$$\int_{-4}^{-2} 6x^2 dt = \frac{6}{2+1} x^{2+1} \Big|_{-4}^{-2} = \frac{6}{3} 2x^3 \Big|_{-4}^{-2} = 2x(-2)^3 - 2x$$

Let $$I = \int x \tan^{-1} x \, dx$$

Take $u = \tan^{-1} x, \qquad dv = x\,dx$

$$du = \frac{1}{1+x^2} dx, \qquad v = \int x\,dx = \frac{x^2}{2}$$

$$I = \tan^{-1} x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{1+x^2} dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{(1+x^2)-1}{1+x^2} dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int dx + \frac{1}{2} \int \frac{dx}{1+x^2}$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{x}{2} + \frac{1}{2} \tan^{-1} x + C$$

$$\int_1^9 \frac{2x^2 + x^2\sqrt{x} - 1}{x^2}\,dx = \int_1^9 \left(2 + x^{1/2} - x^{-2}\right)dx$$

$$= 2x + \frac{x^{3/2}}{3/2} - \frac{x^{-1}}{-1}\Bigg]_1^9$$

$$= 2x + \frac{2}{3}x^{3/2} + \frac{1}{x}\Bigg]_1^9$$

$$= \left[2\cdot 9 + \frac{2}{3}(9)^{3/2} + \frac{1}{9}\right] - \left[2\cdot 1 + \frac{2}{3}(1)^{3/2} + \frac{1}{1}\right]$$

$$= 18 + 18 + \frac{1}{9} - 2 - \frac{2}{3} - 1 = 32\frac{4}{9}$$

$\therefore$

$$I = \int_0^2 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^2 \frac{dx}{(x-1)^{2/3}}$$

$$= \lim_{\epsilon \to 0+} \int_0^{1-\epsilon} \frac{dx}{(x-1)^{2/3}} + \lim_{\delta \to 0+} \int_{1+\delta}^2 \frac{dx}{(x-1)^{2/3}}$$

$$= \lim_{\epsilon \to 0+} \int_0^{1-\epsilon} (x-1)^{-2/3}\,dx + \lim_{\delta \to 0+} \int_{1+\delta}^2 (x-1)^{-2/3}\,dx$$

$$= \lim_{\epsilon \to 0} \left[\frac{(x-1)^{1/3}}{1/3}\right]_0^{1-\epsilon} + \lim_{\delta \to 0} \left[\frac{(x-1)^{1/3}}{1/3}\right]_{1+\delta}^2$$

$$= 3\lim_{\epsilon \to 0}[(1-\epsilon-1)^{1/3} - (-1)^{1/3}] + 3\lim_{\delta \to 0}[1^{1/3} - (1+\delta-1)^{1/3}]$$

$$= 3\lim_{\epsilon \to 0}[(-\epsilon)^{1/3} - (-1)] + 3\lim_{\delta \to 0}(1 - \delta^{1/3}) \qquad [\because (-1)^{1/3} = -1]$$

$$= 3[0+1] + 3[1-0] = 6$$

Let $$I=\int\frac{x^2-3x+2}{x}dx$$

$$=\int\left(x-3+\frac{2}{x}\right)dx \qquad \text{[By term by term divisions]}$$

$$=\int xdx-\int 3dx+\int\frac{2}{x}dx$$

$$=\frac{x^2}{2}-3x+2\log_e|x|+C$$

Evaluate: $\displaystyle\int(5x^3+2\cos x)\,dx.$

$$\begin{aligned}\int(5x^3+2\cos x)\,dx &= \int 5x^3\,dx \;+\; \int 2\cos x\,dx\\ &= 5\int x^3\,dx \;+\; 2\int \cos x\,dx\\ &= 5\cdot\frac{x^4}{4}+2\cdot\sin x+C\\ &= \frac{5x^4}{4}+2\sin x+C.\end{aligned}$$

4.5 Integration by Substitution

U-substitution and definite Integrals

8. $\int_0^1 x\left(x^2+1\right)^3 dx$

$$\int_0^1 (x^2+1)^3 \underbrace{x\,dx} = \frac{1}{2}\int_1^2 u^3\,du$$

$u = 1^2 + 1 = 2$

$u = 0^2 + 1 = 1$

$$\boxed{\begin{aligned} &\text{let } u = x^2+1 \\ &\frac{du}{dx} = 2x \\ &du = 2x\,dx \\ &\tfrac{1}{2}du = \underbrace{x\,dx} \end{aligned}}$$

$$= \frac{1}{2}\cdot\frac{1}{4}u^4 \Big|_1^2 = \frac{1}{8}u^4 \Big|_1^2$$

$$= \frac{1}{8}(2)^4 - \frac{1}{8}(1)^4 = 2$$

Example: To integrate $\int \frac{2}{1-x^2}\,dx$ we can write

$$\frac{2}{1-x^2} = \frac{1}{1-x} + \frac{1}{1+x}$$

and integrate each term

$$\int \frac{2}{1-x^2} = \log(1+x) - \log(1-x)\ .$$

Example: Integrate $\frac{5-2x}{x^2-5x+6}$. **Solution.** The denominator is factored as $(x-2)(x-3)$. Write

$$\frac{5-2x}{x^2-5x+6} = \frac{A}{x-3} + \frac{B}{x-2}\ .$$

Now multiply out and solve for A, B:

$$A(x-2) + B(x-3) = 5-2x\ .$$

This gives the equations $A+B=-2, -2A-3B=5$. From the first equation we get $A=-B-2$ and from the second equation we get $2B+4-3B=5$ so that $B=-1$ and so $A=-1$. We have not obtained

$$\frac{5-2x}{x^2-5x+6} = -\frac{1}{x-3} - \frac{1}{x-2}$$

and can integrate:

$$\int \frac{5-2x}{x^2-5x+6}\,dx = -\log(x-3) - \log(x-2)\ .$$

Example: Integrate $f(x) = \int \frac{1}{1-4x^2}\, dx$. **Solution.** The denominator is factored as $(1-2x)(1+2x)$. Write

$$\frac{A}{1-2x} + \frac{B}{1+2x} = \frac{1}{1-4x^2} .$$

We get $A = 1/4$ and $B = -1/4$ and get the integral

$$\int f(x)\, dx = \frac{1}{4}\log(1-2x) - \frac{1}{4}\log(1+2x) + C .$$

Example: **Marry go round:** Find $I = \int \sin(x)\exp(x)\, dx$. **Solution.** Lets integrate $\exp(x)$ and differentiate $\sin(x)$.

$$= \sin(x)\exp(x) - \int \cos(x)\exp(x)\, dx .$$

Lets do it again:

$$= \sin(x)\exp(x) - \cos(x)\exp(x) - \int \sin(x)\exp(x)\, dx.$$

We moved in circles and are stuck! But wait. Are we really? We have now derived an identity

$$I = \sin(x)\exp(x) - \cos(x)\exp(x) - I$$

which we can solve for I and get $\boxed{I = [\sin(x)\exp(x) - \cos(x)\exp(x)]/2.}$

Example: Find $\int xe^x\, dx$. **Solution.** You want to differentiate x and integrate e^x.

$$\int \mathrm{x}\ \exp(x)\ dx = x\exp(x) - \int 1 \cdot \exp(x)\ dx = x\exp(x) - \exp(x) + C\ dx\ .$$

Example: Find $\int \log(x)\, dx$. **Solution.** While there is only one function here, we need two to use the method. Let us look at $\log(x) \cdot 1$:

$$\int \log(x)\ 1\ dx = \log(x)x - \int \frac{1}{x} x\ dx = x\log(x) - x + C\ .$$

Example: Find $\int x\log(x)\, dx$. **Solution.** Since we know from the previous problem how to integrate log we could proceed by taking $x = u$. We can also take $u = \log(x)$ and $dv = x$:

$$\int \log(x)\ \mathrm{x}\ dx = \log(x)\frac{x^2}{2} - \int \frac{1}{x}\frac{x^2}{2}\ dx$$

which is $\log(x)x^2/2 - x^2/4$.

Example: Find the anti derivative of

$$f(x) = e^{x^4+x^2}(4x^3 + 2x) \ .$$

Solution: It is $e^{x^4+x^2} + C$.
Example: Find

$$\int \sqrt{x^5 + 1} x^4 \ dx \ .$$

Solution. Try $(x^5+1)^{3/2}$ and differentiate. This gives $15/2$ of what we have. Therefore $F(x) = (2/15)(x^5 + 1)^{3/2}$.
Example: Find the anti-derivative of

$$\frac{\log(x)}{x} \ .$$

Solution: We spot that $1/x$ is the derivative of $\log(x)$. The anti-derivative is $\log(x)^2/2 + C$.

Example: Find the anti-derivative of $\int \log(x)/x \; dx$. **Solution:** Pick $u = log(x), du = (1/x)dx$. Because $dx = xdu$, we get $\int udu = u^2/2 + C$. Back substitute to get $\log^2(x)/2 + C$.

Example: Find the anti-derivative

$$\int \log(\log(x)) \frac{1}{\log(x)x} \; dx \; .$$

Solution: Try $u = \log(x)$ and $du = (1/x)dx$, then plug this into the formula. It gives $\int \log(u)/u \; du$. We have just solved this integral before and got $\log(u)^2/2 + C$.

Example: Solve the integral

$$\int \frac{x}{1 + x^4} \; dx \; .$$

Solution: Substitute $u = x^2, du = 2xdx$ gives $(1/2) \int du/(1+u^2) \; du = (1/2) \arctan(u) = (1/2) \arctan(x^2) + C$.

Example: Find the anti-derivative of $f(x) = \sin(4x) + 20x^3 + 1/x$. Solution: We can take the anti-derivative of each term separately. It is $F(x) = -\cos(4x)/4 + 4x^4 + \log(x) + C$.

Example: Find the anti-derivative of $f(x) = 1/\cos^2(x) + 1/(1-x)$. **Solution:** we can find the anti-derivatives of each term separately and add them up. The result is $F(x) = \tan(x) + \log|1-x| + C$.

Example: $\int_0^5 x^7\,dx = \frac{x^8}{8}|_0^5 = \frac{5^8}{8}$. You can always leave such expressions as your final result. It is even more elegant than the actual number 390625/8.

Example: $\int_0^{\pi/2} \cos(x)\,dx = \sin(x)|_0^{\pi/2} = 1$.

Example: Find $\int_0^\pi \sin(x)\,dx$. **Solution**: The answer is 2.

Example: For $\int_0^2 \cos(t+1)\,dt = \sin(x+1)|_0^2 = \sin(2) - \sin(1)$, the additional term $+1$ does not make matter as when using the chain rule, it goes away.

Example: For $\int_{\pi/6}^{\pi/4} \cot(x)\,dx$, the anti-derivative is difficult to spot. It becomes only accessible if we know, where to look: the function $\log(\sin(x))$ has the derivative $\cos(x)/\sin(x)$. So, we know the answer is $\log(\sin(x))|_{\pi/6}^{\pi/4} = \log(\sin(\pi/4)) - \log(\sin(\pi/6)) = \log(1/\sqrt{2}) - \log(1/2) = -\log(2)/2 + \log(2) = \log(2)/2$.

Let us look at two for now more challenging cases:

Example: The example $\int_2^3 2/(t^2-1)\,dt$ is challenging for now. We need a hint and write $2/(x^2-1) = 1/(x-1) - 1/(x+1)$. The function $F(x) = \log|x-1| - \log|x+1|$ has therefore $f(x) = 2/(x^2-1)$ as a derivative. The answer is $\int_2^3 2/(t^2-1)\,dt = F(3) - F(2) = \log(2) - \log(4) - \log(1) + \log(3) = \log(3) - \log(2) = \log(3/2)$.

a) $\int_0^1 2xe^{x^2+3}\,dx$
b) $\int_0^1 4x^3/(1+x^4)\,dx$
c) $\int_0^1 \frac{1}{(x-5)(x-3)}\,dx$
d) $\int_0^1 \log^2(1+x)\,dx$
e) $\int_0^1 xe^{1+x}\,dx$

a) $\int \frac{\log(x)}{x^2}\,dx$
b) $\int \cos^2(x) - 3\sin^2(x)\,dx$
c) $\int \frac{1}{x^2-8x+12}\,dx$
d) $\int (x+1)^3 e^x\,dx$
e) $\int \frac{3x^2}{(1+x^6)}\,dx$

a) $\int_0^1 \frac{1}{(x+3)(x-4))}\,dx$

b) $\int_0^1 2x/(1+x^2)\,dx$

c) $\int_0^1 x^3/(1+x^2)\,dx$

d) $\int_0^1 \log(1+x)\,dx$

e) $\int_0^1 5xe^x\,dx$

a) $\int x^2\log(x)\,dx$

b) $\int \cos^2(x)\sin(x)\,dx$

c) $\int \cos^2(2x)\,dx$

d) $\int x^2\sin(x)\,dx$

e) $\int 3x^2/(1+x^6)\,dx$