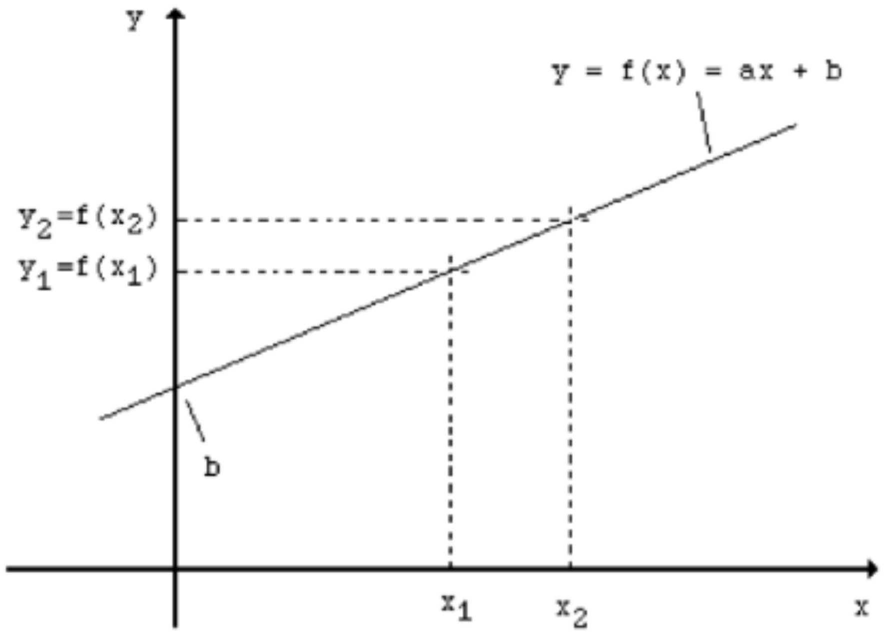
Short Tour of Calculus - A Reminder for those having done calculus before:

Rule y = f(x) = ax + b has graph called a straight line as shown

$$a = \text{ slope of the line } = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$b = \text{ intercept on the y-axis } = f(0)$$



Problem of finding equation of that line which is tangent to function y = f (x) at point (c, d), where d = f (c) is central to CALCULUS.

The most general straight line passing through point (c,d) is given by y–d = m(x – c).

Proof: This certainly represents a straight line since can rewrite it as y = mx+(d–mc),

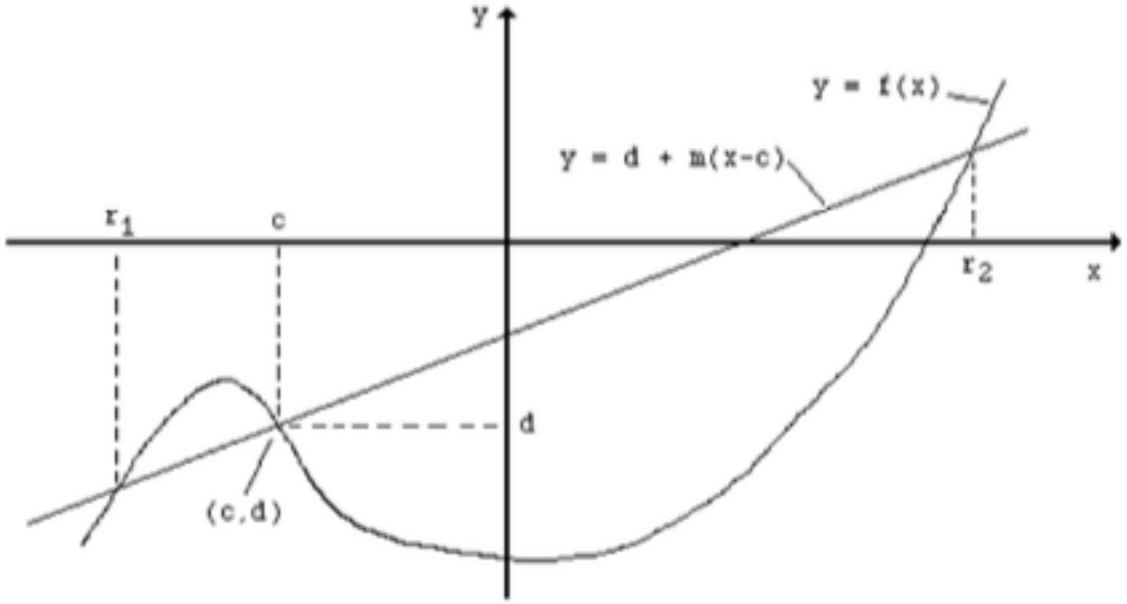
which is standard form of equation of straight line with slope = m and intercept (d – mc).

Line passes through point (c, d)

i.e., choose x = c gives y = mc+(d–mc) = d. This completes the proof.

Represents most general straight line through (c,d) since its slope m is arbitrary.

Plot two graphs as shown:



Have indicated 3 intersections labelled by x–values, namely, r_1, c and r_2.

Intersections r_1, c and r_2 represent zeroes of function p(x) = f(x)– d – m(x – c).

Now, very close to point x = c,

p(x) must have the form p(x)=(x–c)g(x) since it equals 0 at x=c.

Similarly, near x=r_1, p(x) must have the form p(x) = (x – r_1)g(x).

If we rotate (change slope of) line about point (c, d) until straight line

becomes tangent to curve at (c,d)

then, since this means that r_1 approaches c, must have

$$p(x) = (x-c)(x-r_1)d(x) \quad \text{near } x = c = r_1$$

$$= (x-c)(x-c)d(x) = (x-c)g(x)$$

In other words, when line is tangent to curve at (c,d)must have g(c) = 0.

From definition of g(x), then have

$$g(x) = \frac{p(x}{x-c} = \frac{f(x)-d}{x-c} - m = q(x) - m$$

When x=c,

this implies that g(c)=0=q(c)−m or m=q(c)= slope of tangent line to f(x) at (c,d),

where

$$q(x) = \frac{f(x)-d}{x-c} = \frac{f(x)-f(c)}{x-c}$$

For simple functions rule easily applied.

Consider case y = f(x) = x^3.

We have

$$q(x) = \frac{f(x)-f(c)}{x-c} = \frac{x^3-c^3}{x-c} = \frac{(x-c)(x^2+cx+c^2)}{x-c} = x^2+cx+c^2$$

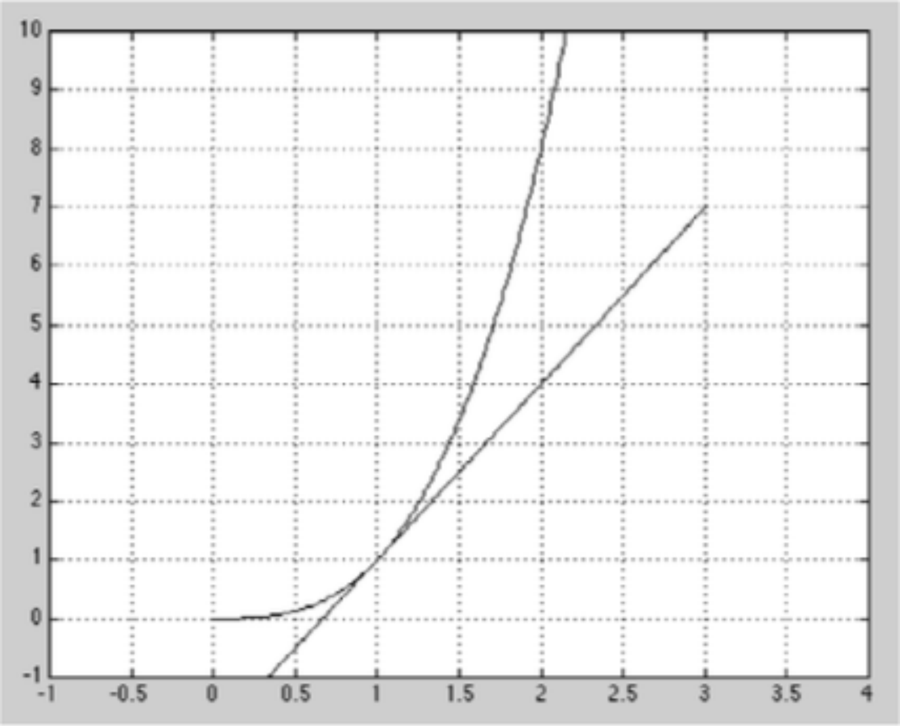
This implies that m = slope of line tangent to f(x) = x^3 at point (c, c^3) = q(c) = $3c^2$.

Then, equation

$$y - c^3 = 3c^2(x-c) \rightarrow y = 3c^2x - 2c^2$$

represents tangent line!

Case c = 1 is plotted:



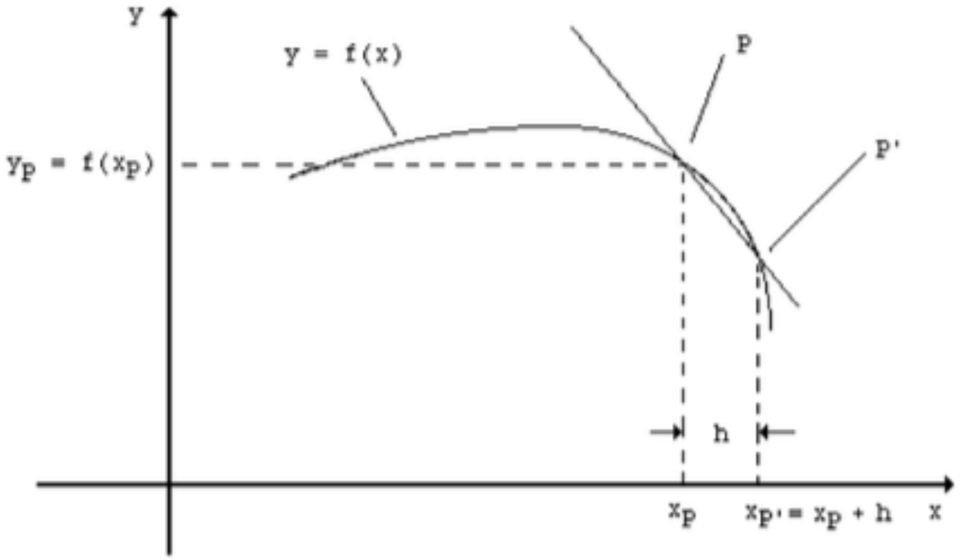
In this case, tangent line at point (1, 1) given by line $y = 3x - 2$.

The Derivative

Above procedure, while transparent, hard to apply for more complicated functions.

Now we develop an alternative approach (called the "derivative") which enables us to find slope of tangent line for arbitrary functions.

Consider figure:

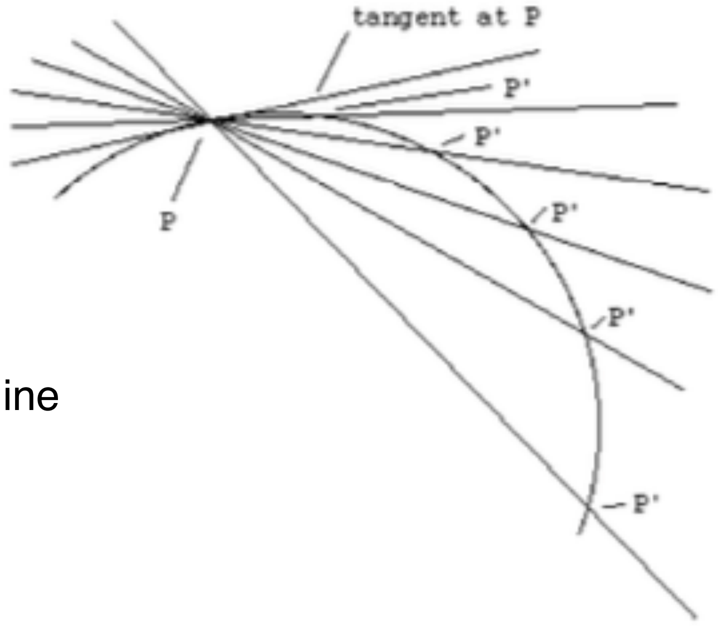


Now follow this procedure:

1. Choose a point P corresponding to $(x_P , f(x_P))$
2. Choose a second point P' such that $x_{P'} = x_P + h$

3. Find the equation of the straight line through P and P$'$
4. As P$'$ approaches P , the slope of the line PP$'$ approaches a limiting value equal to the slope of the tangent line at point P.

   This is shown schematically in the diagram:



Limit approaches tangent line

Approximating tangent line

Now slope of PP$'$ is

$$m_h = \frac{f(x_{P'}) - f(x_P)}{x_{P'} - x_P} = \frac{f(x_P + h) - f(x_P)}{h}$$

and the slope of the tangent line at P is

$$m_P = \lim_{h \to 0} m_h = \lim_{h \to 0} \frac{f(x_P + h) - f(x_P)}{h}$$

To illustrate these ideas, return to our previous example f(x) = x^3. We then have

$$m_P = \lim_{h\to 0} \frac{(x_P+h)^3 - x_P^3}{h} = \lim_{h\to 0} \frac{x_P^3 + 3hx_P^2 + 3h^2x_P + h^3 - x_P^3}{h}$$

$$= \lim_{h\to 0} \frac{3hx_P^2 + 3h^2x_P + h^3}{h} = \lim_{h\to 0}(3x_P^2 + 3hx_P + h^2) = 3x_P^2$$

For point x_P = c, have m = $3c^2$ as before

In general, derivative of function f(x) at arbitrary point x defined by

$$f'(x) = \frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Derivative also called “rate of change”,

i.e., df/dx = rate of change of f(x) with respect to x

= slope of tangent line to the graph y = f (x) at the point (x, f (x)).

Simple Example: Let f(x) = cx^2. Then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c(x+h)^2 - cx^2}{h}$$

$$= \lim_{h\to 0} \frac{2cxh + ch^2}{h} = \lim_{h\to 0}(2cx + h) = 2cx$$

This is slope of line tangent to f (x) = cx^2 at (x, f (x)).

Integration

Now, if we can write $$h(x) = \frac{d}{dx}(g(x))$$

then quantity g(x) + c

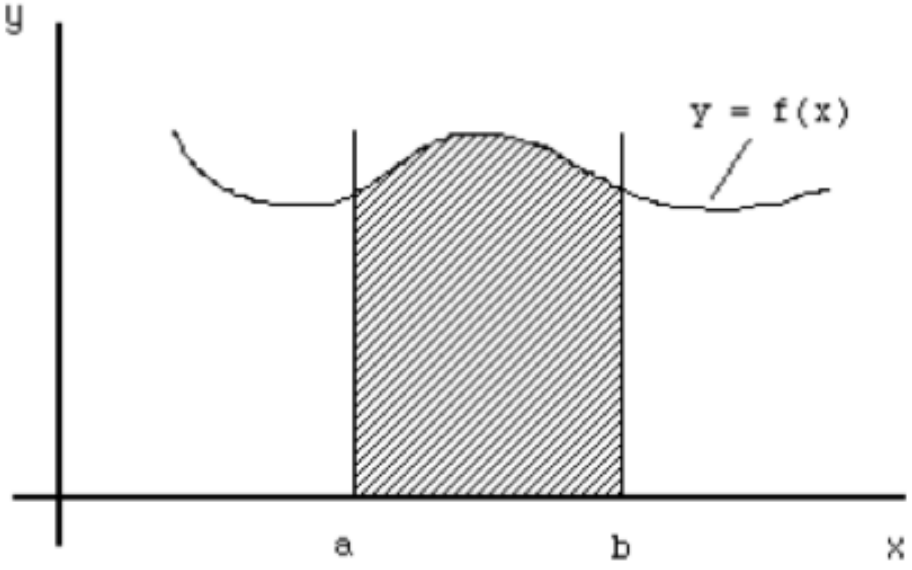
where c = arbitrary constant is called **antiderivative** of h(x),

i.e., g(x) + c is function whose derivative is h(x).

Suppose now we ask the following question:

what is area(shaded region in figure) under curve y = f(x) between x = a and x = b?

Good approximation to area given by following procedure:



1. divide interval a ≤ x ≤ b into N equal segments each of length

$$\Delta = \frac{b-a}{N}$$

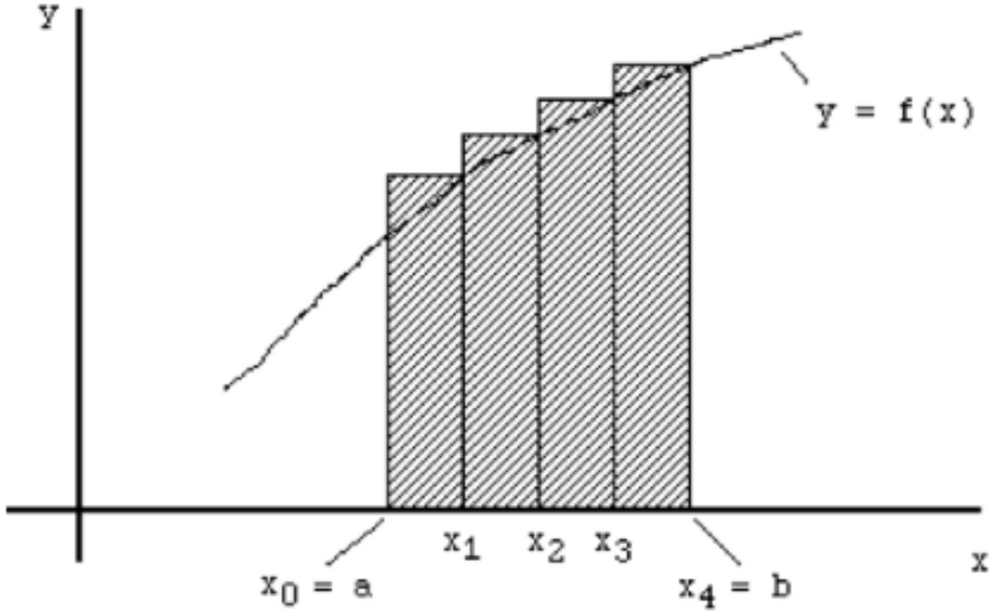
2. define $x_k = a + k\Delta$ for k = 1,2,3,4,.....,N

3. calculate corresponding values of f(x), namely, $f(x_k) = f(a + k\Delta) \;\; k = 1, 2, 3, 4, ..., N$

4. then approximation to area given by

$$\text{AREA } = \sum_{k=1}^{N} f(x_k)\Delta$$

as shown in figure (choosing N=4)



As can be seen from figure,

approximation for area equals sum of shaded rectangles.

In case shown, calculated area is greater than actual area.

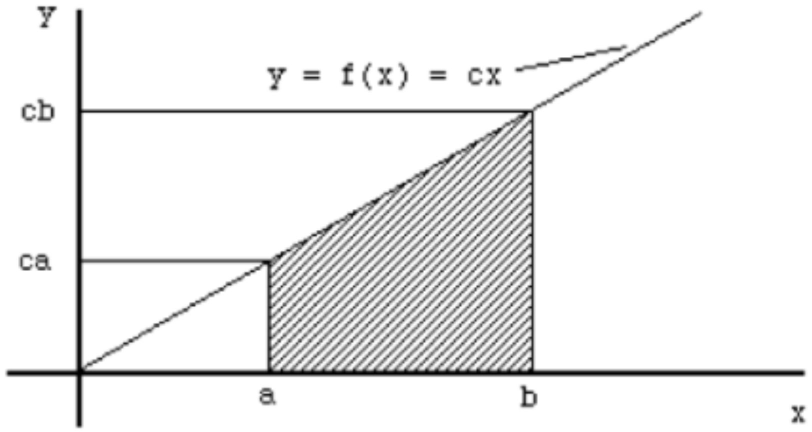
In limit N → ∞, approximation = actual area under curve.

Limit N → ∞ usually written as

$$\text{AREA } = \int_a^b f(x)dx = \text{ integral of } f(x) \text{ from } a \text{ to } b:$$

and a and b are called the limits of integration.

Simple Integral: Let f(x) = cx (straight line) as shown in figure:



In this case we have

$$\int_a^b f(x)dx = \int_a^b cxdx = \lim_{N\to\infty} \Delta f(x_k) = \lim_{N\to\infty} cx_k\Delta$$

$$= c\lim_{N\to\infty} \frac{b-a}{N}\sum_{k=1}^{N}(a+k\Delta) = c\lim_{N\to\infty} \frac{b-a}{N}\sum_{k=1}^{N} a + c\lim_{N\to\infty} \frac{b-a}{N}\frac{b-a}{N}\sum_{k=1} k$$

$$= c\frac{b-a}{N}aN + c\lim_{N\to\infty} \frac{b-a}{N}\frac{b-a}{N}\frac{N(N-1)}{2}$$

$$= c(b-a)a + \frac{1}{2}c(b-a)^2 \lim_{N\to\infty}\left(1+\frac{1}{N}\right)$$

$$= c(b-a)a + \frac{1}{2}c(b-a)^2 = \frac{1}{2}c(b^2-a^2)$$

Shaded area is easy to calculate directly in this case and is given by

$$(b-a)ca = \frac{1}{2}(b-a)(cb-ca) = \frac{1}{2}c(b^2-a^2)$$

So it works!

In this manner, could evaluate any integral (find area under corresponding curve).

Procedure quickly becomes very cumbersome and tedious, however.

Better method is to realize that there is connection between integrals and derivatives.

The Fundamental Theorem of Calculus

If

$$\frac{dF}{dx} = f(x)$$

i.e., if F (x) + c is the antiderivative of f (x), then

$$\int_a^b f(x)dx = F(b) - F(a) = \left. F(x)\right|_{x=a}^{x=b} = \text{ definite integral (a number)}$$

Alternatively, another way of saying same thing is to use definition

$$\int f(x)dx = F(x) + c = \text{ indefinite integral (a function of x)}$$

Indefinite integral represents most general antiderivative of f(x).