute *Sf*(x) for *f*(x) = 1.

Solution. We have *Sf*(x) = 0 for x ≤ h, and *Sf*(x) = h for h ≤ x < 2h and *Sf*(x) = 2h for 2h ≤ x < 3h.

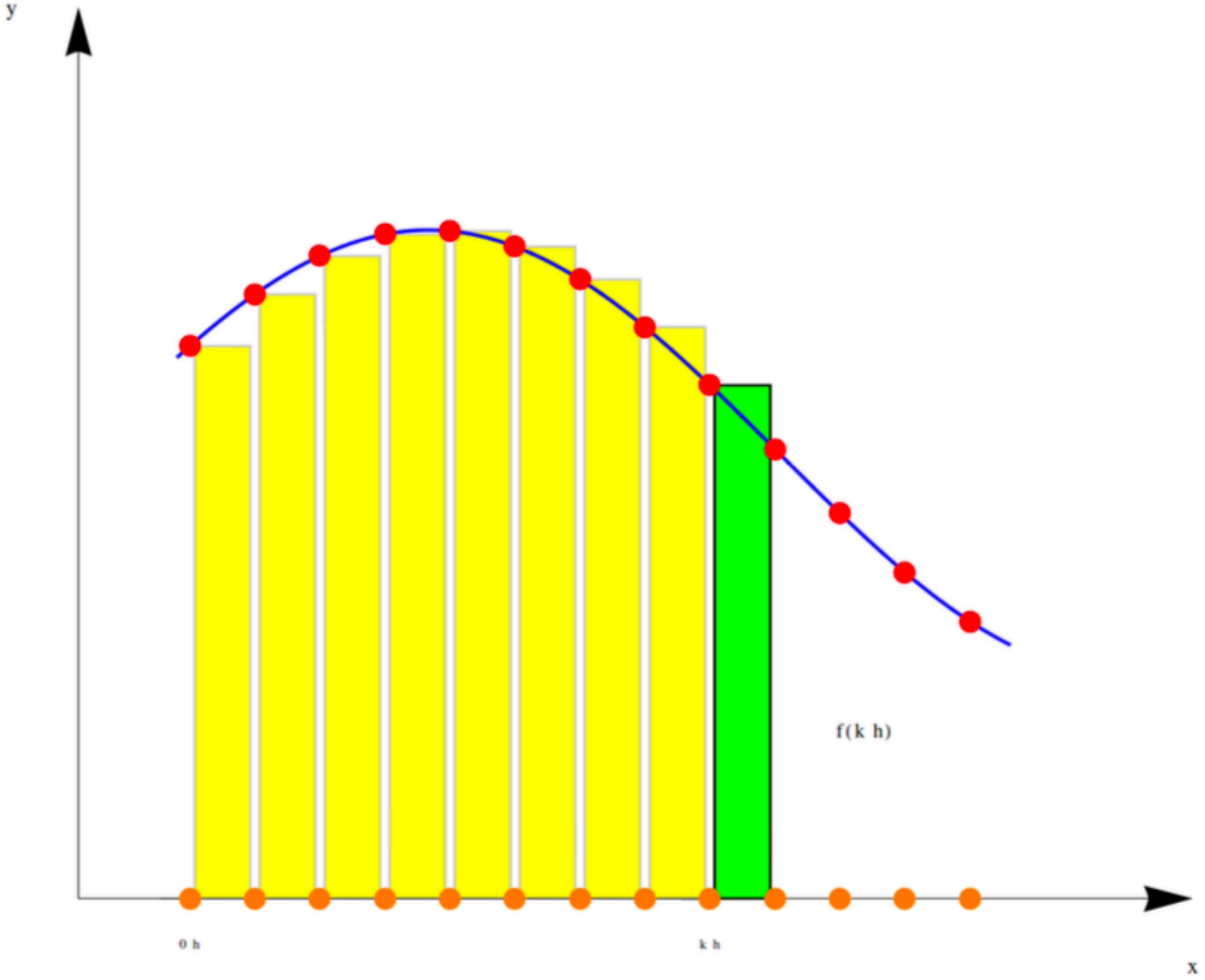
In general *S1*(jh) = j and *S1*(x) = kh where k is the largest integer such that kh < x.

The difference *Df*(x) will become the derivative *f* ′(x).

The sum *Sf*(x) will become the integral $\int_0^x f(t)dt$



Df means rise over run and is close to the slope of the graph of *f*.



Sf means areas of rectangles and is close to the area under the graph of *f*.

6.8. Here is the **quantum fundamental theorem of calculus**

Theorem: Sum the differences gives

$$SDf(kh) = f(kh) - f(0)$$

Theorem: Difference the sum gives

$$DSf(kh) = f(kh)$$

Example: For f(x) = $[x]_h^m$ = x(x− h)(x− 2h)...(x− mh + h) we have

$$f(x+h) - f(x) = (x(x-h)(x-2h)\dots(x-kh+2h))\,((x+h)-(x-mh+h)) = [x]^{m-1}hm$$

and so $D[x]_h^m = m[x]_h^{(m-1)}$.

We have obtained the important formula $D[x]^m = m[x]^{m-1}$.

6.9. This leads to differentiation formulas for **polynomials**.

We will leave away the square brackets later to make it look like the calculus we will do later on.

6.10. If $f(x) = [x] + [x]^3 + 3[x]^5$ then $Df(x) = 1 + 3[x]^2 + 15[x]^4$.

The fundamental theorem allows us to integrate and get $Sf(x) = [x]^2/2 + [x]^4/4 + 3[x]^6/6$.

Definition: Define $\exp_h(x) = (1 + h)^{x/h}$.

It is equal to 2^x for h = 1 and morphs into the function e^x when h goes to zero.

As a rescaled exponential, it is continuous and monotone.

Indeed, using rules of the logarithm we can see $\exp_h(x) = e^{x(\log(1+h)/h)} = e^{xA}$.

It is actually a classical exponential with some constant A.

6.11. The function $\exp_h(x) = (1 + h)^{x/h}$ has the property that its derivative is the function again.

We also have $\exp_h(x + y) = \exp_h(x) \exp_h(y)$.

More generally, define $\exp(a \cdot x) = (1 + ah)^{x/h}$.

It satisfies $D \exp_h(a \cdot x) = a \exp_h(a \cdot x)$.

We write a dot because $\exp_h(ax)$ is not equal to $\exp_h(a \cdot x)$.

For now, only the differentiation rule for this function is important.

6.12. If a is replaced with ai where $i = \sqrt{-1}$, we have $\exp(1 + ia)(1 + aih)^{x/h}$ and still $D \exp_h^{ai} = ai \exp_h^{ai}$

Taking real and imaginary parts define new trig functions $\exp_h^{ai}(x) = \cos_h(a \cdot x) + i \sin_h(a \cdot x)$.

These functions are real and morph into the familiar cos and sin functions for h → 0.

For any $h > 0$ and any a, we have now $D\cos_h(a \cdot x) = -a \sin_h(a \cdot x)$ and $D\sin_h(a \cdot x) = a \cos_h(a \cdot x)$.

We will later derive these identities for the usual trig functions.

6.13.

Definition: Define $\log_h(x)$ as the inverse of $\exp_h(x)$ and $1/[x + a]_h = D\log_h(x + a)$.

6.14. We have directly from the definition $S1/[x + 1]_h = \log_h(x + 1)$

As a consequence we can compute things like

$$S\frac{1}{[3x+3]} = \frac{1}{3}S\frac{1}{[x+1]} = \frac{1}{3}\log_h(x+1)\ .$$

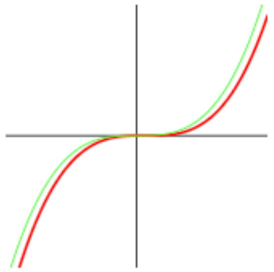
More generally $S(1/[x + a]) = \log(x + a) - \log(a)$.

Fundamental theorem of Calculus: $DSf(x) = f(x)$ and $SDf(x) = f(x) - f(0)$.

Differentiation rules	**Integration rules** (**for** $x = kh$)
$D\, x^n = nx^{n-1}$	$S\, x^n = x^{n+1}/(n + 1)$
$D\, e^{a\cdot x} = ae^{a\cdot x}$	$S\, e^{a\cdot x} = (e^{a\cdot x} - 1)/a$
$D\cos(a \cdot x) = -a\sin(a \cdot x)$	$S\cos(a \cdot x) = \sin(a \cdot x)/a$
$D\sin(a \cdot x) = a\cos(a \cdot x)$	$S\sin(a \cdot x) = (1 - \cos(a \cdot x))/a$
$D\log(1 + xa) = a/(1 + ax)$	$S\, 1/(1+ax) = \log(1 + ax)/a$

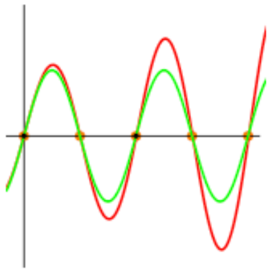
Fermat's extreme value theorem: If *Df*(x) = 0 and *f* is continuous, then *f* has a local maximum or minimum in the open interval (x, x + h).

Pictures

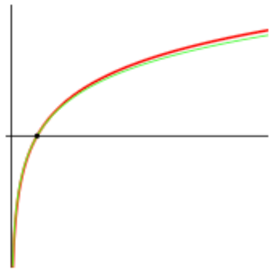


$[x]^3_h$ for $h = 0.1$

$\exp_h(x)$ for $h = 0.1$



$\sin_h(x)$ for $h = 0.1$



$\log_h(x)$ for $h = 0.1$

7: Rate of Change

7**.1.** Given a function *f* and a constant h > 0, we can look at the new function

$$Df(x) = \frac{f(x+h) - f(x)}{h} .$$

It is the average rate of change of the function with step size h.

When changing x to x + h and then *f*(x) changes to *f*(x + h).

The quotient *Df*(x) is a slope and “rise over run”.

Now, we take the limit h → 0.

It is called the instantaneous rate of change.

We derive the important formulas $\frac{d}{dx} x^n = nx^{n-1}$, $\frac{d}{dx} \exp(ax) = a \exp(ax)$, $\frac{d}{dx} \sin(ax) = a \cos(ax)$, $\frac{d}{dx} \cos(ax) = -a \sin(ax)$ which we have seen already before in a discrete setting.

But now we see them also to in the limit $h \to 0$:

Definition: If the limit $\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists, we say *f* is differentiable at the point x.

The value is called the **derivative** or **instantaneous rate of change** of the function *f* at x.

We denote the limit also with $f'(x)$.

7.2. Example. For $f(x) = 30 - x^2$ we have

$$f(x+h) - f(x) = [30 - (x+h)^2] - [30 - x^2] = -2xh - h^2$$

Dividing this by h gives $-2x - h$.

The limit $h \to 0$ gives $-2x$.

We have just seen that for $f(x) = x^2$, we get $f'(x) = -2x$.

For $x = 3$, this is -6.

Example. For $f(x) = |x|$, we have $(f(x + h) - f(x))/h = 1$ if $x > 0$ and $(f(x + h) - f(x))/h= -1$ if x is smaller than $-h$.

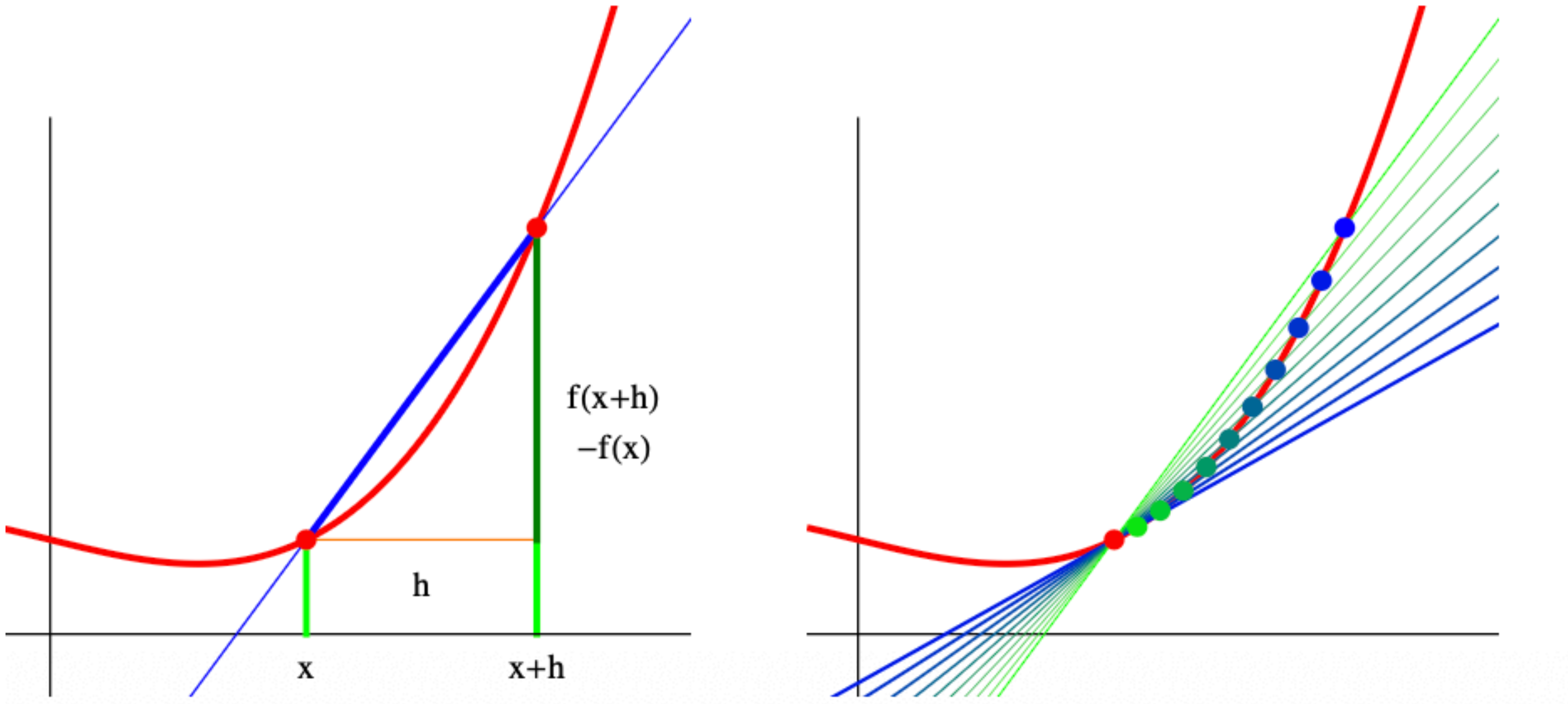
The limit $h \to 0$ does not exist at $x = 0$!

The derivative $f\,'(x)$ has a geometric meaning.

It is the slope of the tangent at x.

This is an important geometric interpretation.

Often, x is “time” and the derivative as the rate of change of the quantity $f\,(x)$ in time



For $f(x) = x^n$, we have $f'(x) = nx^{n-1}$.

Proof: We can use our discrete calculus set-up and note that $[x]^n$ goes to x^n for $h \to 0$.

More traditional is an expansion $f(x+h) - f(x) = (x+h)^n - x^n = (x^n + nx^{n-1}h + a_2h^2 + \cdots + h^n) - x^n = nx^{n-1}h + a_2h^2 + \ldots + h^n)$.

If we divide by h, we get $nx^{n-1} + h(a_2 + \cdots + h^{n-2})$ for which the limit $h \to 0$ exists: it is nx^{n-1}.

This example is important because many functions can be approximated very well with polynomials.

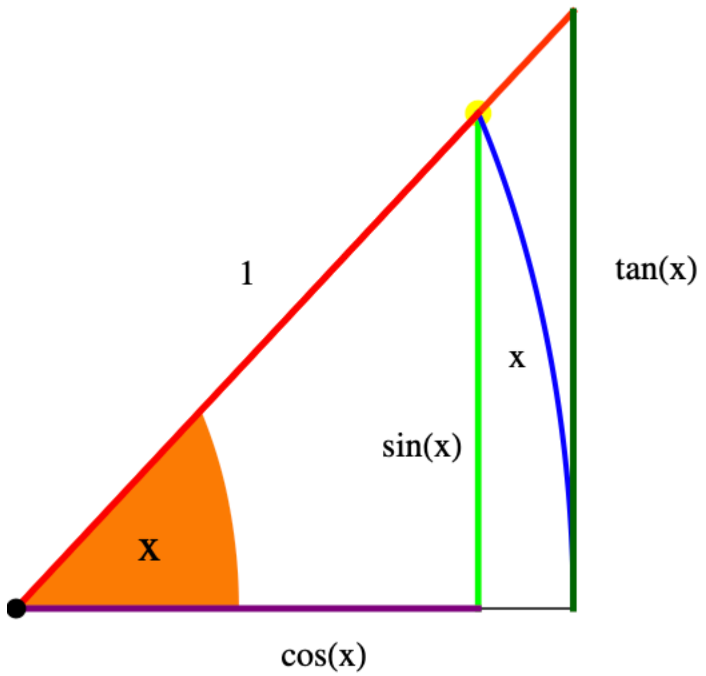
For $f(x) = \sin(x)$ we have $f'(0) = 1$ because the differential quotient is $[f(0 + h) - f(0)]/h = \sin(h)/h = \mathrm{sinc}(h)$.

We have already mentioned that the limit is 1 before.

Lets look at it again geometrically.

For all $0 < x < \pi/2$ we have $\sin(x) \leq x \leq \tan(x)$.

Now divide everything by sin(x).



Because $\tan(x)/\sin(x) = 1/\cos(x) \to 1$ for $x \to 0$, the value of $\mathrm{sinc}(x) = \sin(x)/x$ must go to 1 as $x \to 0$.

Renaming the variable x with the variable h, we have verified the fundamental theorem of trigonometry

7.3. For $f(x) = \cos(x)$ we have $f'(0)$ $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$

To see this, look at $[f(0 + h) - f(0)]/h = [\cos(h) - 1]/h$.

From 2– 2 cos(h) = $\sin^2$(h) + (1– cos(h)$)^2$ which is less than h_2 (geometry!) we have (1– cos(h) ≤ $h^2/2$ so that (1– cos(h))/h ≤ h/2.

We have now

$$\lim_{h\to 0} \frac{(1-\cos(h)}{h} = 0$$

The interpretation is that the tangent is horizontal for the cos function at x = 0.

7.4. From the previous two examples and trig identities we get

$$\cos(x+h)-\cos(x) = \cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x) = \cos(x)(\cos(h)-1)-\sin(x)\sin(h)\ .$$

Now use the just established (cos(h)– 1)/h → 0 and the fundamental theorem of trigonometry sin(h)/h → 1 to see that [cos(x + h)– cos(x)]/h → – sin(x).

For *f*(x) = cos(ax) we have *f* ′(x) = −a sin(ax).

7.5. Similarly,

$$\sin(x+h)-\sin(x) = \cos(x)\sin(h)+\sin(x)\cos(h)-\sin(x) = \sin(x)(\cos(h)-1)+\cos(x)\sin(h)$$

because (cos(h)– 1)/h → 0 and sin(h)/h → 1, we see that [sin(x + h)– sin(x)]/h → cos(x).

for *f*(x) = sin(ax), we have *f* ′(x) = a cos(ax).

$$e^x = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx}$$

7.6. Like π, the Euler number $e = e^1$ is irrational.

Here are the first digits: 2.718281828459045235‌4.

If you want to find an approximation, just pick a large n, like $n = 100$ and compute $(1 + 1/n)^n$.

For $n = 100$ for example, we see $101^{100}/100^{100}$.

We only need to compute the integer 101^{100} and then put a comma after the first digit to get a decent approximation of e.

7.7. To see why the limit exists, verify that the fractions $A_n = (1 + 1/n)^n$ increase and $B_n = (1 + 1/n)^{(n+1)}$ decrease.

Since $B_n/A_n = (1 + 1/n)$ which goes to 1 for $n \to \infty$, the limit exists.

The same argument shows that $(1 + 1/n)^{xn} = \exp_{1/n}(x)$ increases and $\exp_{1/n}(x)(1 + 1/n)$ decreases.

The limiting function $\exp(x) = e^x$ is called the exponential function.

Remember that if we write $h = 1/n$, then $(1 + 1/n)^{nx} = \exp_h(x)$ considered earlier in the class.

We can sandwich the exponential function between $\exp_h(x)$ and $(1 + h) \exp_h(x)$:

$$\exp_h(x) \leq \exp(x) \leq \exp_h(x)(1 + h), \quad x \geq 0.$$

For $x < 0$, the inequalities are reversed.

7.9. It follows from the properties of taking limits that $(f(x) + g(x))' = f'(x) + g'(x)$.

We also have $(a f(x))' = a f'(x)$.

From this, we can now compute many derivatives

7.10. Find the slope of the tangent of $f(x) = \sin(3x) + 5 \cos(10x) + e^{5x}$ at the point $x = 0$.

Solution: $f'(x) = 3 \cos(3x) - 50 \sin(10x) + 5e5x$.

Now evaluate it at $x = 0$ which is $3 + 0 + 5 = 8$.

Finally, lets mention an example of a function which is not everywhere differentiable.

7.11. The function $f(x) = |x|$ has the properties that $f'(x) = 1$ for $x > 0$ and $f'(x) =-1$ for $x = < 1$.

The derivative does not exist at $x = 0$ even though the function is continuous there.

You see that the slope of the graph jumps discontinuously at the point $x = 0$.

7.12. For a function which is discontinuous at some point, we don’t even attempt to differentiate it there.

For example, we would not even try to differentiate $\sin(4/x)$ at $x = 0$ nor $f(x) = 1/x^3$ at $x = 0$ nor $\sin(x)/|x|$ at $x = 0$.

Remember these bad guys?

To the end, you might have noticed that sometimes, more general results have appeared, where x is replaced by ax.

We will look at this again but in general, the relation $f'(ax) = af(ax)$ holds (”if you drive twice as fast, you climb twice as fast”).

8: Derivative Function

8.1. The derivative $f'(x) = \frac{d}{dx}f(x)$ was defined as a limit of $(f(x + h) \to f(x))/h$ for $h \to 0$.

We have seen that $\frac{d}{dx} x^n = nx^{n-1}$ holds for integer n.

We also know already that $\sin'(ax) = a \cos(x)$, $\cos'(ax) = \to a \sin(x)$ and $\exp'(ax) = a \exp(ax)$.

We can now differentiate already a lot of functions and evaluate the derivative $f'(x)$ at a given point x and compute the slope of the graph of f at x.

8.2. Example: Find the derivative $f'(x)$ of $f(x) = \sin(4x) + \cos(5x) - \sqrt{x} + 1/x + x^4$ and evaluate it at $x = 1$.

Solution: $f'(x) = 4 \cos(4x) -5 \sin(5x) - 1/(2 \sqrt{x}) - 1/x^2 +4x^3$.

Plugging in $x = 1$ gives $-\pi - 1/2 - 1 + 4$.

8.3. The differentiation process produces also a rule which assigns to a function f a new function f', the derivative function.

For example, for $f(x) = \sin(x)$, we get $f'(x) = \cos(x)$.

In this lecture, we want to understand the new function and its relation with f.

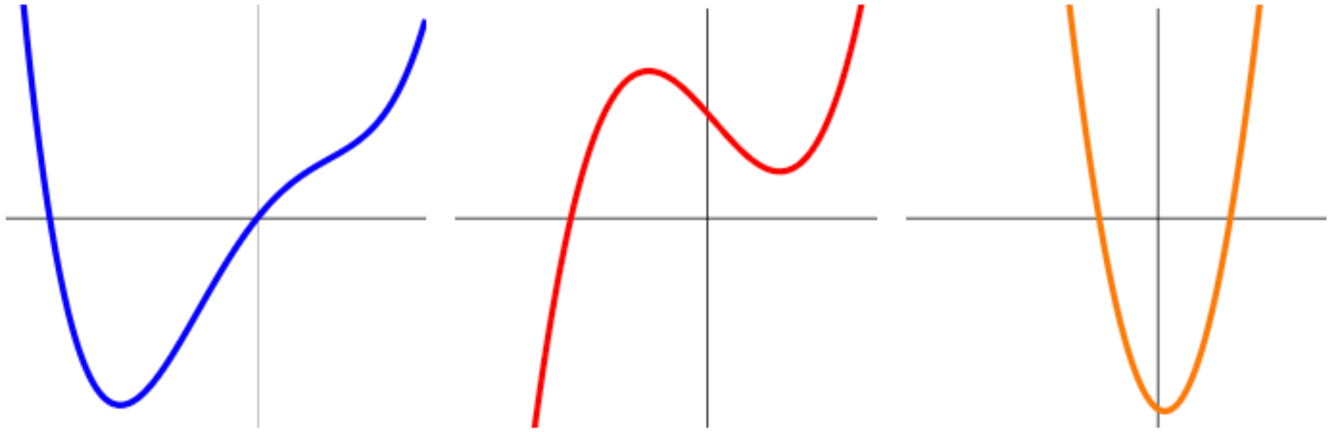
What does it mean if $f'(x) > 0$?

What does it mean that $f'(x) < 0$?

Do the roots of f tell about f' or do the roots of f' tell about f?

8.4. Here is an example of a function f, its derivative f' and the derivative of the derivative f''.

Can you see the relation?



To understand this, it is good to distinguish intervals, where $f'(x)$ is increasing or decreasing.

This are the intervals where $f'(x)$ is positive or negative.

> **Definition:** A function is called **strictly monotonically increasing** on an interval $I = (a, b)$ if $f'(x) > 0$ for all $x \in (a, b)$. It is **strictly monotonically decreasing** if $f'(x) < 0$ for all $x \in (a, b)$.

Monotonically increasing functions “go up” when you “increase x”.

Lets color that:

Example: Can you find a function f on the interval $[0, 1]$ which is bounded $|f(x)| \leq 1$ but such that $f'(x)$ is unbounded?

Definition: Given f(x), we can define g(x) = f'(x) and then take the derivative g' of g. This second derivative f''x) is called the **acceleration**. It measures the rate of change of the tangent slope. For f(x) = x^4, for example we have f''(x) = $12x^2$. If f''(x) > 0 on some interval the function is called **concave up**, if f''(x) < 0, it is **concave down**.

9: Product Rule

9.1. The product rule gives the derivative of a product of functions in terms of the functions and the derivatives of each function.

It is also called **Leibniz rule** named after **Gottfried Leibniz**, who found it in 1684.

It is important because it allows us to differentiate many more functions.

We will be able to compute so the derivative of f(x) = x sin(x) for example without having to take the limit the limit lim(f(x + h) - f(x))/h.

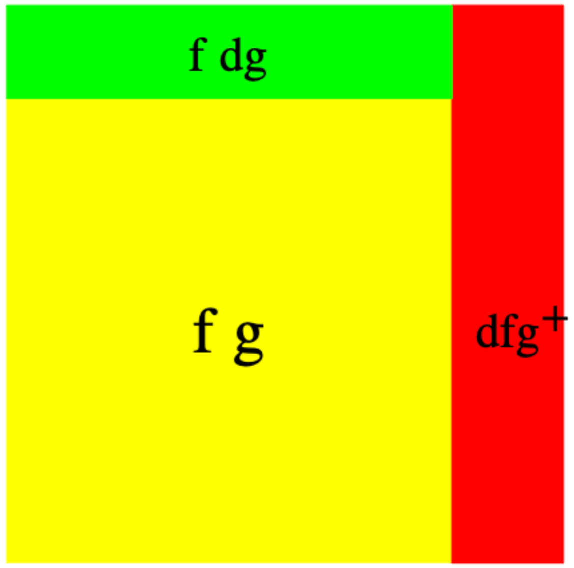
Let us start with an identity which is a discrete Leibniz rule which holds in the **Babylonian calculus** where we do not take limits.

$$f(x+h)g(x+h) - f(x)g(x) = [f(x+h) - f(x)] \cdot g(x+h) + f(x) \cdot [g(x+h) - g(x)] \ .$$

$\frac{d}{dx}$ f f' f'' the the the the

9.2. It can be written as $D(f\,g) = Df\,g^{+} + f\,Dg$ with a shifted function $g^{+}(x) = g(x + h)$ and $Df(x) = [f(x + h) - f(x)]/h$.

This quantum Leibniz rule can also be seen geometrically: the rectangle of area $(f + df)(g + dg)$ is the union of rectangles with area $f \cdot g$, $f \cdot dg$ and $df \cdot g^{+}$.



Now take the limit h → 0:

$$\frac{[f(x+h)-f(x)]}{h} \cdot g(x+h) \to f'(x) \cdot g(x)$$
$$f(x) \cdot \frac{[g(x+h)-g(x)]}{h} \to f(x) \cdot g'(x)$$

9.3. We get the extraordinarily important product rule:

$$\frac{d}{dx}(f(x)g(x)) = f\,'(x)g(x) + f(x)g(x).$$

Example: Find the derivative function $f\,'(x)$ for $f(x) = x^3\sin(x)$.

Solution: We know how to differentiate x^3 and $\sin(x)$ so that $f\,'(x = 3x^2 \sin(x) + x^3 \cos(x)$.

9.4. The **quotient rule** allows to differentiate $f(x)/g(x)$.

Because we can write this as $f(x) \cdot 1/g(x)$, we need only to differentiate $1/g(x)$.

This gives the **reciprocal rule**:

If $g(x) \neq 0$, then

$$\frac{d}{dx}\frac{1}{g(x)} = \frac{-g'(x)}{g(x)^2}$$

9.5. In order to see this, write $h = 1/g$ and differentiate the equation $1 = g(x)h(x)$ on both sides.

The product rule gives $0 = g'(x)h(x) + g(x)h'(x)$ so that $h'(x) = -h(x)g'(x)/g(x) = -g'(x)/g^2(x)$.

This implies that the formula $\frac{d}{dx} x^n = nx^{n-1}$ holds for **all** integers n.

9.6. The **quotient rule** is obtained by applying the product rule to $f(x)\cdot(1/g(x))$ and using the reciprocal rule.

If $g(x) \neq 0$, then $$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{[g(x)f'(x) - f(x)g'(x)]}{g^2(x)}$$

10: Chain rule

10.1. In order to take the derivative of a composition of functions like $f(x) = \sin(x^7)$, the product rule does not work.

The functions are not multiplied but are “chained” in the sense that we evaluate first x^7 then apply the sin function to it.

In order to differentiate, we take the derivative of the x^7 then multiply this with the derivative of the function sin evaluated at x^7 ——- the answer is $7x^6 \cos(x^7)$.

10.2.

$$\frac{f(g(x+h))-f(g(x))}{h}=\frac{[f(g(x)+(g(x+h)-g(x)))-f(g(x))]}{[g(x+h)-g(x)]}\cdot\frac{[g(x+h)-g(x)]}{h}.$$

Write H(x) = g(x+h)-g(x) in the first part on the right hand side

$$\frac{f(g(x+h))-f(g(x))}{h}=\frac{[f(g(x)+H)-f(g(x))]}{H}\cdot\frac{g(x+h)-g(x)}{h}.$$

As h → 0, we also have H → 0 and the first part goes to *f*(g(x)) and the second factor has g (x) as a limit.

10.3. Let us look at some examples.

Example: Find the derivative of *f*(x) = (4x^2– 1)^{17}.

Solution The inner function is g(x) = 4x^2– 1 with derivative 8x.

We get therefore *f*(x) = 17(4x– 1)6 ·8x.

Remark. We could have expanded out the power (4x2– 1)^{17} first and avoided the chain rule.

Try it. You will see that the rule of avoiding the chain rule is called the **pain rule**.

Example: Find the derivative of *f*(x) = sin(π cos(x)) at x = 0.

Solution: applying the chain rule gives cos(π cos(x)) · (–π sin(x)).

10.4. One of the cool applications of the chain rule is that we can compute derivatives of inverse functions:

Example: Find the derivative of the natural logarithm function log(x).

Solution: Differentiate the identity exp(log(x)) = x.

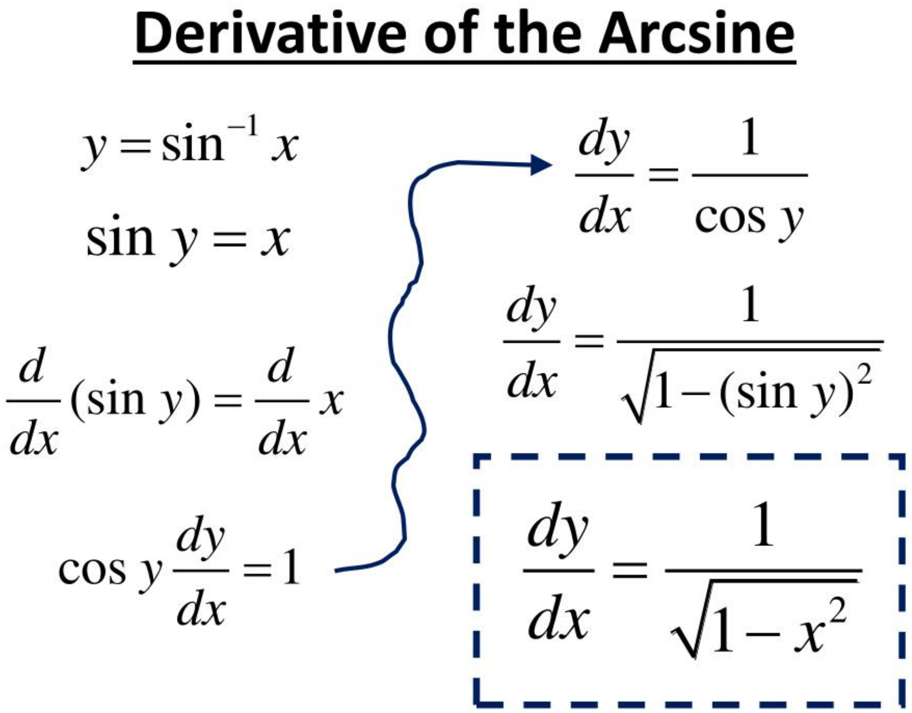
On the right hand side we have 1.

On the left hand side the chain rule gives exp(log(x)) log (x) = x log (x) = 1.

Therefore log′(x) = 1/x.

$$\frac{d}{dx}\log(x) = \frac{1}{x}$$

Definition: Denote by arccos(x) the inverse of cos(x) on [0, π] and with arcsin(x) the inverse of sin(x) on [−π/2, π/2].



Derivative of the Arcsine

$$y = \sin^{-1} x$$

$$\sin y = x$$

$$\frac{d}{dx}(\sin y) = \frac{d}{dx} x$$

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(\sin y)^2}}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}}$$

Thus

$$\frac{d}{dx} arcsin(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx} arccos(x) = -\frac{1}{\sqrt{1-x^2}}$$

Example: $f(x) = \sin(x^2 + 3)$ —> $f'(x) = \cos(x^2 + 3)2x$.

11: Critical Points

11.1. We like to maximize nice quantities and minimize unpleasant ones.

Optimizing quantities is also an important principle which nature follows: laws in physics like Newton's law describing the motion of planets, or the Maxwell's equations describing the propagation of light, or the equations written down by Einstein to describe how matter influences geometric space are based on the principle of extremization.

An important intuitive insight is that at maxima or minima of a function *f* , the tangent to the graph must be horizontal.

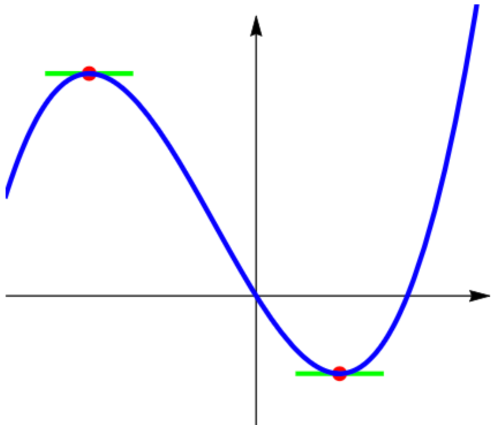
This leads to the following notion for differentiable functions:

Definition: A point x_0 is a *critical point* of *f* if $f'(x_0) = 0$.

Example: Find the critical points of the function $f(x) = x^3 + 3x^2 - 24x$.

Solution: we compute the derivative as $f(x) = 3x^2 + 6x - 24$.

The roots of *f* are 2, –4.



Definition: A point is called a **local maximum** of *f*, if there exists an interval $U = (p - a, p + a)$ around p, such that $f(p) \geq f(x)$ for all $x \in U$. A **local minimum** is a local maximum of $-f$. Local maxima and minima together are called **local extrema**.

Example: The point $x = 0$ is a local maximum for $f(x) = \cos(x)$.

The reason is that $f(0) = 1$ and $f(x) < 1$ nearby.

Example: The point $x = 1$ is a local minimum for $f(x) = (x- 1)^2$.

The function is zero at $x = 1$ and positive everywhere else.

Fermat: If f is differentiable and has a local extremum at x, then $f'(x) = 0$.

11.2. Why is this so?

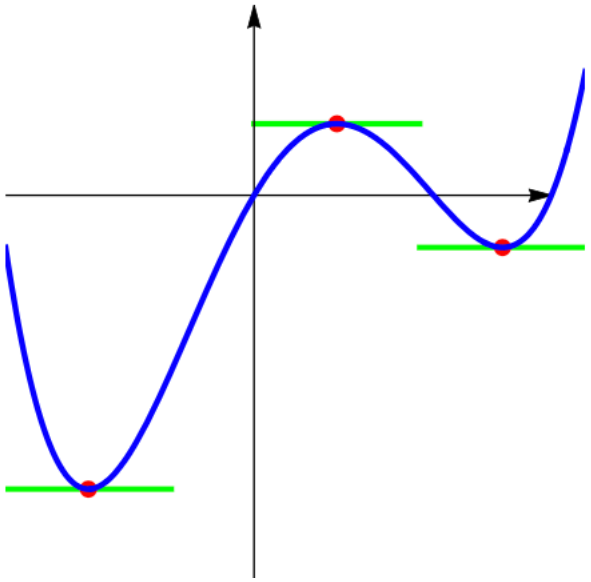
Assume the derivative $f'(x) = c$ is non-zero.

We can assume $c > 0$ otherwise replace *f* with $-f$.

By the definition of limits, for some small enough h, we have $f(x + h)- f(x)/h \geq c/2$.

But this means $f(x + h) \geq f(x) + hc/2$ and x cannot be a local maximum.

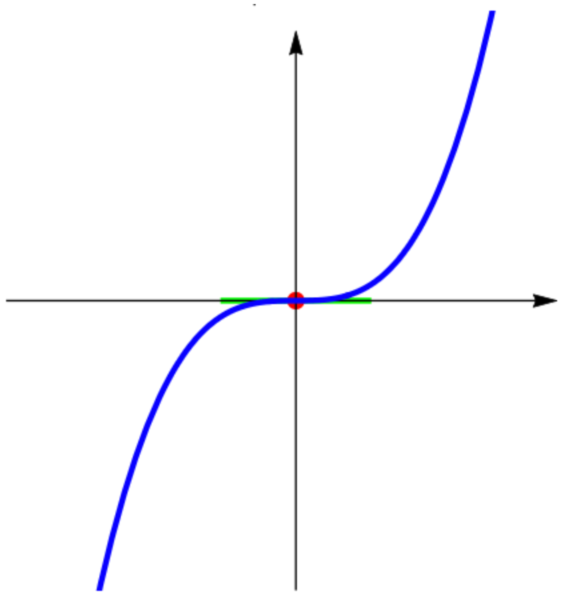
Example: The derivative of $f(x) = 72x - 30x^2 - 8x^3 + 3x_4$ is
$f'(x) = 72 - 60x - 24x^2 + 12x^3$.



By plugging in integers (calculus teachers like integer roots because students like integer roots!) we can guess the roots $x = 1$, $x = 3$, $x = -2$ and see $f(x) = 12(x - 1)(x + 2)(x - 3)$.

The critical points are $1, 3, -2$.

Example: We have already seen that $f'(x) = 0$ does not necessarily imply that x is a local maximum or minimum.

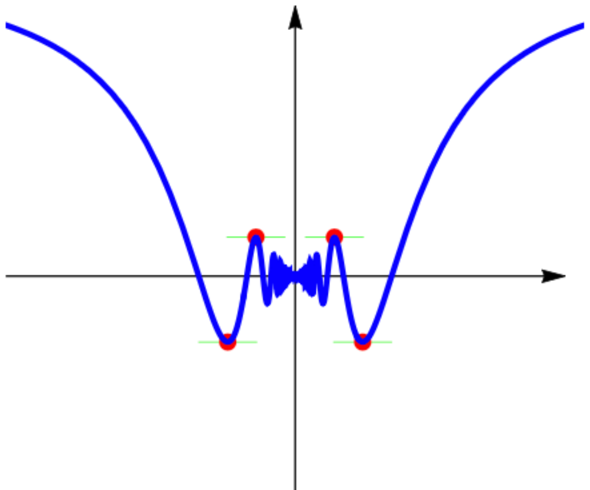


The function $f(x) = x^3$ is a counter example.

It satisfies $f'(0) = 0$ but 0 is neither a minimum nor maximum there.

It is an example of an **inflection** point, which is a point, where the second derivative f changes sign.

Example: The function $f(x) = x \sin(1/x)$ is continuous at $x = 0$ but there are infinitely many critical points near 0.



The function f is not differentiable at 0, the derivative $\sin(1/x) - \cos(1/x)/x$ not only oscillates like crazy at $x = 0$, it also blows up at $x = 0$.

11.3. If $f''(x) > 0$, then the graph of the function is **concave up**.

If $f''(x) < 0$ then the graph of the function is **concave down**.

Second derivative test. If x is a critical point of *f* and $f''((x) > 0$, then *f* is a local minimum.

If $f''(x) < 0$, then *f* is a local maximum.

11.4. If $f''(x_0) > 0$ then $f'(x)$ is negative for $x < x_0$ and positive for $f'(x) > x_0$.

This means that the function decreases left from the critical point and increases right from the critical point.

The point x_0 is a local minimum.

Similarly, if $f''(x_0) < 0$ then $f'(x)$ is positive for $x < x_0$ and $f'(x)$ is positive for $x > x_0$.

This means that the function increases left from the critical point and increases right from the critical point.

The point is a local maximum.

Example: The function $f(x) = x^2$ has one critical point at $x = 0$.

Its second derivative is 2 there.

Example: Find the local maxima and minima of the function $f(x) = x^3 - 3x$ using the second derivative test.

Solution: $f'(x) = 3x^2 - 3$ has the roots $1, -1$.

The second derivative $f''(x) = 6x$ is negative at $x = -1$ and positive at $x = 1$.

The point $x = -1$ is therefore a local maximum and the point $x = 1$ is a local minimum.

Example: Find the critical points of $f(x) = 4 \arctan(x) + x^2$.

Solution. The derivative is

$$f'(x) = \frac{4}{1+x^2} + 2x = \frac{2x + 2x^3 + 4}{1+x^2} \; .$$

We see that $x = -1$ is a critical point.

There are no other roots of $2x + 2x^3 + 4 = 0$.

How do we get the derivative of arctan again?

Differentiate: $\tan(\arctan(x)) = x$ and write $u = \arctan(x)$:

$$\frac{1}{\cos^2(u)} \arctan'(x) = 1 \; .$$

Use the identity $1 + \tan^2(u) = 1/\cos^2(u)$ to write this as

$$(1 + \tan^2(u)) \arctan(x) = 1.$$

But $\tan(u) = \tan(\arctan(x)) = x$ so that $\tan^2(u) = x^2$.

And we have $(1+x^2) \arctan(x) = 1$.

Solving for arctan(x) gives $\arctan(x) = 1/(1+x^2)$.

Fun and games!!!!!!!!!!!

12: Global extrema

12.1. Now we are interested in global maxima.

These are points where the function is maximal overall.

These global extrema can occur either at critical points of *f* or at the boundary of the domain, where both *f* and *f '* are defined.

Definition: A point a is called a **global maximum** of *f* if $f(a) \geq f(x)$ for all x. It is called a **global minimum** of *f* if $f(a) \leq f(x)$ for all x.

12.2. How do we find global maxima?

The answer is simple: make a list of all local extrema and boundary points, then pick the largest.

Global maxima or minima do not need to exist however.

The function $f(x) = x^2$ has a global minimum at $x = 0$ but no global maximum.

The function $f(x) = x^3$ has no global maximum nor minimum at all.

We can however look at global maxima on finite intervals.

To decide about global maxima, just look at the critical points and boundary points and pick the maximal.

Example: Find the global maximum of $f(x) = x^2$ on the interval $[-1, 2]$.

Solution. We look for local extrema at critical points and at the boundary.

Then we compare all, these extrema to find the maximum or minimum.

The critical points are $x = 0$.

The boundary points are $-1, 2$.

Comparing the values $f(-1) = 1, f(0) = 0$ and $f(4) = 4$ shows that f has a global maximum at 2 and a global minimum at 0.

Extreme value theorem of Bolzano: A continuous function *f* on a closed finite interval [a, b] attains a global maximum and a global minimum.

12.3. Note that the global maximum or minimum can also be on the boundary or points where the derivative does not exist:

13: Hospital's rule

13.1. Hospital's rule allows to compute limits - it is a miracle procedure:

Hospital's rule. If *f*, g are differentiable and *f*(p) = g(p) = 0 and g′(p) ≠ 0, then

$$\lim_{x \to p} \frac{f(x)}{g(x)} = \lim_{x \to p} \frac{f'(x)}{g'(x)}$$

Lets see how it works in examples:

Example: Lets prove the fundamental theorem of trigonometry again:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = 1 .$$

In order to apply l'Hospital, we had to know the derivative.

Our work to establish the limit was not in vain.

13.2. The proof of the rule is very simple: since $f(p) = g(p) = 0$ we have $Df(p) = (f(p+h) - f(p))/h = f(p+h)/h$ and $Dg(p) = (g(p+h) - g(p))/h = g(p+h)/h$ so that for every $h > 0$ with $g(p+h) \neq 0$ the **quantum l'Hospital rule** holds:

$$\frac{f(p+h)}{g(p+h)} = \frac{Df(p)}{Dg(p)} \ .$$

Now take the limit $h \to 0$ - done!

13.3. Sometimes, we have to administer l'Hospital twice:

If $f(p) = g(p) = f'(p) = g'(p) = 0$ then $\lim_{x \to p} \frac{f(x)}{g(x)} = \lim_{x \to p} \frac{f'(x)}{g'(x)}$ if the limit to the right exists.

Example: Find the limit $\lim_{x \to 0} (1 - \cos(x))/x^2$.

This limit had been pivotal to compute the derivatives of trigonometric functions.

Solution: differentiation gives

$$\lim_{x \to 0} -\sin(x)/2x \ .$$

Now apply l'Hospital again

$$\lim_{x \to 0} -\sin(x)/(2x) = \lim_{x \to 0} -\cos(x)/2 = -\frac{1}{2} \ .$$

Example: What is 0^0?

What is the limit $\lim_{x\to 0} x^x$?

This will provide the best answer to the question — What is 0^0?

Solution: Because $x^x = e^{x\log(x)}$, it is enough to understand the limit $x\log(x)$ for $x \to 0$.

Now the limit can be seen a $\lim_{x\to 0} \frac{\log(x)}{1/x}$. $1/x^2) = -x$ which goes to 0.

Therefore $\lim_{x\to 0} x^x = 1$.

We assume that $x > 0$ in order to have real values x^x.

If we want a function defined everywhere take $|x|^x$.)

13.5. Hospital's rule always works in calculus situations, where functions are differentiable.

14: Newton method

We were able to find roots of functions using a "divide and conquer" technique: start with an interval [a, b] for which $f(a) < 0$ and $f(b) > 0$.

If $f((a + b)/2)$ is positive, then use the interval $[a, (a+b)/2]$ otherwise $[(a+b)/2, b]$.

After n steps, we are $(b-a)/2n$ close to the root.

If the function *f* is differentiable, we can do better and use the value of the derivative to get closer to a point y= T(x).

Lets find this point y.

If we draw a tangent at (x, *f*(x)) and intersect it with the x-axes, then

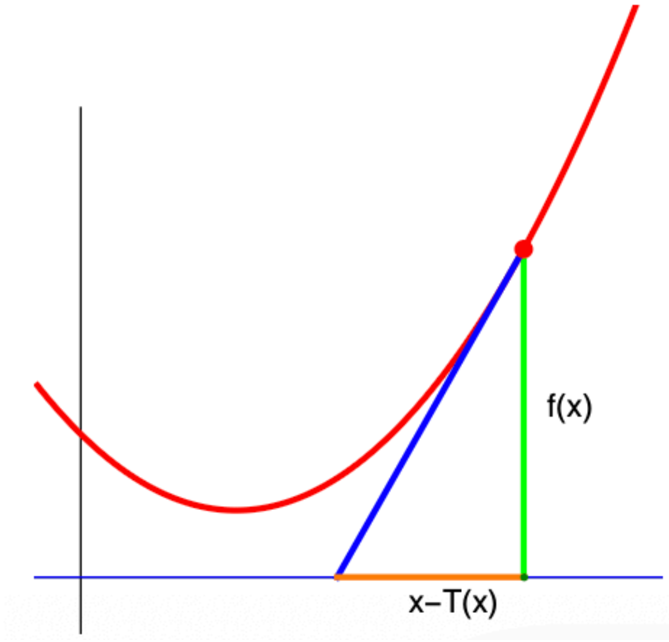
$$f'(x) = \frac{f(x) - 0}{x - T(x)} \, .$$

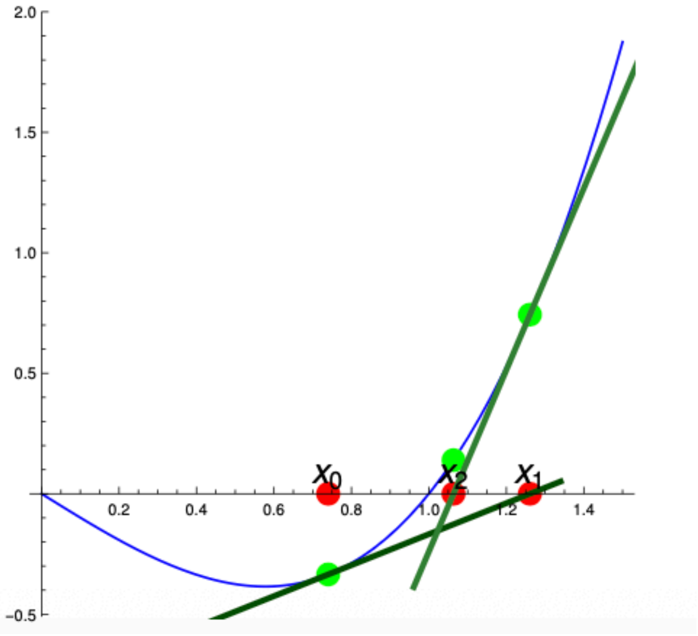
Now, $f'(x)$ is the slope of the tangent and the right hand side is ”rise” over ”run” (see the picture).

If we solve for T(x), we get

Definition: The **Newton map** is defined as

$$T(x) = x - \frac{f(x)}{f'(x)}$$





14.2. The Newton's method applies this map a couple of times until we are sufficiently close to the root: start with a point x, then compute a new point $x_1 = T(x)$, then $x_2 = T(x_1)$ etc.

If p is a root of f such that $f(p) \neq 0$, and x_0 is close enough to p, then $x_1 = T(x)$, $x_2 = T^2(x)$ converges to the root p.

Example: If $f(x) = ax + b$, we reach the root in one step.

Example: If $f(x) = x^2$ then $T(x) = x - x^2/(2x) = x/2$.

We get exponentially fast to the root 0.

In general, the method is much better:

The Newton method converges extremely fast to a root $f(p) = 0$ if $f(p) \neq 0$.

In general, the number of correct digits double in each step.

In 4 steps we expect to have $2^4 = 16$ digits correct.

Having a fast method to compute roots is useful.

For example, in computer graphics, where things can not be fast enough.

We will explore a bit in the lecture how fast the method is.

14.3. If we have several roots, and we start at some point, to which root will the Newton method converge?

Does it at all converge?

This is an interesting question.

It is also historically intriguing because it is one of the first cases, where ”chaos” was observed at the end of the 19’th century.

Example: Lets compute √2 to 12 digits accuracy.

We want to find a root $f(x) = x^2 - 2$.

The Newton map is $T(x) = x - (x^2 - 2)/(2x)$.

Lets start with $x = 1$

$$\begin{aligned} T(1) &= 1 - (1-2)/2 = 3/2 \\ T(3/2) &= 3/2 - ((3/2)^2 - 2)/3 = 17/12 \\ T(17/12) &= 577/408 \\ T(577/408) &= 665857/470832\ . \end{aligned}$$

This is already 1.6×10^{-12} close to the real root!

12 digits, by hand!

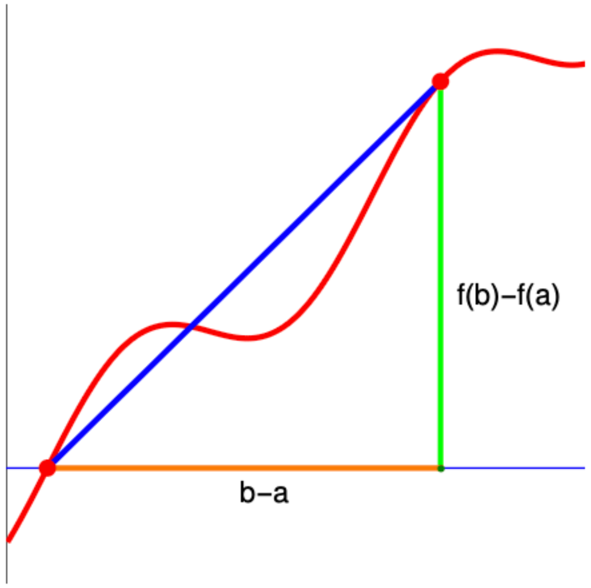
14.5. Unlike the intermediate value theorem which applied for continuous functions, the mean value theorem involves derivatives.

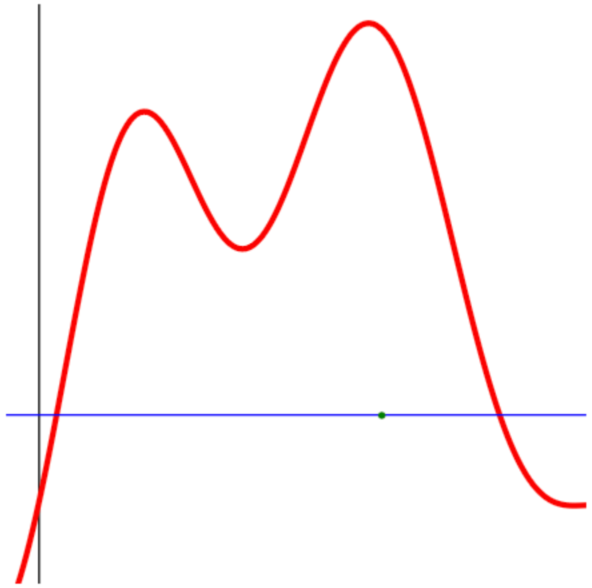
We assume therefore today that all functions are differentiable unless specified.

The mean value theorem can be seen as a consequence of the intermediate value theorem.

Mean value theorem: Any interval (a, b) contains a point x such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$





Rolle's theorem: If *f*(a) = *f*(b) then *f* has a **critical point** in (a, b).

the the the the

15: Review

Major points

f is **continuous** at a if there is $b = f(a)$ such that $\lim_{x \to a} f(x) = b$ for every a. The intermediate value theorem: $f(a) > 0, f(b) < 0$ implies f having a root in (a, b).

$f'(x) = 0, f''(x) > 0$ then x is **local min**. $f'(x) = 0,''(x) < 0$ then x is **local max**. For **global minima or maxima**, compare local extrema and boundary values.

If f changes sign we have a **root** $f = 0$, if f' changes sign, we have a **critical point** $f' = 0$ if f'' changes sign, we have an **inflection points**. A function is **even** if $f(-x) = f(x)$, and **odd** if $f(-x) = -f(x)$.

If $f' > 0$ then f is increasing, if $f' < 0$ it is decreasing. If $f''(x) > 0$ it is **concave up**, if $f''(x) < 0$ it is **concave down**. If $f'(x) = 0$ then f has a horizontal tangent.

Hosptal's theorem applies for indeterminate forms $0/0$ or ∞/∞. In that case, $\lim_{x \to a} f(x)/g(x)$, where $f(a) = g(a) = 0$ or $f(a) = g(a) = \infty$ with $g'(a) \neq 0$ are given by $f'(a)/g'(a)$.

With $Df(x) = (f(x+h) - f(x))/h$ and $S(x) = h(f(0) + f(2h) + \cdots + f((k-1)h))$ we have a **preliminary fundamental theorem of calculus** $SDf(kh) = f(kh) - f(0)$ and $DS(f(kh)) = f(kh)$.

Roots of $f(x)$ with $f(a) < 0, f(b) > 0$ can be obtained by the dissection method by applying the **Newton map** $T(x) = x - f(x)/f'(x)$ again and again.

Algebra reminders

Healing: $(a+b)(a-b) = a^2 - b^2$ or $1 + a + a^2 + a^3 + a^4 = (a^5 - 1)/(a - 1)$
Denominator: $1/a + 1/b = (a+b)/(ab)$
Exponential: $(e^a)^b = e^{ab}$, $e^a e^b = e^{a+b}$, $a^b = e^{b \log(a)}$
Logarithm: $\log(ab) = \log(a) + \log(b)$. $\log(a^b) = b \log(a)$
Trig functions: $\cos^2(x) + \sin^2(x) = 1, \sin(2x) = 2\sin(x)\cos(x), \cos(2x) = \cos^2(x) - \sin^2(x)$
Square roots: $a^{1/2} = \sqrt{a}$, $a^{-1/2} = 1/\sqrt{a}$

Important functions

Polynomials	$x^3 + 2x^2 + 3x + 1$	Exponential	$5e^{3x}$
Rational functions	$(x+1)/(x^3 + 2x + 1)$	Logarithm	$\log(3x)$
Trig functions	$2\cos(3x)$	Inverse trig functions	$\arctan(x)$

Important derivatives

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$f(x) = x^n$	nx^{n-1}	$f(x) = \sin(ax)$	$a\cos(ax)$
$f(x) = e^{ax}$	ae^{ax}	$f(x) = \tan(x)$	$1/\cos^2(x)$
$f(x) = \cos(ax)$	$-a\sin(ax)$	$f(x) = \log(x)$	$1/x$
$f(x) = \arctan(x)$	$1/(1+x^2)$	$f(x) = \sqrt{x}$	$1/(2\sqrt{x})$

Differentiation rules

Addition rule	$(f+g)' = f' + g'$.	Quotient rule	$(f/g)' = (f'g - fg')/g^2$.
Scaling rule	$(cf)' = cf'$.	Chain rule	$(f(g(x)))' = f'(g(x))g'(x)$.
Product rule	$(fg)' = f'g + fg'$.	Easy rule	simplify before deriving

Extremal problems

To maximize or minimze f on an interval $[a,b]$, find all critical points inside the interval, evaluate f on the boundary $f(a), f(b)$ and then compare the values to find the global maximum. No second derivative test at the boundary.

Limit examples

$\lim_{x\to 0}\sin(x)/x$	l'Hospital 0/0	$\lim_{x\to 1}(x^2-1)/(x-1)$	heal
$\lim_{x\to 0}(1-\cos(x))/x^2$	l'Hospital 0/0 twice	$\lim_{x\to\infty}\exp(x)/(1+\exp(x))$	l'Hospital
$\lim_{x\to 0}(1/x)/\log(x)$	l'Hospital ∞/∞	$\lim_{x\to 0}(x+1)/(x+5)$	no work necessary

Important things

Summation and rate of change are at the heart of calculus.
The 3 major types of discontinuities are jump, oscillation, infinity.
Dissection and Newton methods are algorithms to find roots.
The fundamental theorem of trigonometry is $\lim_{x\to 0}\sin(x)/x=1$.
The derivative is the limit $Df(x)=[f(x+h)-f(x)]/h$ as $h\to 0$.
The rule $D(1+h)^{x/h}=(1+h)^{x/h}$ leads to $\exp'(x)=\exp(x)$.
If you forget a derivative like of $\arctan(x)$, use the chain rule.

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16: Riemann Integral

16.1. In this part, we define the definite integral $\int_0^x f(t)dt$ if f is a differentiable function.

We then compute it for some basic functions.

We have previously defined the **Riemann sums**

$$Sf(x) = h[\ f(0) + f(h) + f(2h) + \cdots + f(kh)]\ ,$$

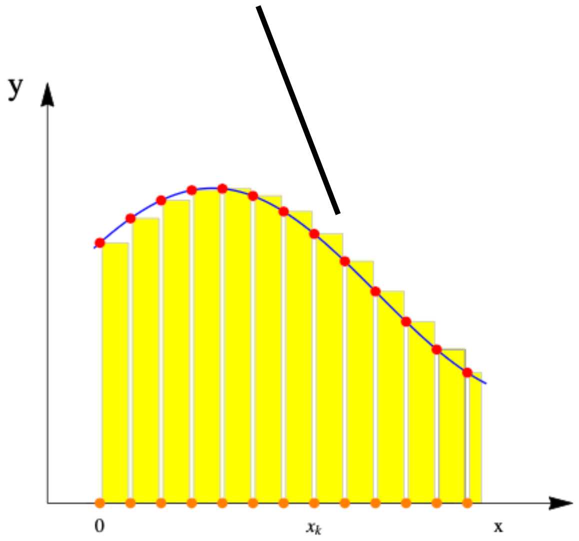
where k is the largest integer such that kh < x.

Lets write S_n if we want to stress that the parameter h = 1/n was used in the sum.

We define the Riemann integral as the limit of these sums $S_n f$, when the mesh size h = 1/n goes to zero.

Definition: Define

$$\int_0^x f(t)dt = \lim_{n \to 0} S_n f(x)$$



16.2. A very important result is that

For any continuous function, the limit exists.

It is easier to see when f is differentiable as one can then estimate the error.

There are n little pieces which are each of area ≤ M/n, where M is the maximal slope that f can have in the given interval.

For non-negative f, the value $\int_0^x f(t)dt$ is the a**rea between the x-axis and the graph** of f. For general f, it is a **signed area**, the difference between two areas.

16.3. The Riemann integral is the limit $h \sum_{x_k = kh \in [0,x)} f(x_k)$.

It converges to the area under the curve for all continuous functions.

16.4. Riemann also looked at points $x_0 < x_1 < \cdots < x_n$ $[0, x]$ such that the maximal distance $(x_{k+1} - x_k)$ between neighboring x_j goes to zero.

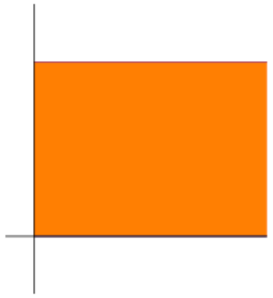
The Riemann sum is then $S_n f = \sum_k f(y_k)(x_{k+1} - x_k)$ (see image), where y_k is arbitrarily chosen inside the interval (x_k, x_{k+1}).

For continuous functions, the limiting result is the same the $Sf(x)$ sum done here.

There are numerical reasons to allow more general partitions because it allows to adapt the mesh size: use more points where the function is complicated.

Example: If $f(x) = c$ is constant, then $\int_0^x f(t)dt = cx$.

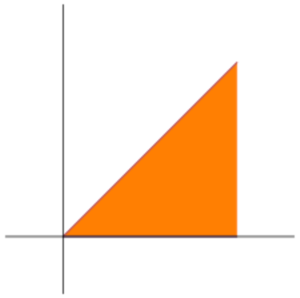
We can see also that $cnx/n \leq S_n f(x) \leq c(n+1)x/n$.



Example: Let $f(x) = cx$.

The area is half of a rectangle of width x and height cx so that the area is $cx^2/2$.

Adding up the Riemann sum is more difficult.



Let k be the largest integer smaller than $xn = x/h$.

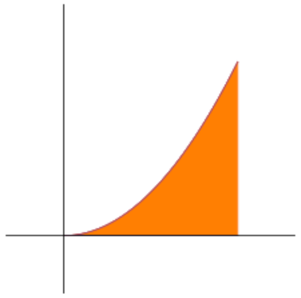
Then

$$S_n f(x) = \frac{1}{n} \sum_{j=1}^{k} \frac{cj}{n} = \frac{ck(k+1)/2}{n^2} .$$

Taking the limit $n \to \infty$ and using that $k/n \to x$ shows that $\int_0^x f(t)\,dt = cx^2/2$.

Example: Let $f(x) = x^2$.

In this case, we can not see the numerical value of the area geometrically.



But since we have computed $S[x^2]$ earlier in this class and seen that it is $[x^3]/3$ and since we have defined $S_h f(x) \to \int_0^x f(t)\,dt$ for $h \to 0$ and $[x^k] \to x^k$ for $h \to 0$, we know that

$$\int_0^x t^2 \, dt = \frac{x^3}{3} \, .$$

This example actually computes the **volume of a pyramid** which has at distance t from the top an area t^2 cross section.

Think about t^2dt as a slice of the pyramid of area t_2 and height dt.

Adding up the volumes of all these slices gives the volume.

Linearity of the integral

$$\int_0^x (f(t) + g(t))dt = \int_0^x f(t)dt + \int_0^x g(t)dt$$

$$\int_0^x \lambda f(t)dt = \lambda \int_0^x f(t)dt$$

Upper bound:

If $0 \le f(x) \le M$ for all x, then $\int_0^x f(t)dt \le Mx$

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16.5 We see that if two functions are close then their difference is a function which is included in a small rectangle and therefore has a small integral:

If f and g satisfy $|f(x) - g(x)| \leq c$, then $\int_0^x |f(x) - g(x)|dt \leq cx$

16.6 It is easy to show identities like $S_n[x]_h^n = \frac{[x]_h^{n+1}}{n+1}$ and $S_n \exp_h(x) = \exp_h(x)$.

Since $[x]_h^k - [x]^k \to 0$ we have $S_n[x]_h^k - S_n[x]^k \to 0$ and from $S_n[x]_h^k = [x]_h^{k+1}/(k+1)$.

The other equalities are the same since $\exp_h(x) = \exp(x) \to 0$

This gives us:

$$\int_0^x t^n dt = \frac{x^{n+1}}{n+1} \qquad \int_0^x \cos(t)dt = \sin(x)$$

$$\int_0^x e^t dt = e^x - 1 \qquad \int_0^x \sin(t)dt = 1 - \cos(x)$$

17: Fundamental theorem

17.1 The fundamental theorem of calculus for differentiable functions will allow us to compute many integrals nicely.

Earlier in the class, we saw that if $Sf(x) = h(f(0) + \cdots + f(kh))$ and $Df(x) = (f(x + h) - f(x))/h$ then we have $SDf = f(x) - f(0)$ and $DSf(x) = f(x)$ if $x = nh$.

This now becomes the **fundamental theorem.**

It assumes that f' must be continuous.

$$\int_0^x f'(t)dt = f(x) - f(0) \quad \text{and} \quad \frac{d}{dx}\int_0^x f(t)dt = f(x)$$

Proof. Using notation of Euler, we write $A \sim B$.

We say "A and B are close" and mean that $A - B \to 0$ for $h \to 0$.

From $DSf(x) = f(x)$ for $x = kh$ we have $DSf(x) \sim f(x)$ for $kh < x < (k + 1)h$ because f is continuous.

We also know $\int_0^x Df(t)\,dt \sim \int_0^x f'(t)\,dt$ because $Df(t) \sim f'(t)$ uniformly for all $0 \leq t \leq x$ by the definition of the derivative and the assumption that f' is continuous on the bounded interval.

We also know $SDf(x) = f(x) - f(0)$ for $x = kh$.

By definition of the Riemann integral, $Sf(x) \sim \int_0^x f(t)\,dt$ and so $SDf(x) \sim \int_0^x Df(t)\,dt$.

$$f(x) - f(0) \sim SDf(x) \sim \int_0^x Df(t)\,dt \sim \int_0^x f'(t)\,dt$$

as well as

$$f(x) \sim DSf(x) \sim D\int_0^x f(t)\,dt \sim \frac{d}{dx}\int_0^x f(t)\,dt\ .$$

Example: $\int_0^5 x^7\,dx = \frac{x^8}{8}\Big|_0^5 = \frac{5^8}{8}$.

You can always leave such expressions as your final result.

It is even more elegant than the actual number 390625/8.

Example: $\int_0^{\pi/2} \cos(x)\,dx = \sin(x)|_0^{\pi/2} = 1$.

Example:

The example $\int_2^3 2/(t^2-1)\,dt$ is challenging for now.

We need a hint and write $2/(x^2-1) = 1/(x-1) - 1/(x+1)$.

The function $F(x) = \log|x-1| - \log|x+1|$ has therefore $f(x) = 2/(x^2 - 1)$ as a derivative.

The answer is

$$\int_2^3 2/(t^2 - 1)\ dt = F(3) - F(2) = \log(2) - \log(4) - \log(1) + \log(3) = \log(3) - \log(2) = \log(3/2).$$

We give reformulations of the fundamental theorem in ways in which it is mostly used: If *f* is the derivative of a function *F* then

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$$

Here is a version of the fundamental theorem, where the boundaries are functions of x.

Given functions g, h and if F is a function such that $F' = f$, then

$$\int_{h(x)}^{g(x)} f(t)dt = F(g(x)) - F(h(x))$$

The function *F* is called an *anti-derivative*.

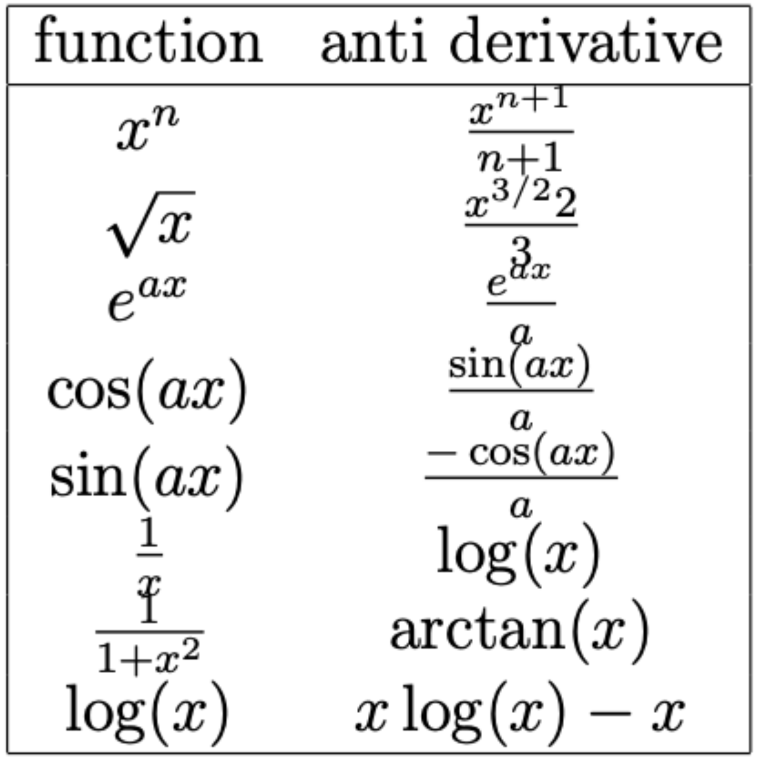
It is not unique but the above formula does always give the right result.

Lets make a list

You should have as many anti-derivatives “hard wired” in your brain.

It really helps.

Here are the core functions you should know.



function	anti derivative
x^n	$\frac{x^{n+1}}{n+1}$
$\sqrt{x}$	$\frac{x^{3/2}2}{3}$
e^{ax}	$\frac{e^{ax}}{a}$
$\cos(ax)$	$\frac{\sin(ax)}{a}$
$\sin(ax)$	$\frac{-\cos(ax)}{a}$
$\frac{1}{x}$	$\log(x)$
$\frac{1}{1+x^2}$	$\arctan(x)$
$\log(x)$	$x\log(x) - x$

18: Anti-derivatives

18.1. The definite integral $\int_a^b f(t)\ dt$ represents a signed area under the curve.

We say “signed” because the area of the region below the curve is counted negatively.

There is something else to mention:

Definition: For every C, the function $F(x) = \int_0^x f(t)dt + C$ is called an **anti-derivative** of g. The constant C is arbitrary and not fixed.

18.2. The fundamental theorem of calculus assured us that

> The anti derivative gives us from a function *f* a function *F* which has the property that $F' = f$. Two different anti derivatives *F* differ only by a constant.

18.3. Finding the anti-derivative of a function is in general harder than finding the derivative.

We will learn some techniques but it is in general not possible to give antiderivatives for a function, if it looks simple.

Example: Find the anti-derivative of $f(x) = \sin(4x) + 20x^3 + 1/x$.

Solution: We can take the anti-derivative of each term separately.

It is $F(x) = -\cos(4x)/4 + 4x_4 + \log(x) + C$.

Example: Find the anti-derivative of $f(x) = 1/\cos^2(x) + 1/(1 - x)$.

Solution: we can find the anti-derivatives of each term separately and add them up.

The result is $F(x) = \tan(x) + \log|1 - x| + C$.

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19: Area

19.1. If *f*(x) ≥ 0, then $\int_a^b f(x)dx$ is the area under the graph of *f*(x) and above the interval [a, b] on the x-axis.

If the function is negative, then $\int_a^b f(x)dx$ is negative too and the integral is minus the area below the curve:

Therefore, $\int_a^b f(x)dx$ is the difference of the area above the graph minus the area below the graph.

We call it a **signed area**.

19.2. More generally we can also look at areas sandwiched between two graphs *f* and g.

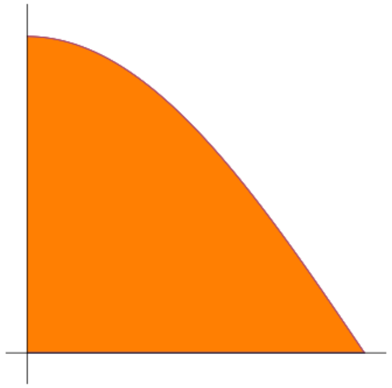
The area of a region G enclosed by two graphs *f* ≤ g and bounded by a ≤ x ≤ b is

$$\int_a^b (g(x) - f(x))dx$$

19.3. Make sure that if you have to compute such an integral that g ≥ *f* before giving it the interpretation of an area.

Example: Find the area of the region bound by the cos function and the x and y axes.

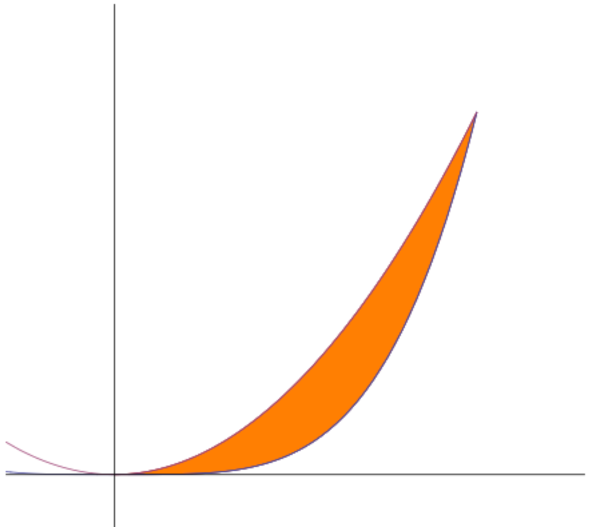
Solution: $\int_0^{\pi/2} \cos(x)dx = 1$



Example: Find the area of the region enclosed by the graphs $f(x) = x^2$ and $f(x) = x^4$.

Solution:

$$\int_0^1 (x^2 - x^4)dx = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$$

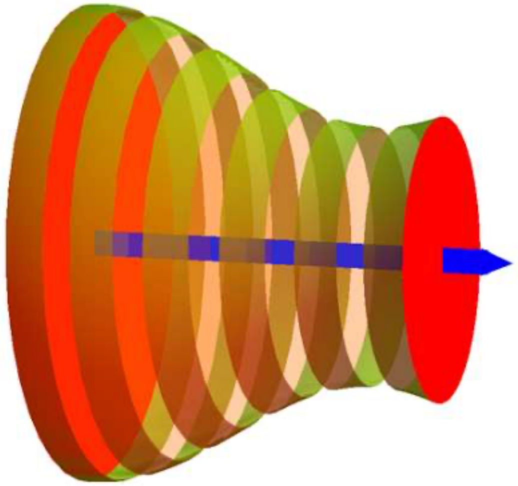


20: Volume

20.1. To compute the volume of a solid, one can cut it into slices, so that each slice is perpendicular to a given line x.

If *A*(x) is the area of the slice and the body is enclosed between a and b then

$$V = \int_a^b A(x)\, dx$$



is the **volume** of the body.

The integral adds up *A*(x)dx, the volume of the slices.

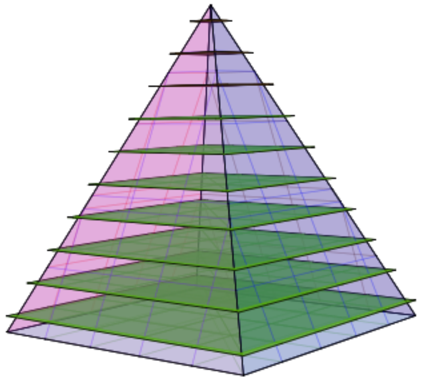
Example: Compute the volume of a pyramid with square base length 2 and height 2.

Solution: we can assume the pyramid is built over the square $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$.

The cross section area at height h is *A*(h) = (2− h)2.

Therefore,

$$V = \int_0^2 (2-h)^2 \, dh = \frac{8}{3} \, .$$



This is base area 4 times height 2 divided by 3.

Definition: A **solid of revolution** is a surface obtained by rotating the graph of a function *f*(x) around the x-axis.

The area of the cross section at x of a solid of revolution is $A(x) = \pi f(x)^2$.

The volume of the solid is $\int_a^b \pi f(x)^2 dx$

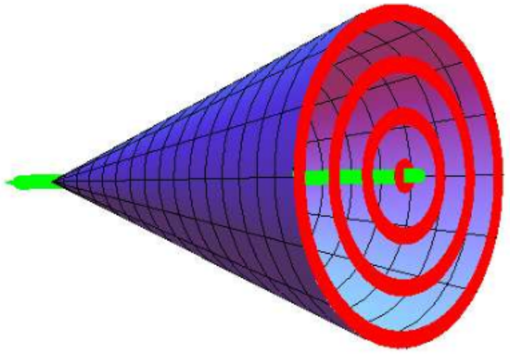
Example: Find the volume of a round cone of height 2 and where the circular base has the radius 1.

Solution. This is a solid of revolution obtained by rotation the graph of $f(x) = x/2$ around the x axis.

The area of a cross section is $\pi x^2/4$.

Integrating this up from 0 to 2 gives

$$\int_0^2 \pi x^2/4 \, dx = \frac{x^3}{4 \cdot 3}|_0^2 = \frac{2\pi}{3} \; .$$



This is the height 2 times the base area π divided by 3.

21: Improper Integrals

21.1. Integrals on infinite intervals or integrals with a function becoming infinite at some point are called **improper integrals.**

The area under the curve can either remain finite or become infinite.

Here is an example, where the value is finite:

Example: What is the integral $\int_1^\infty \frac{1}{x^4} \, dx$?

Since the anti-derivative is $-1/(3x^3)$, we have

$$\frac{-1}{3x^3}|_1^\infty = -1/(3\infty) + (1/3) = 1/3 \; .$$

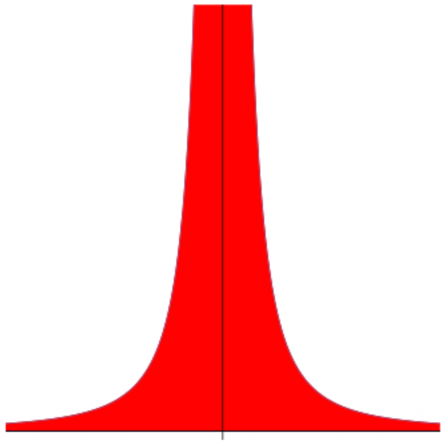
To justify this, compute the integral $\int_1^b 1/x^4 dx = (1/3 - 1/(3b^3))$ and see that in the limit b → ∞, the value 1/3 is achieved.

21.2. A shocking example is the following.

It is a case where things go wrong.

Example:

$$\int_{-1}^{1} \frac{1}{x^2}\, dx = -\frac{1}{x}\Big|_{-1}^{1} = -1 - 1 = -2\ .$$



This does not make any sense because the function is positive so that the integral should be a positive area.

The problem is this time not at the boundary −1, 1.

The sore point is x = 0 over which we have carelessly integrated over.

22: Integration by parts

22.1. Integrating the product rule (uv)′ = u′v + uv′ gives the method **integration by parts**.

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$$

Example: To see how integration by parts work, lets try to find $\int x\sin(x)\ dx$.

First identify what you want to differentiate and call it u, the part to integrate is called v.

Now, write down uv and subtract a new integral which integrates u′v :

$$\int x \, \sin(x) \; dx = x \, (-\cos(x)) - \int 1 \, (-\cos(x) \,) \; dx = -x\cos(x) + \sin(x) + C \; dx \; .$$

Example: Find $\int xe^x \; dx$.

Solution. You want to differentiate x and integrate e^x.

$$\int x \, \exp(x) \; dx = x\exp(x) - \int 1 \cdot \exp(x) \; dx = x\exp(x) - \exp(x) + C \; dx \; .$$

Example: Find $\int \log(x) \; dx$.

Solution. While there is only one function here, we need two to use the method.

Let us look at log(x) · 1:

$$\int \log(x) \; 1 \; dx = \log(x)x - \int \frac{1}{x} x \; dx = x\log(x) - x + C \; .$$

23: Trig Substitution

23.1. A trig substitution is a substitution, where x is a trigonometric function of u or u is a trigonometric function of x.

Here is an important example:

Example: The area of a half circle of radius 1 is given by the integral

$$\int_{-1}^{1} \sqrt{1-x^2}\, dx \ .$$

Solution. Write x = sin(u) so that cos(u) = $\sqrt{1-x^2}$. dx = cos(u)du.

We have sin(−π/2) =−1 and sin(π/2) = 1 the answer is

$$\int_{-\pi/2}^{\pi/2} \cos(u)\cos(u) du = \int_{-\pi/2}^{\pi/2} (1+\cos(2u))/2 = \frac{\pi}{2} \ .$$

23.2. Let us do the same computation for a general radius r:

Example: Compute the area of a half disc of radius r which is given by the integral

$$\int_{-r}^{r} \sqrt{r^2-x^2}\, dx \ .$$

Solution. Write x = r sin(u) so that r cos(u) = $\sqrt{r^2-x^2}$ and dx = r cos(u) du and r sin(−π/2) = −r and and r sin(π/2) = r.

The answer is

$$\int_{-\pi/2}^{\pi/2} r^2 \cos^2(u)\, du = r^2\pi/2 \ .$$

Example: Evaluate the following integral

$$\int x^2/\sqrt{1-x^2}\,dx\ .$$

Solution: Substitute x = cos(u), dx =− sin(u) du and get

$$\int -\frac{\cos^2(u)}{\sin(u)}\ \sin(u)du = -\int \cos^2(u)\ du = -\frac{u}{2}-\frac{\sin(2u)}{4}+C = -\frac{\arcsin(x)}{2}+\frac{\sin(2\arcsin(x))}{4}+C\ .$$

That is enough.

You have seen all the main ideas of calculus .

Further knowledge is accumulated by using calculus.

There exists (also on the website) a shorter review of calculus ideas for those already having a class before (it is named Calculus.pdf)..

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