Problems

1.1 Show that the Earth's rotation speed at the latitude of Lisbon (about 39^o) is approximately 1,300 kilometers per hour (radius of the Earth: approximately 6,400 kilometers). Show that this speed is greater than the speed of sound (about 340 meters per second).

1.2 Show that the speed of the Earth with respect to the Sun is about 30 kilometers per second. (Distance from the Earth to the Sun: approximately 8.3 light-minutes).

1.3 A machine gun mounted in the rear of a bomber which is flying at 900 kilometers per hour is firing bullets also at 900 kilometers per hour but in the direction opposite to the flight direction. What happens to the bullets?

1.4 A boy throws a tennis ball at 50 kilometers per hour towards a train which is approaching at 100 kilometers per hour. Assuming that the collision is perfectly elastic, at what speed does the ball come back?

1.5 Relativity of simultaneity: The starship *Enterprise* travels at 80% of the speed of light with respect to the Earth. Exactly midship there is a floodlight. When it is turned on, its light hits the bow and the stern at exactly the same time as seen by observers on the ship. What do observers on Earth see?

1.6 Check that the inverse Lorentz transformation formulas are correct.

1.7 Length contraction: Consider a ruler of length ℓ' at rest in the frame S'. The ruler is placed along the x'-axis, so that its ends satisfy $x' = 0, x' = \ell'$ for all t'. Write the equations for the motion of the ruler's ends in the frame S, and show that in this frame the ruler's length is

$$\ell = \frac{\ell'}{\gamma} < \ell'$$

1.8 Check that the relativistic velocity addition formula can be written as

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$

1.9 A rocket which is flying at 50% of the speed of light with respect to the Earth fires a missile at 50% of the speed of light (with respect to the rocket). What is the velocity of the missile with respect to the Earth when the missile is fired

(a) Forwards?

(b) Backwards?

1.10 Two rockets fly at 50% of the speed of light with respect to the Earth, but in opposite directions. What is the velocity of one of the rockets with respect to the other?

1.11 The time dilation formula can be derived directly from the fact that the speed of light is the same on any inertial frame by using a light clock (figure below)

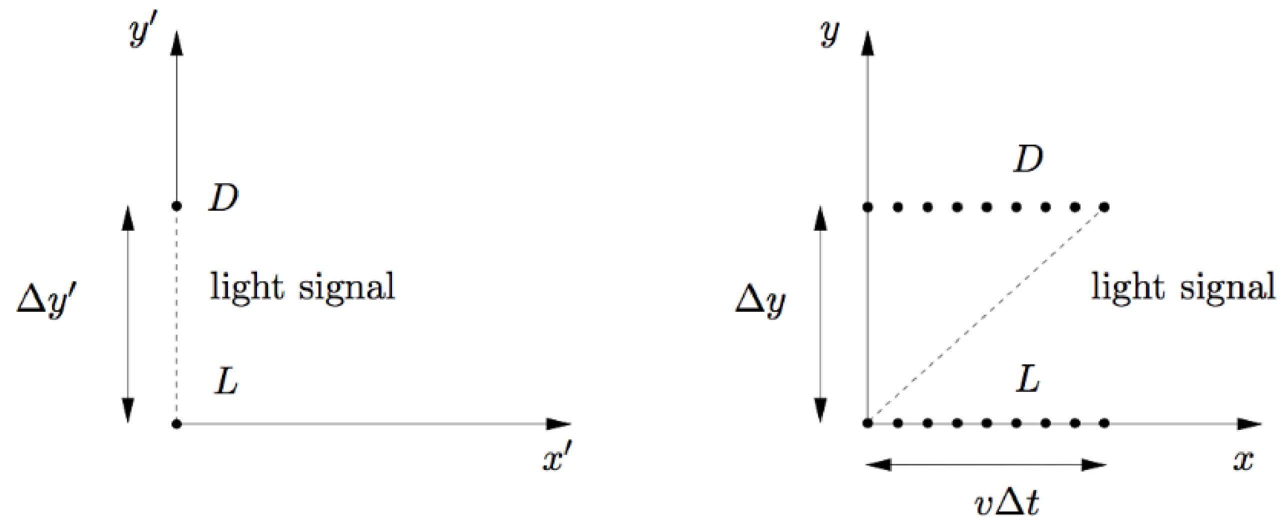

Figure: Light clock.

Consider a light signal propagating in S' along the y'-axis between a laser emitter L and a detector D. If the signal travel time is $\Delta t'$ (as measured in S'), then the distance between L and D (as measured in S') is $\Delta y' = c\Delta t'$. On the other hand, as seen from S the detector D moves along the x-axis with velocity v. Therefore during the time interval Δt (as measured in S) between the emission and the detection of the signal the detector moves by $v\Delta t$ along the x-axis.

Assuming that the distance Δy measured in S between the laser emitter and the detector is the same as in S', $\Delta y = \Delta y'$, derive the time dilation formula.

1.12 **Useful approximations**: Show that if $|\varepsilon| \ll 1$ then

(a) $\frac{1}{1\varepsilon} \simeq 1 - \varepsilon$

(b) $\sqrt{1+\varepsilon} \simeq 1 + \frac{\varepsilon}{2}$

with error of order $\varepsilon^2 \ll |\varepsilon|$.

1.13. The time dilation formula was experimentally verified in 1971 by comparing the readings of two very precise atomic clocks. One of the clocks was kept at rest on the surface of the Earth, whereas the other was flown once around the Earth, along the parallel of latitude 39^o, at an average speed of 900 kilometers per hour.

(a) What was the difference in the readings of the two clocks? Does it make a difference whether the clock was flown eastwards or westwards?

(b) Show that even if the moving atomic clock were transported very slowly along the parallel the two clocks would be desynchronized at the end of the journey (**Sagnac effect**).

(Recall that the Earth is **not** a perfect inertial frame because it is spinning; use the approximation $\sqrt{1-\frac{v^2}{c^2}} \simeq 1-\frac{v^2}{2c^2}$ or velocities v much smaller than c).

1.14. When cosmic rays hit the Earth?s atmosphere they produce (among others) particles called muons, typically at an altitude of 10 kilometers. These elementary particles are unstable, and decay in about 2.2×10^{-6} seconds. However, a large percentage of these muons is detected on the ground. What is the minimum speed at which the detected particles must be moving?

1.15. Twin paradox: Two twins, Alice and Bob, part on their 20th birthday. While Alice remains on Earth (which is an inertial frame to a very good approximation), Bob departs at 80% of the speed of light towards Planet X, 8 light-years away from Earth. Therefore Bob reaches his destination 10 years later (as measured on the Earth?s frame). After a short stay, he returns to Earth, again at 80% of the speed of light. Consequently Alice is 40 years old when she sees Bob again.

(a) How old is Bob when they meet again?

(b) How can the asymmetry in the twins? ages be explained? Notice that from Bob?s point of view he is at rest in his spaceship and it is the Earth which moves away and then back again.

2.1 Check that a light-year is about 9.5×10^{12} kilometers, and that a light-meter is about $3.3 \times 10^{?9}$ seconds.

2.2 **Lucas problem**: By the end of the 19th century there was a regular transatlantic service between Le Havre and New York. Every day at noon (GMT) a transatlantic liner would depart from Le Havre and another one would depart from from New York. The voyage took exactly 7 days, so that the liners would also arrive at noon (GMT). Thus a liner departing from Le Havre would see a liner arriving from New York, and a liner arriving in New York would see a liner departing to Le Havre. Besides these two, how many other liners would a passenger doing the Le Havre-New York voyage meet? At what times? How many liners were needed in total?

2.3 Show that the coordinates (x, y) and (x', y') of the same point P in two systems of orthogonal axes S and S', with S' rotated by an angle α with respect to S, satisfy

$$
\begin{aligned}
x' &= x\cos\alpha + y\sin\alpha \\
y' &= -x\sin\alpha + y\cos\alpha
\end{aligned}
$$

2.4 Twin paradox (again): Recall the setup of Exercise 15 in the previous chapter: Two twins, Alice and Bob, part on their 20th birthday. While Alice remains on Earth (which is an inertial frame to a very good approximation), Bob departs at 80% of the speed of light towards Planet X, 8 light-years away from Earth. Therefore Bob reaches his destination 10 years later (as measured on the Earth's frame). After a short stay, he returns to Earth, again at 80% of the speed of light. Consequently Alice is 40 years old when she sees Bob again. Bob, however, is only 32 years old.

(a) Represent these events in the inertial frame S' in which Bob is at rest during the first leg of the journey, and check that the ages of the twins at the reunion are correct.

(b) Do the same in the inertial frame S'' in which Bob is at rest during the return leg of the journey.

(c) Imagine that each twin watches the other trough a very powerful telescope. What do they see? In particular, how much time do they experience as they see one year elapse for their twin?

2.5 **Doppler effect**: Use the space-time diagram in the figure below

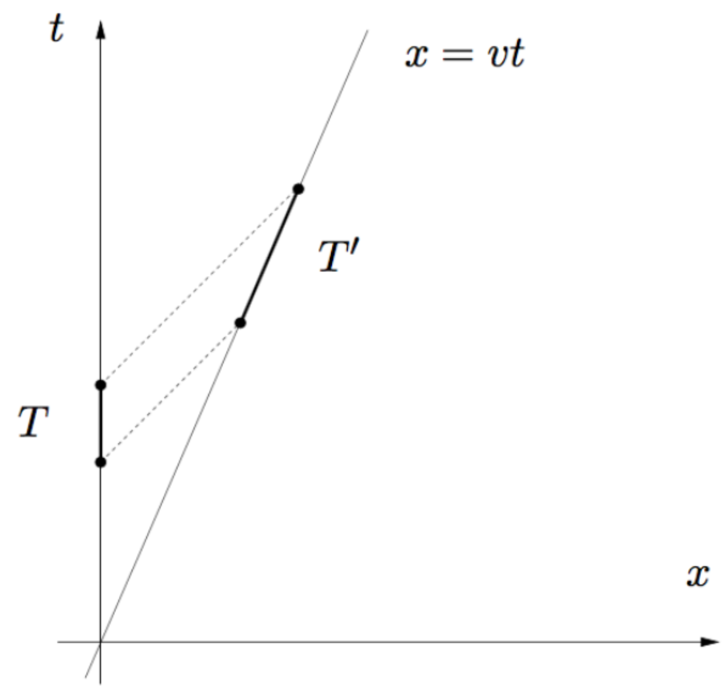

Figure: Doppler effect.

to show that if a light signal has period T in S then its period as measured in S' is

$$T' = T\sqrt{\frac{1+v}{1-v}}$$

Moreover, show that for speeds much smaller than the speed of light, $|v| \ll 1$, this formula becomes

$$T' = T(1 + v)$$

This effect can be used to measure the velocity of an approaching (or receding) light source (e.g. a star).

2.6 A Klingon spy commandeers the Einstein, Earth's most recent starship, and escapes towards his home planet at maximum speed, 60% of the speed of light. In despair, Starfleet Command decides to install a highly experimental engine aboard the Enterprise, which theoretically will allow it to reach 80% of the speed of light. The installation takes 1 year, but the new engine works perfectly. The Enterprise departs in chase of the spy and captures him some time later during an exciting space battle.

(a) How long does it take between the theft and the recapture of the Einstein:

  (i) According to an observer on the Earth's (inertial) frame?
  (ii) According to the Klingon spy?
  (iii) According to the Enterprise crew?

(b) Starfleet has decided that if the Enterprise fails to recapture the Einstein then a radio signal will be emitted from Earth activating a secret self-destruct device aboard the Einstein. How long after the theft can the engineers on Earth expect to know whether to emit the radio signal?

(c) When could the engineers expect confirmation of the Einstein's destruction if the signal was emitted?

2.7 **The faster-than-light missile**: During a surveillance mission on the home planet of the devious Klingons, the Enterprise discovers that they are preparing to build a faster-than-light missile to attack the peaceful planet Organia, 12 light-years away.

Alarmed, Captain Kirk orders the Enterprise to depart at its maximum speed ($\frac{12}{13}$ of the speed of light) to the threatened planet. At the same time a radio signal is sent to warn the Organians about the imminent attack. Unfortunately, it is too late: eleven years later (in the frame of both planets) the Klingons finish the missile and launch it at 12 times the speed of light. So the warning reaches Organia at the same time as the missile, twelve years after being emitted, and the Enterprise arrives at the planet's debris one year later.

(a) How long does the Enterprise take to get to Organia according to its crew?

(b) In the planets' frame, using years and light-years as units of time and length, let $(0, 0)$ be the (t, x) coordinates of the event in which the Enterprise uncovers the plot, $(11, 0)$ the coordinates of the missile launch, $(12, 12)$ the coordinates of the destruction of Organia and $(13, 12)$ the coordinates of the arrival of the Enterprise at the debris. Compute the coordinates (t', x') of the same events in the Enterprise's frame.

(c) Plot the histories of the Enterprise, the planets, the radio warning and the missile in the Enterprise's frame. How do events unfold in this frame?

3.1 Lisbon and New York are approximately at the same latitude (40^o). However, a plane flying from Lisbon to New York does not depart from Lisbon heading west. Why not?

3.2 Consider two points in the sphere at the same latitude $\theta = \frac{\pi}{4}$ but with longitudes differing by π. Show that the distance between these two points is smaller than the length of the parallel arc between them.

3.3 Decide whether the following statements are true of false for the plane's geometry and for the sphere's geometry:

(a) All geodesics intersect.

(b) If two distinct geodesics intersect, they do so at a unique point.

(c) Given a geodesic γ and a point $p \notin \gamma$, there exists a unique geodesic γ' containing p which does not intersect γ (Euclid's' fifth postulate).

(d) There exists a geodesic containing any two distinct points.

(e) There exists a unique geodesic containing any two distinct points.

(f) There exist points arbitrarily far apart.

(g) All geodesics are closed curves.

(h) The sum on a triangle's internal angles is π.

(i) The length of a circle with radius r is $2\pi r$.

(j) The area enclosed by a circle of radius r is πr^2.

3.4 Show that the average curvature of a spherical triangle in which one of the vertices is the north pole and the other two vertices are on the equator is $\frac{1}{R^2}$.

3.5 What is the relation between the sum of the internal angles of a 2-sided polygon on the sphere and its area?

3.6 Check that the cylindrical projection preserves the area of spherical triangles with one vertex on the north pole and the other two vertices on the equator.

3.7 Show that neither the stereographic projection nor the sphere's map using latitude and longitude as coordinates preserve areas.

3.8 Show that no map of the sphere can simultaneously preserve areas and be conformal.

4.1 Compute the mass of the Earth in geometrized units from the gravitational acceleration at the Earth's surface (9.8 meters per squared second).

4.2 Compute the escape velocity from the Earth's surface.

4.3 The orbit of Halley's comet is a very elongated ellipse, which makes its venture away from the Sun almost as far as Pluto's orbit. Therefore Halley's comet (like most comets) moves almost at the Solar System's escape velocity. Knowing that the perihelion of its orbit is about 4.9 light-minutes from the Sun, calculate the Halley comet's speed at that point.

4.4 Referring to Figure 4.1 in text, show that when the coordinate θ varies by a small amount $\Delta\theta$, the line segment joining the center of M to the point particle m sweeps an approximate area $\Delta A = \frac{1}{2} r^2 \Delta\theta$. Conclude that the law of conservation of angular momentum can be geometrically interpreted as the statement that the velocity at which this area is swept is constant.

4.5 Use the period of the Earth's orbit to compute the mass of the Sun in geometrized units.

4.6 Calculate the period of a (circular) low Earth orbit.

4.7 Determine the radius of a geostationary orbit (that is, a circular orbit around the Earth with period equal to 24 hours). Do the same for the GPS satellites' orbits, whose period is about 12 hours.

4.8 Compute the period of the Moon's orbit. (Distance from the Earth to the Moon: approximately 1.3 light-seconds).

4.9 The figure below shows the orbits of a few stars about Sagittarius A^*, the black hole at the center of our galaxy, which is approximately 26,000 light-years away. The orbits were deduced from the observations plotted in the figure, done during the period 1995-2003. Use this data to estimate the mass of Sagittarius A^*.

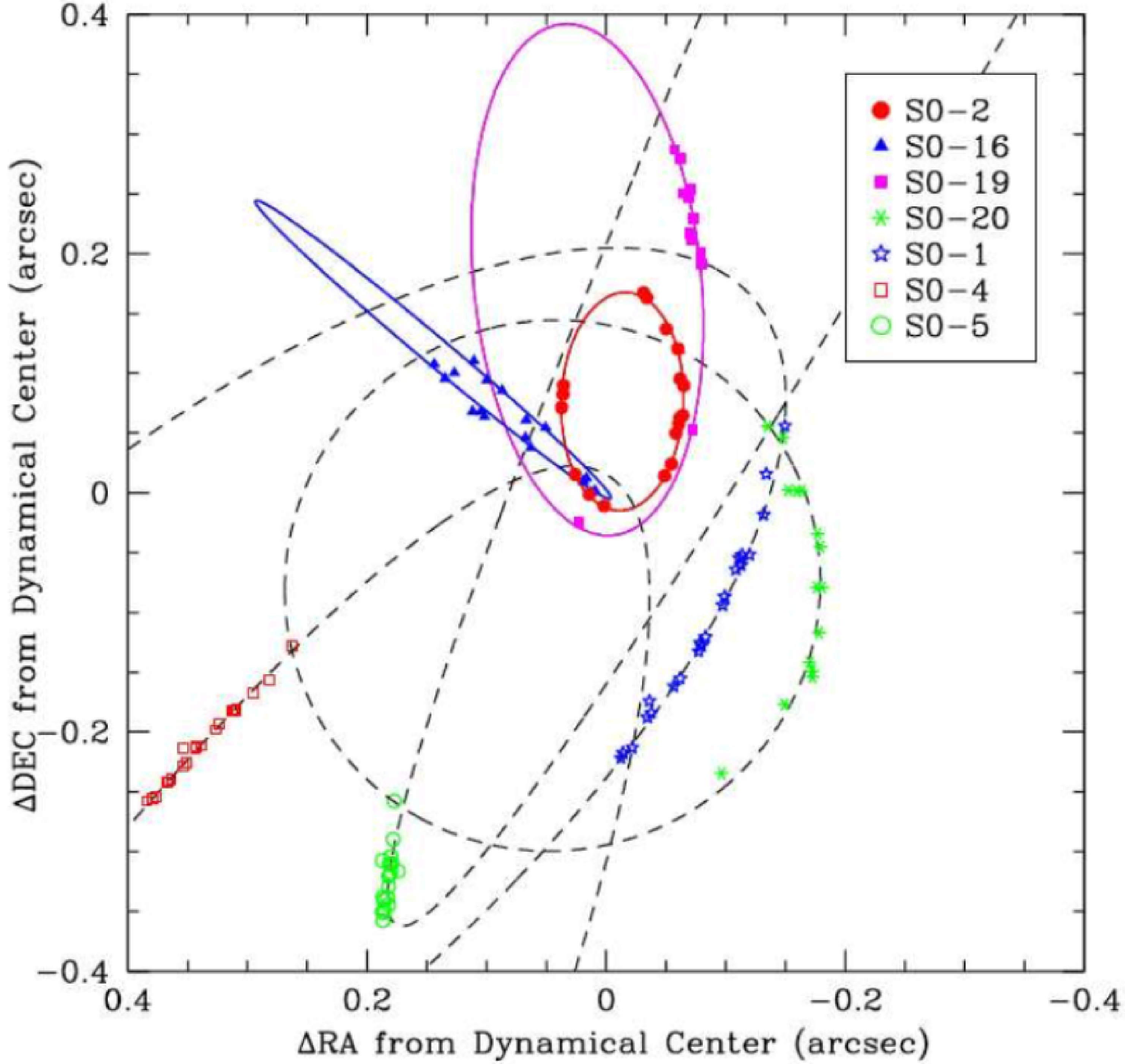

Figure: 4.3: Stars orbiting Sagittarius.

5.1 Since in everyday situations the gravitational redshift is very small, this effect was only experimentally confirmed in the beginning of the sixties. The first experiment was carried out in an elevator shaft of one of the Harvard university buildings, which was 23 meters high. Compute the percentual variation of the period measured in this experiment.

5.2 Sometimes light behaves as if composed of particles, called **photons**, with energy given by the **Planck-Einstein relation**

$$E = \frac{h}{T}$$

where h is Planck's constant and T is the light's period. From the mass-energy equivalence relation

$$E = mc^2$$

(which Einstein derived as a consequence of special relativity), one would expect a photon going up in a gravitational field to lose an amount of energy

$$\Delta E = \frac{E}{c^2}\Delta\phi$$

Show that if this is the case then the Planck-Einstein relation implies the gravitational redshift formula.

5.3 Correct the result of Problem 13 in Chapter 1 to include the effect of the gravitational field. Assume that the airplane flew at an average altitude of 10 kilometers.

5.4 The Global Positioning System (GPS) uses satellites on 12-hour orbits carrying very precise atomic clocks. It is very important that these clocks are synchronized with the clocks on the ground tracking stations, since any desynchronization will result in positional errors of the same magnitude (using $c = 1$). Show that if one failed to apply relativistic corrections then the desynchronization with respect to a ground station on the equator would be about 12 light-kilometers after just one day.

5.5 Due to its rotation motion the Earth is not a perfect sphere, being flattened at the poles. Because of this the Earth's gravitational potential is given by a more complicated expression than simply $\phi = -\frac{M}{r}$ (which is however a very good approximation). Actually, $\phi = -\frac{v^2}{2}$ has the same value at all points on the surface of the Earth, where ϕ is the Earth's gravitational potential and v is the Earth's rotation speed at that point. Show that as a consequence of this formula all clocks on the surface of the Earth tick at the same rate.

5.6 Twin paradox (yet again): Recall once more the setup of Problem 15 in Chapter 1: Two twins, Alice and Bob, part on their 20th birthday. While Alice remains on Earth (which is an inertial frame to a very good approximation), Bob departs at 80% of the speed of light towards Planet X, 8 light-years away from Earth.

Therefore Bob reaches his destination 10 years later (as measured on the Earth's frame). After a short stay, he returns to Earth, again at 80% of the speed of light. Consequently Alice is 40 years old when she sees Bob again, whereas Bob is only 32 years old.

(a) In both legs of his journey Bob is on inertial frames, and the time dilation formula applies. How much time does he expect Alice to experience?

(b) Bob is then forced to conclude that Alice experienced the missing time during the very short (according to him) acceleration phase of his journey. Check that this is consistent with the gravitational redshift formula.

6.1 Show that if $\frac{M}{r}, \frac{M}{r'} \ll 1$ then the Schwarzschild gravitational redshift formula reduces to the approximate formula

$$T' = (1 + \Delta\phi)T$$

6.2 Check that the precession of the perihelion of Mercury's orbit due to general relativistic effects is about 43 arcseconds per century (distance from Mercury to the Sun: approximately 3.1 light-minutes). What is the precession of the perihelion of Earth's orbit due to these effects?

6.3 Compute the period of a circular orbit of radius $r > 3M$:

(a) For an observer at infinity.

(b) As seen by a stationary observer. What is the orbital velocity measured by these observers? What happens as r approaches $3M$?

(c) As measured by an orbiting observer. What happens as r approaches $3M$?

(d) How is it possible that the free-falling observers in orbit measure a smaller period than the (accelerated) stationary observers?

6.4

(a) Show that the fact that the sum of the internal angles of a Euclidean triangle is π is equivalent to the statement that the angles α, β and γ in the figure below satisfy $\alpha + \beta = \gamma$.

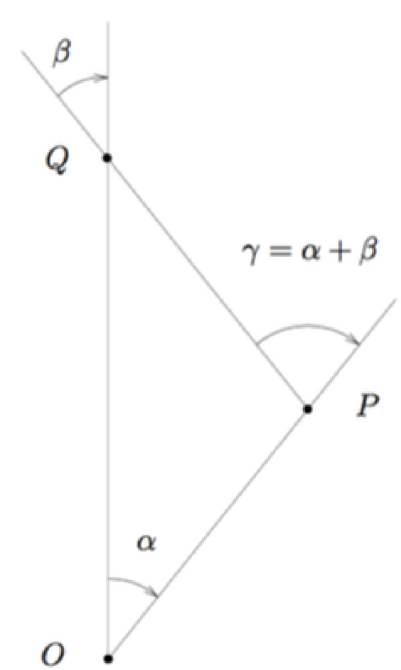

Figure: 6.9: The sum of the internal angles of an Euclidean triangle is π.

(b) For speeds much smaller than the speed of light, the **angle** between two causal curves is just the relative velocity of the corresponding observers. Show that for these speeds the above relation is still valid in the Minkowski geometry.

(c) Two circular orbits with the same radius r, traversed in opposite directions, form a two-sided polygon in the Schwarzschild geometry. Approximating the polygon's area by half the area of the cylinder of radius r and height equal to half the period of the orbit(Note that the map corresponding to the Schwarzschild coordinates preserves areas), estimate the curvature of the Schwarzschild solution for $\frac{M}{r} \ll 1$.

6.5

(a) What is the escape velocity for a stationary observer? What happens when r approaches $2M$?

(b) For which value of r is this velocity equal to the velocity of the circular orbit? What is the corresponding velocity?

6.6 A typical **neutron star** has a mass of about 1.4 times the mass of the Sun compressed into a sphere with a 10 kilometer radius.

(a) Compute the escape velocity for a stationary observer on the neutron star's surface.

(b) How many Earth gravities does the stationary observer measure?

6.7 Show that the deflection of a light ray just grazing the Sun's surface is about 1.75 arcseconds (radius of the Sun: approximately 2.3 light-seconds).

6.8 Draw a space-time diagram describing the observation of the Einstein Cross from Earth at a given time.

6.9 During an exciting space battle, the Enterprise and a Klingon warship fall into the same circular orbit around a black hole, in diametrically opposite positions (see figure). Where should Captain Kirk aim his lasers?

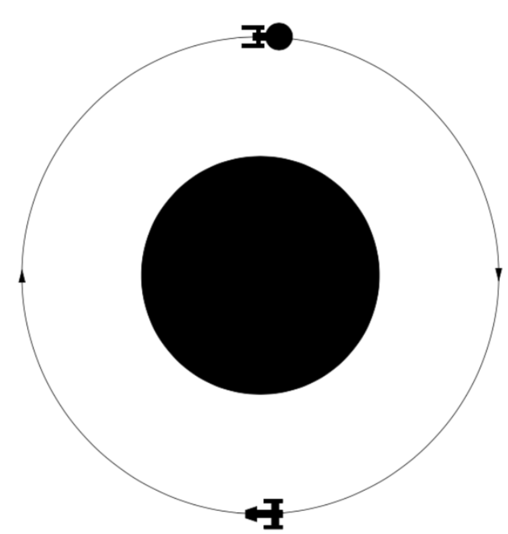

Figure: 6.11: Where should Captain Kirk aim his lasers?

6.10 Show that Painleve's time coordinate t' satisfies

$$\Delta t' = \Delta t + \frac{\sqrt{\frac{2M}{r}}}{1 - \frac{2M}{r}} \Delta r$$

6.11 Show that the observers who in the Painleve coordinates satisfy

$$\frac{\Delta r}{\Delta t'} = -\sqrt{\frac{2M}{r}} \text{ and } \frac{\Delta \theta}{\Delta t'} = 0$$

are free-falling, and that t' is their proper time.

6. 12 A particle falling into a black hole is observed by a stationary observer at infinity. What does she see?

6.13

(a) What causes the tides? How many high tides are there per day?

(b) Compute the approximate mass of the Moon from the fact that the Moon's tides are about twice as high as the Sun's tides.

6.14 Compute the radius of the event horizon of Sagittarius A^* in light-seconds and in solar radii. 6.15 Compute the tidal forces at the event horizon in Earth gravities per meter:

(a) For Cygnus X-1.

(b) For Sagittarius A^*.

Solutions

Section 1

1. The radius for the parallel through Lisbon is about $6,400 \times \cos(39°) \simeq 5,000$ kilometers (see Figure 3.5), yielding a circumference of about $2\pi \times 5,000 \simeq 31,000$ kilometers. Therefore a point at this latitude travels $31,000$ kilometers each 24 hours, corresponding to a speed of about $1,300$ kilometers per hour. This speed is greater than the speed of sound, which is approximately $0.34 \times 3,600 \simeq 1,200$ kilometers per hour.

2. One light-minute is the distance travelled by light (whose speed is $300,000$ kilometers per second) in one minute. The circumference of the Earth's orbit is therefore $2\pi \times 8.3 \times 60 \times 300,000$ kilometers, and the Earth travels this distance in one year, that is, $365 \times 24 \times 3,600$ seconds. Dividing these two numbers yields the result.

3. By the (Galileo) velocity addition formula, the speed of the bullets with respect to the ground is $900 - 900 = 0$ kilometers per hour. Consequently, the bullets simply fall down vertically.

4. Relative to the train the ball is travelling at $50 + 100 = 150$ kilometers per hour. Since the collision is perfectly elastic, it comes back with the same speed. Relative to the boy, it then comes back at $150 + 100 = 250$ kilometers per hour. This trick is often used by space probes in a maneuver called **gravitational assist**. In this maneuver, the probe plays the role of the tennis ball and the planet plays the role of the train.

   Instead of colliding elastically with the planet, the probe performs a close flyby, being strongly deflected by the planet's gravitational field. By conservation of energy, this deflection behaves much like an elastic collision, allowing the probe to increase its speed considerably.

5. As seen by observers on Earth, the light hits the stern first, as the stern is moving towards the emission point (whereas the bow is moving away from the emission point). Quantitatively, let $2L$ be the *Enterprise*'s length, and assume that the floodlight is placed at $x' = 0$ and is lit at time $t' = 0$. Then its light hits the stern ($x' = -L$) and the bow ($x' = L$) at time $t' = \frac{L}{c}$. Since $\frac{v}{c} = 0.8$, and hence $\sqrt{1 - \frac{v^2}{c^2}} = 0.6$, the Lorentz transformation formulas tell us that in the Earth's frame the light hits the stern at time

$$t = \frac{t' + 0.8\frac{x'}{c}}{0.6} = \frac{\frac{L}{c} - 0.8\frac{L}{c}}{0.6} = \frac{L}{3c},$$

and hits the bow at time

$$t = \frac{t' + 0.8\frac{x'}{c}}{0.6} = \frac{\frac{L}{c} + 0.8\frac{L}{c}}{0.6} = \frac{3L}{c}.$$

Thus from the point of view of observers on Earth, the light takes 9 times longer to hit the bow than to hit the stern.

6. We just have to check that

$$\gamma\left(t' + \frac{vx'}{c^2}\right) = \gamma^2\left(t - \frac{vx}{c^2}\right) + \gamma^2\frac{v}{c^2}(x - vt) = \gamma^2\left(1 - \frac{v^2}{c^2}\right)t = t$$

and

$$\gamma(x' + vt') = \gamma^2(x - vt) + v\gamma^2\left(t - \frac{vx}{c^2}\right) = \gamma^2\left(1 - \frac{v^2}{c^2}\right)x = x.$$

7. The ruler's ends move according to the equations

$$x' = 0 \Leftrightarrow \gamma(x - vt) = 0 \Leftrightarrow x = vt$$

and

$$x' = l' \Leftrightarrow \gamma(x - vt) = l' \Leftrightarrow x = \frac{l'}{\gamma} + vt.$$

Therefore the ruler's length in the frame S is

$$l = \frac{l'}{\gamma} + vt - vt = \frac{l'}{\gamma} = l'\sqrt{1 - \frac{v^2}{c^2}}$$

(which is always less than l').

8. We just have to see that

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}} \Leftrightarrow u - v = u' - \frac{u'uv}{c^2} \Leftrightarrow \left(1 + \frac{u'v}{c^2}\right) u = u' + v \Leftrightarrow u = \frac{u' + v}{1 + \frac{u'v}{c^2}}.$$

9. In both cases the velocity of the rocket with respect to the Earth is $v = 0.5\,c$. Therefore:

(a) When the missile is fired forwards its velocity with respect to the rocket is $u' = 0.5\,c$. Thus its velocity with respect to the Earth will be

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}} = \frac{0.5\,c + 0.5\,c}{1 + 0.5 \times 0.5} = \frac{c}{1.25} = 0.8\,c.$$

(b) When the missile is fired backwards its velocity with respect to the rocket is $u' = -0.5\,c$. Thus its velocity with respect to the Earth will be

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}} = \frac{-0.5\,c + 0.5\,c}{1 - 0.5 \times 0.5} = 0.$$

10. The frame of the rocket which is moving from left to right has velocity $v = 0.5\,c$. In the Earth's frame, the rocket which is moving from right to left has velocity $u = -0.5\,c$. With respect to the first rocket, its velocity is then

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}} = \frac{-0.5\,c - 0.5\,c}{1 + 0.5 \times 0.5} = -\frac{c}{1.25} = -0.8\,c.$$

11. By the Pythagorean theorem, the distance traveled by the light signal in the frame S is

$$c^2\Delta t^2 = v^2\Delta t^2 + \Delta y^2 = v^2\Delta t^2 + \Delta y'^2 = v^2\Delta t^2 + c^2\Delta t'^2.$$

Solving for $\Delta t'$ we obtain

$$\Delta t' = \Delta t\sqrt{1 - \frac{v^2}{c^2}}.$$

12. If $|\varepsilon| \ll 1$ then $\varepsilon^2 \ll |\varepsilon|$ (for instance if $\varepsilon = 0.01$ then $\varepsilon^2 = 0.0001$). Therefore with an error on order ε^2 (hence negligible) we have:

    (a) $(1-\varepsilon)(1+\varepsilon) = 1 - \varepsilon^2 \simeq 1$, whence $\frac{1}{1+\varepsilon} \simeq 1 - \varepsilon$;

    (b) $\left(1 + \frac{\varepsilon}{2}\right)^2 = 1 + \varepsilon + \frac{\varepsilon^2}{4} \simeq 1 + \varepsilon$, whence $\sqrt{1+\varepsilon} \simeq 1 + \frac{\varepsilon}{2}$.

    For instance,

    (a) $\frac{1}{1.01} = 0.99009900... \simeq 0.99 = 1 - 0.01$;

    (b) $\sqrt{1.01} = 1.00498756... \simeq 1.005 = 1 + \frac{0.01}{2}$.

13. In this problem the fact that the frame attached to the surface of the Earth is not an inertial frame (because of the Earth's rotation) is relevant. Consequently, we must use the inertial frame attached to the center of the Earth. As we saw in the answer to Exercise 1, the length of the parallel of latitude 39° is $L = 31,000$ kilometers. Consequently the airplane journey took approximately $\frac{31{,}000}{900} \simeq 31.4$ hours, that is, about $124,000$ seconds.

(a) As was seen in Exercise 1, the clock on the surface of the Earth is moving at about $1,300$ kilometers per hour in the frame attached to the center of the Earth. Thus when the airplane is flying eastwards it is moving at $1,300 + 900 = 2,200$ kilometers per hour in this frame, whereas when it flies westwards it is moving at $1,300 - 900 = 400$ kilometers per hour. The difference in the readings of the two clocks when the airplane has flown eastwards is then

$$124,000\left(\sqrt{1-\frac{1,300^2}{(3,600\times 300,000)^2}}-\sqrt{1-\frac{2,200^2}{(3,600\times 300,000)^2}}\right) \text{ seconds.}$$

Using the approximation $\sqrt{1-\frac{v^2}{c^2}} \simeq 1-\frac{v^2}{2c^2}$, we obtain

$$124,000\times\frac{2,200^2-1,300^2}{2\times(3,600\times 300,000)^2} \simeq 170\times 10^{-9} \text{ seconds.}$$

The difference in the readings of the two clocks when the airplane has flown westwards is

$$124,000\left(\sqrt{1-\frac{1,300^2}{(3,600\times 300,000)^2}}-\sqrt{1-\frac{400^2}{(3,600\times 300,000)^2}}\right) \text{ seconds,}$$

that is, approximately

$$124,000\times\frac{400^2-1,300^2}{2\times(3,600\times 300,000)^2}\simeq -80\times 10^{-9} \text{ seconds.}$$

Thus if the clock was flown eastwards it was about 170 nanoseconds late with respect to the stationary clock, whereas if it was flown westwards it was about 80 nanoseconds early. These differences were indeed measured in the experiment, together with the corrections due to the gravitational field (see Exercise 3 in Chapter 5).

(b) Let $V \simeq 1,300$ kilometers per hour be the rotation speed of the Earth at latitude 39°, and suppose that the clock is transported at a very small velocity v along the parallel. Then the journey duration will be $\frac{L}{v}$. If the clock is transported eastwards then the desynchronization between the fixed and the moving clock will be

$$\frac{L}{v}\sqrt{1-\frac{V^2}{c^2}}-\frac{L}{v}\sqrt{1-\frac{(V+v)^2}{c^2}}\simeq\frac{L}{v}\frac{(V+v)^2-V^2}{2c^2}=\frac{L}{v}\frac{(2V+v)v}{2c^2}\simeq\frac{VL}{c^2},$$

that is,

$$\frac{\frac{1,300}{3,600}\times 31,000}{300,000^2}\simeq 120\times 10^{-9} \text{ seconds.}$$

Thus the moving clock will be about 120 nanoseconds late with respect to the fixed clock. If the clock is transported westwards then the desynchronization will have the same absolute value but opposite sign, that is, the moving clock will be about 120 nanoseconds early with respect to the fixed clock.

The GPS satellite navigation system relies on ground stations which track the satellites' motions with great accuracy. These stations have very precise atomic clocks, which must be synchronized to the nanosecond. To synchronize the clocks the Sagnac effect must be taken into account.

14. If the muons are moving at a speed of v kilometers per second, it will take them at least $\frac{10}{v}$ seconds to reach the ground. In the muons' frame, however, the elapsed time is

$$\frac{10}{v}\sqrt{1-\frac{v^2}{c^2}},$$

because of time dilation. For the muons to reach the ground this time interval must be less than 2.2×10^{-6} seconds:

$$\frac{10}{v}\sqrt{1-\frac{v^2}{c^2}} < 2.2 \times 10^{-6} \Leftrightarrow \frac{100}{v^2}\left(1-\frac{v^2}{c^2}\right) < 4.84 \times 10^{-12} \Leftrightarrow \frac{1}{v^2}-\frac{1}{c^2} < 4.84 \times 10^{-14}$$
$$\Leftrightarrow \frac{c^2}{v^2} < 1 + 4.4 \times 10^{-3} \Leftrightarrow \frac{v}{c} > \frac{1}{\sqrt{1+4.4 \times 10^{-3}}} \simeq \frac{1}{1+2.2 \times 10^{-3}} \simeq 1 - 2.2 \times 10^{-3}.$$

So the muons detected on the ground must be moving at more than 99.998% of the speed of light.

15. (a) Since 20 years have gone by for Alice, during which Bob was mostly moving at 80% of the speed of light, Bob must have experienced

$$20\sqrt{1-0.8^2} = 20\sqrt{0.36} = 20 \times 0.6 = 12 \text{ years},$$

and so he will by 32 years old when they meet again.

(b) The asymmetry in the twins' ages is due to the fact that only Alice is on an inertial frame, since Bob must slow down as he reaches Planet X and then speed up again to return to Earth. Although velocity is a relative concept, being an inertial observer or an accelerated observer is an absolute concept.

Section 2

1. A light-year is about

$$365 \times 24 \times 3,600 \times 300,000 \simeq 9.5 \times 10^{12} \text{ kilometers.}$$

A light-meter is about

$$\frac{0.001}{300,000} \simeq 3.3 \times 10^{-9} \text{ seconds,}$$

that is, about 3.3 nanoseconds.

2. The solution of the problem becomes trivial when one draws the histories of the liners on a space-time diagram (Figure 2.9). Thus the passenger would meet 13 other liners, at noon and at midnight. Assuming that each liner would need one day to unload and reload, the service could be assured by 16 liners.

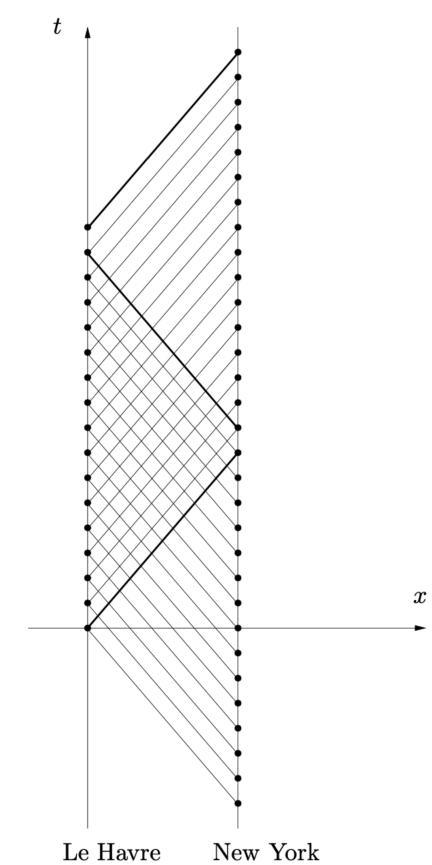

Figure 2.9: Space-time diagram for the Lucas problem.

3. By elementary trigonometry, it is easy to see from Figure 2.2 that

$$x' = \frac{x}{\cos\alpha} + \frac{y'}{\cos\alpha}\sin\alpha \Leftrightarrow x = x'\cos\alpha - y'\sin\alpha$$

and

$$y = \frac{y'}{\cos\alpha} + \frac{x}{\cos\alpha}\sin\alpha \Leftrightarrow y' = y\cos\alpha - x\sin\alpha.$$

Substituting the second equation in the first yields

$$x = x'\cos\alpha - y\cos\alpha\sin\alpha + x\sin^2\alpha \Leftrightarrow x'\cos\alpha = x\cos^2\alpha + y\cos\alpha\sin\alpha,$$

whence

$$x' = x\cos\alpha + y\sin\alpha.$$

4. Let $(0,0)$ be the coordinates of the event O corresponding to Bob's departure in the Earth's frame S. Then the event P in which Bob arrives on Planet X has coordinates $(10,8)$, and the event Q in which Bob arrives on Earth has coordinates $(20,0)$.

(a) The inertial frame S' in which Bob is at rest during the first leg of the journey is moving with velocity $v = 0.8$ with respect to S. Therefore $\sqrt{1-v^2} = 0.6$, and so the coordinates (t', x') of an event in S' are related to the coordinates (t, x) of an event in S by the Lorentz transformation formulas

$$t' = \frac{t - vx}{\sqrt{1 - v^2}} = \frac{t - 0.8x}{0.6} \quad \text{and} \quad x' = \frac{x - vt}{\sqrt{1 - v^2}} = \frac{x - 0.8t}{0.6}.$$

Thus in S' event O has coordinates $(0, 0)$, event P has coordinates $(6, 0)$ (as it should), and event Q has coordinates $(\frac{100}{3}, -\frac{80}{3})$. These events are plotted in Figure 2.10. In we compute in this frame the time measured by Alice between events O and Q we obtain

$$\sqrt{\left(\frac{100}{3}\right)^2 - \left(\frac{80}{3}\right)^2} = \sqrt{400} = 20 \text{ years.}$$

In the same way, the time measured by Bob between events O and P is clearly 6 years, and the time measured by Bob between events P and Q is

$$\sqrt{\left(\frac{100}{3} - 6\right)^2 - \left(\frac{80}{3}\right)^2} = \sqrt{36} = 6 \text{ years.}$$

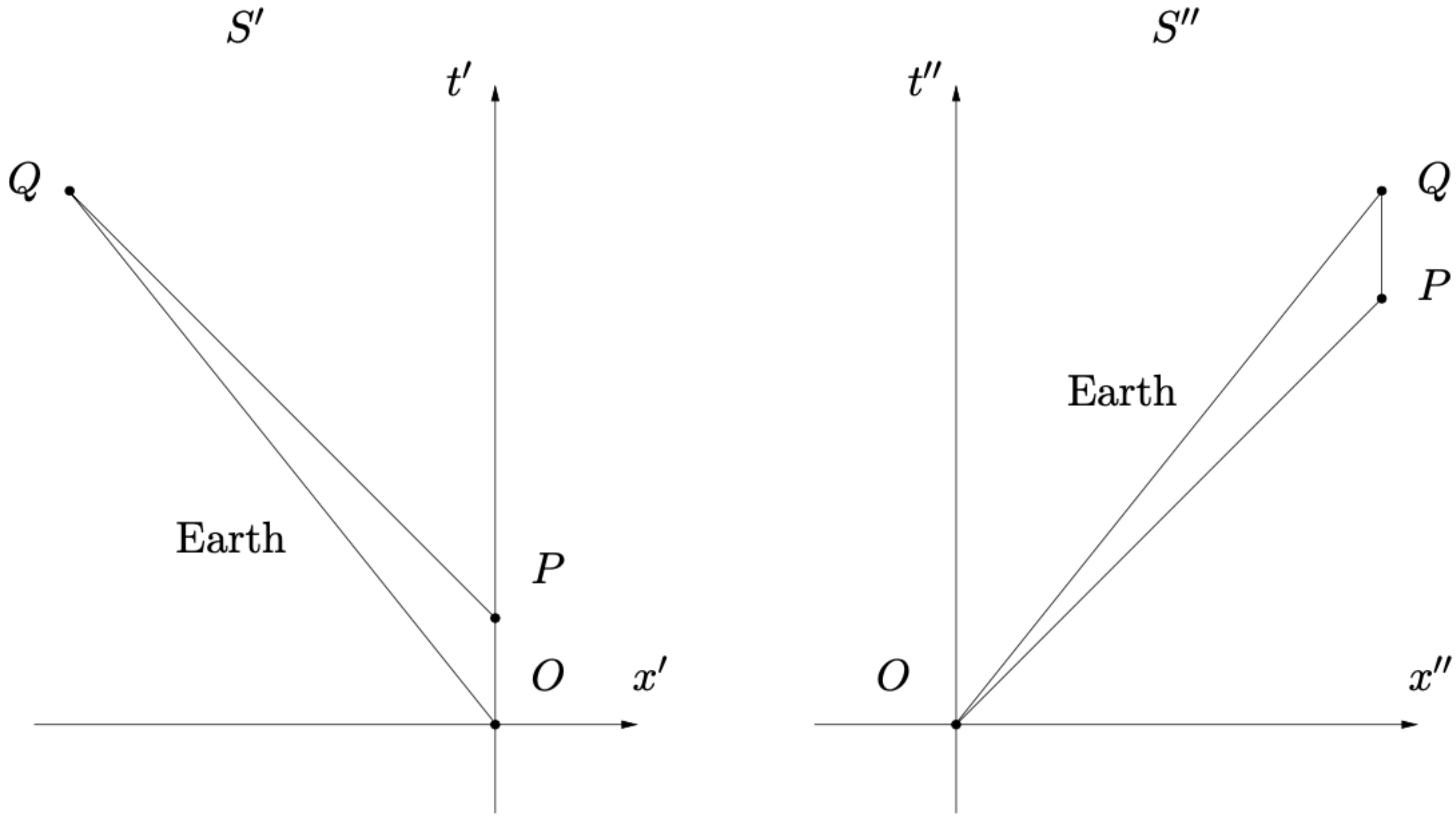

Figure 2.10: Space-time diagrams for the twin paradox in the frames S' and S''.

(b) The inertial frame S'' in which Bob is at rest in the return leg of the journey is moving with velocity $v = -0.8$ with respect to S. Consequently, $\sqrt{1 - v^2} = 0.6$, and so the coordinates (t'', x'') of an event in S'' are related to the coordinates (t, x) of an event in S by the Lorentz transformation formulas

$$t'' = \frac{t - vx}{\sqrt{1 - v^2}} = \frac{t + 0.8x}{0.6} \quad \text{and} \quad x'' = \frac{x - vt}{\sqrt{1 - v^2}} = \frac{x + 0.8t}{0.6}.$$

Therefore in S'' event O has coordinates $(0, 0)$, event P has coordinates $\left(\frac{82}{3}, \frac{80}{3}\right)$, and event Q has coordinates $\left(\frac{100}{3}, \frac{80}{3}\right)$. These events are plotted in Figure 2.10. If we compute in this frame the time measured by Alice between events O and Q we obtain again

$$\sqrt{\left(\frac{100}{3}\right)^2 - \left(\frac{80}{3}\right)^2} = \sqrt{400} = 20 \text{ years.}$$

In the same way, the time measured by Bob between events O and P is

$$\sqrt{\left(\frac{82}{3}\right)^2 - \left(\frac{80}{3}\right)^2} = \sqrt{36} = 6 \text{ years.}$$

Finally, the time measured by Bob between events P and Q is clearly $\frac{100}{3} - \frac{82}{3} = \frac{18}{3} = 6$ years.

(c) At event P Bob is receiving light which left Earth in $t = 2$, meaning that in the 6 years of the first leg of the journey Bob saw only 2 years of Alice's life (Figure 2.11).

Consequently, in the first leg of the journey Bob saw Alice moving in slow motion, at a rate 3 times slower than normal. In the 6 years of the return leg, Bob will see the remaining 18 years which Alice will experience, and so he will see her moving at a rate 3 times faster than normal.

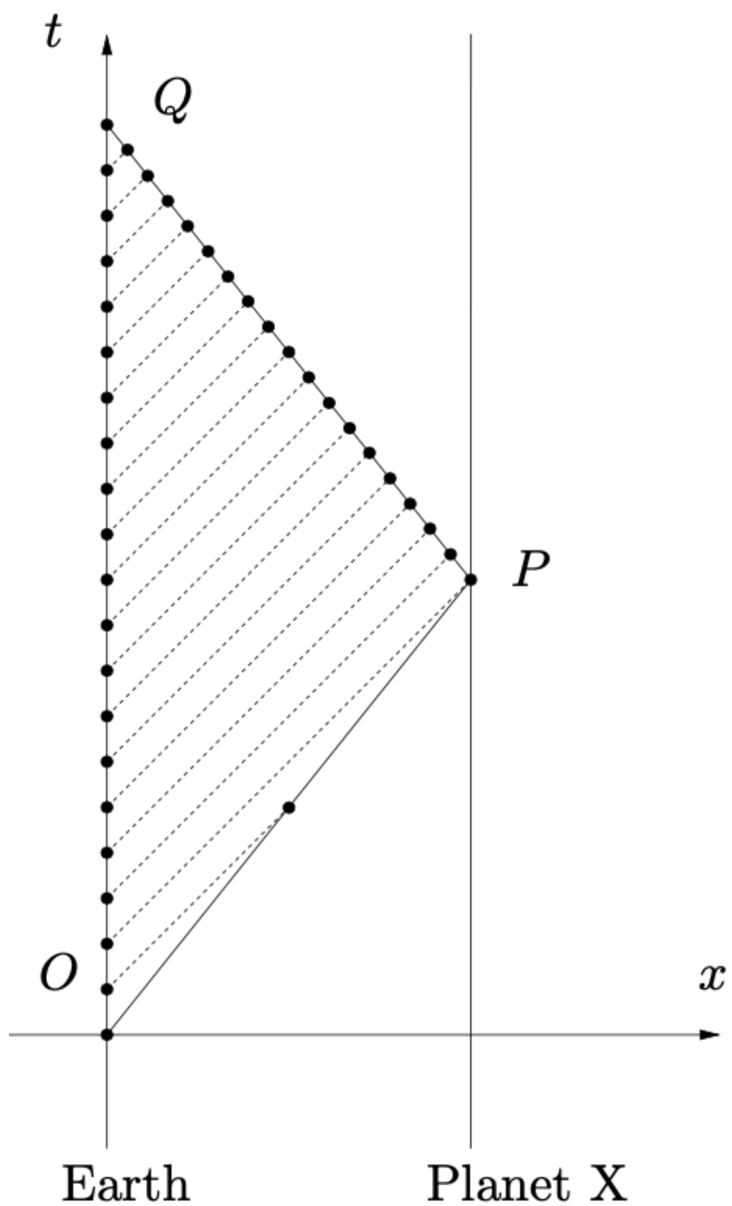

Figure 2.11: Space-time diagram for the twin paradox.

On the other hand, light emitted at event P reaches Alice at $t = 18$ (Figure 2.11), so that she spends the first 18 years watching the 6 years of the first leg of Bob's journey. Therefore she saw Bob moving in slow motion, at a rate 3 times slower than normal. In the remaining 2 years Alice will see the 6 years of the return leg, and so she will see Bob moving at a rate 3 times faster than normal.

5. We can model a periodic light signal as a sequence of flashes. If the first flash occurs at time $t = t_0$, its history is the line with equation $t = t_0 + x$. Therefore the observer in S' detects the flash in the event with coordinates

$$\begin{cases} t = t_0 + x \\ x = vt \end{cases} \quad \Leftrightarrow \begin{cases} t = \dfrac{t_0}{1-v} \\ \\ x = \dfrac{vt_0}{1-v} \end{cases}$$

Similarly, the second flash is emitted at time $t = t_0 + T$, its history is the line with equation $t = t_0 + T + x$, and it is detected in S' in the event with coordinates

$$\begin{cases} t = \dfrac{t_0 + T}{1-v} \\ \\ x = \dfrac{v(t_0 + T)}{1-v} \end{cases}$$

Consequently, the time interval measured in S' between the two flashes is

$$T' = \sqrt{\left(\frac{t_0+T}{1-v} - \frac{t_0}{1-v}\right)^2 - \left(\frac{v(t_0+T)}{1-v} - \frac{vt_0}{1-v}\right)^2} = \sqrt{\frac{T^2}{(1-v)^2} - \frac{v^2T^2}{(1-v)^2}}$$
$$= T\sqrt{\frac{1-v^2}{(1-v)^2}} = T\sqrt{\frac{(1-v)(1+v)}{(1-v)^2}} = T\sqrt{\frac{1+v}{1-v}}.$$

If $v = 0.8$, say, we have

$$\sqrt{\frac{1+v}{1-v}} = \sqrt{\frac{1.8}{0.2}} = \sqrt{9} = 3.$$

This is consistent with what the twins in Exercise 4 see through their telescopes during the first leg of Bob's journey: each year for the faraway twin is observed during a 3-year period. If, on the other hand, $v = -0.8$, we have

$$\sqrt{\frac{1+v}{1-v}} = \sqrt{\frac{0.2}{1.8}} = \sqrt{\frac{1}{9}} = \frac{1}{3}.$$

Indeed, during the return leg of the journey each twin observes a 3-year period for the faraway twin per year.

For $|v| \ll 1$ we can apply the formula

$$\frac{1}{1-v} \simeq 1+v$$

to obtain the approximation

$$T' = T\sqrt{\frac{1+v}{1-v}} \simeq T\sqrt{(1+v)^2} = T(1+v).$$

6. (a) (i) We assume that the theft of the *Einstein* is the event with coordinates $(0,0)$ in the Earth's frame S. Then the history of the *Einstein* from that event on is represented by the line $x = 0.6\,t$. The *Enterprise* leaves Earth in the event $(1,0)$, and its history from that event on is the line $x = 0.8(t-1)$. Therefore the *Enterprise* reaches the *Einstein* in the event with coordinates

$$\begin{cases} x = 0.6\,t \\ x = 0.8(t-1) \end{cases} \Leftrightarrow \begin{cases} x = 0.6\,t \\ 0.2t = 0.8 \end{cases} \Leftrightarrow \begin{cases} t = 4 \\ x = 2.4 \end{cases}$$

From the point of view of an observer in the Earth's inertial frame, the *Einstein* is then captured 4 years after the theft.

(ii) According to the Klingon spy, the time interval between commandeering the *Einstein* and being captured is

$$\sqrt{4^2 - 2.4^2} = 3.2 \text{ years.}$$

(iii) According to the *Enterprise* crew, the chase takes

$$\sqrt{(4-1)^2 - 2.4^2} = 1.8 \text{ years,}$$

and so they experience $1+1.8 = 2.8$ years between the theft and the recapture of the *Einstein*.

(b) Since the information about the outcome of the battle will propagate at most at the speed of light, only after $4 + 2.4 = 6.4$ years can the engineers expect to know whether to send the self-destruct signal.

(c) In the worst case the *Einstein* keeps moving away from the Earth at 60% of the speed of light, corresponding to the history $x = 0.6\,t$. The history of the self-destruct signal is the line $t = 6.4 + x$. Consequently, the *Einstein*'s self-destruction would occurs at the event with coordinates

$$\begin{cases} x = 0.6\,t \\ t = 6.4 + x \end{cases} \Leftrightarrow \begin{cases} x = 9.6 \\ t = 16 \end{cases}$$

Light from this event would reach the Earth at $t = 16 + 9.6 = 25.6$ years. So the engineers could expect confirmation of the *Einstein*'s self-destruction within 25.6 years after the theft.

7. In the planets' frame S, let $(0,0)$ be the coordinates of the event O where the plot is uncovered, $(11,0)$ the coordinates of the event L where the missile is launched, $(12,12)$ the coordinates of the event D in which Organia is destroyed, and $(13,12)$ the coordinates of the event A where the *Enterprise* arrives at the debris. These events are represented in Figure 2.12.

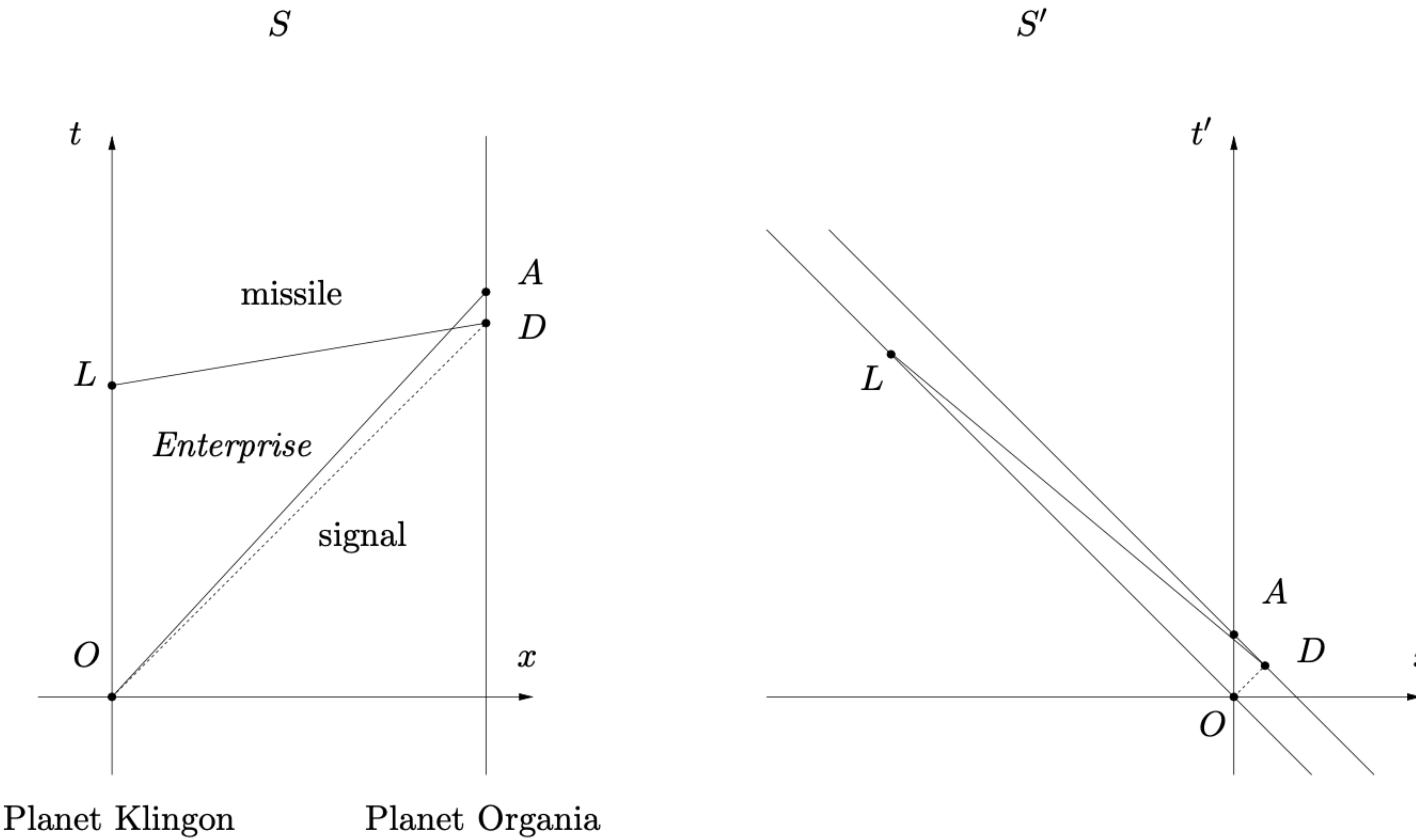

Figure 2.12: Space-time diagrams for the faster-than-light missile.

(a) The duration of the journey for the *Enterprise* crew is simply the interval $\overline{OA}$, that is,

$$\sqrt{13^2 - 12^2} = \sqrt{25} = 5 \text{ years.}$$

(b) The *Enterprise*'s frame S' moves with velocity $v = \frac{12}{13}$ with respect to S. Consequently, $\sqrt{1-v^2} = \frac{5}{13}$, and thus the coordinates (t', x') of an event in S' are related to the coordinates (t, x) of an event in S by the Lorentz transformations

$$t' = \frac{t - vx}{\sqrt{1-v^2}} = \frac{13t - 12x}{5} \quad \text{and} \quad x' = \frac{x - vt}{\sqrt{1-v^2}} = \frac{13x - 12t}{5}.$$

Thus in S' event O has coordinates $(0, 0)$, event L has coordinates $(28.6, -26.4)$, event D has coordinates $(2.4, 2.4)$, and event A has coordinates $(5, 0)$ (as it should).

(c) These events are plotted in the *Enterprise*'s frame S' in Figure 2.12. In S' the sequence of events is surreal: planet Organia explodes for no reason; the faster-than-light missile jumps from the debris and flies backwards towards planet Klingon, where an exact replica is being built; finally, the two missiles disappear simultaneously at event L. This illustrates the kind of absurd situations which can occur if faster-than-light speeds are permitted.

Section 3

1. Because the shortest distance between Lisbon and New York is not measured along the parallel through the two cities, but along the great circle which contains them. This circle is the intersection of the Earth's surface with the plane defined by Lisbon, New York and the center of the Earth. Therefore the shortest path between Lisbon and New York is a curve whose initial heading is northwest.

2. The meridians of the two points form a great circle, and the angle between them is $\frac{\pi}{2}$. The distance between the two points is then $\frac{\pi R}{2}$. The radius of the parallel through the two points is $R\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}R}{2}$, and the difference in latitudes is π. The length of the parallel arc between them is then $\frac{\pi\sqrt{2}R}{2} > \frac{\pi R}{2}$.

3. (a) Plane: false (there exist parallel lines). Sphere: true.
   (b) Plane: true. Sphere: false (they always intersect at two points).
   (c) Plane: true. Sphere: false.
   (d) Plane: true. Sphere: true.
   (e) Plane: true. Sphere: false (for example any meridian contains the poles).
   (f) Plane: true. Sphere: false (the maximum distance between two points is πR).
   (g) Plane: false. Sphere: true.
   (h) Plane: true. Sphere: false.
   (i) Plane: true. Sphere: false (for instance the equator is a circle centered at the north pole with radius $r = \frac{\pi}{2}R$, but its length is $2\pi R = 4r \neq 2\pi r$).
   (j) Plane: true. Sphere: false (for example the equator encloses an area $2\pi R^2 = \frac{8}{\pi}r^2 \neq \pi r^2$).

4. Let α be the angle at the north pole. Since the angles on the equator are right angles, the spherical excess is exactly α. On the other hand, the triangle contains a fraction $\frac{\alpha}{2\pi}$ of the northern hemisphere's area $2\pi R^2$, that is, αR^2. We conclude that the triangle's average curvature is
$$\frac{\alpha}{\alpha R^2} = \frac{1}{R^2}.$$

5. Let α be the common value of the polygon's internal angles. Then it contains a fraction $\frac{\alpha}{2\pi}$ of the sphere's area $4\pi R^2$, that is, $2\alpha R^2$. We conclude that the ratio between the sum of the polygon's internal angles and its area is
$$\frac{2\alpha}{2\alpha R^2} = \frac{1}{R^2}.$$

6. If α is the angle at the north pole then the triangle contains a fraction $\frac{\alpha}{2\pi}$ of the northern hemisphere's area $2\pi R^2$, that is, αR^2. It is easy to see that the triangle projects to a rectangle with the same base length αR and height equal to the sphere's radius R. Therefore the area of the projection is also αR^2.

7. In the case of the stereographic projection, the product of the coefficients of Δx^2 and Δy^2 is

$$16\left(1+\frac{x^2}{R^2}+\frac{y^2}{R^2}\right)^{-4} \neq 1.$$

For the sphere's map using latitude and longitude as coordinates, the product of the coefficients of $\Delta\theta^2$ and $\Delta\varphi^2$ is

$$R^4 \cos^2\theta \neq 1.$$

8. A conformal map of the sphere is of the form

$$\Delta s^2 = A(x,y)\Delta x^2 + A(x,y)\Delta y^2$$

for some coefficient $A(x,y)$. If it preserved areas, we would have

$$A^2 = 1 \Rightarrow A = 1.$$

But then the sphere's metric in this coordinate system would be

$$\Delta s^2 = \Delta x^2 + \Delta y^2,$$

which is the metric of the Euclidean plane in Cartesian coordinates. In particular, the sphere's geodesics would be represented by straight lines on the map, and the sum of a triangle's internal angles would always be π, which we know not to be true for the sphere. Thus no conformal map of the sphere can preserve areas.

1. In geometrized units, the gravitational acceleration on the Earth's surface is

$$9.8 \times \frac{1 \text{ meter}}{(1 \text{ second})^2} \simeq 9.8 \times \frac{1 \text{ meter}}{(3 \times 10^8 \text{ meters})^2} \simeq 1.1 \times 10^{-16} \text{ meters}^{-1}.$$

   On the other hand, we know that the Earth's radius is $r \simeq 6,400$ kilometers. From the gravitational acceleration formula, $g = \frac{M}{r^2}$, we conclude that the Earth's mass in geometrized units is

$$M = gr^2 \simeq 1.1 \times 10^{-16} \times (6.4 \times 10^6)^2 \simeq 4.5 \times 10^{-3} \text{ meters},$$

   that is, about 4.5 millimeters.

2. In geometrized units, the escape velocity from the Earth's surface is

$$v = \sqrt{\frac{2M}{r}} \simeq \sqrt{\frac{9.0 \times 10^{-3}}{6.4 \times 10^6}} \simeq 3.8 \times 10^{-5}.$$

   We can obtain this velocity in kilometers per second by multiplying by the speed of light:

$$v \simeq 3.8 \times 10^{-5} \times 3 \times 10^5 \simeq 11 \text{ kilometers per second}.$$

3. The Halley comet's speed will be approximately the escape velocity at 4.9 light-minutes from the Sun. Since the mass of the Sun is about 1.5 kilometers, this speed will then be

$$v = \sqrt{\frac{2M}{r}} \simeq \sqrt{\frac{3.0}{4.9 \times 60 \times 300,000}} \simeq 1.8 \times 10^{-4}.$$

To convert to kilometers per second we multiply by the speed of light:

$$v \simeq 1.8 \times 10^{-4} \times 3 \times 10^{5} \simeq 54 \text{ kilometers per second.}$$

4. When the coordinate θ varies by a small amount $\Delta\theta$ in Figure 4.1, the length r of the line segment joining the center of M to the point particle m practically does not change. So the swept area ΔA is well approximated by the area of the circular sector of radius r and central angle $\Delta\theta$. The area of this sector is a fraction $\frac{\Delta\theta}{2\pi}$ of the area of the circle of radius r, that is, $\Delta A = \frac{\Delta\theta}{2\pi}\pi r^2 = \frac{1}{2}r^2\Delta\theta$. So the orbit's angular momentum can be written as $L = 2\frac{\Delta A}{\Delta t}$, and the law of conservation of angular momentum is equivalent to the statement that the velocity at which the area is swept is constant.

5. Since the Earth's orbit is approximately circular, its orbital speed is

$$v = \sqrt{\frac{M}{r}},$$

where M is the Sun's mass and r is the distance from the Earth to the Sun. On the other hand, the period of the Earth's orbit is $T = 1$ year, and the distance from the

Earth to the Sun is $r \simeq 8.3$ light-minutes. Using minutes and light-minutes as units, we see that the Earth's orbital speed is

$$v = \frac{2\pi r}{T} \simeq \frac{2\pi \times 8.3}{365 \times 24 \times 60} \simeq 1.0 \times 10^{-4}.$$

Hence the Sun's geometrized mass is

$$M = v^2 r \simeq 10^{-8} \times 8.3 \times 60 \times 3 \times 10^5 \simeq 1.5 \text{ kilometers}.$$

6. For a circular low Earth orbit we have $r \simeq 6,400$ kilometers. The period of the orbit is

$$T = \frac{2\pi r}{v} = 2\pi\sqrt{\frac{r^3}{M}}.$$

Since $\frac{M}{r^2} = g \simeq 9.8$ meters per squared second, we have

$$T \simeq 2\pi \times \sqrt{\frac{6.4 \times 10^6}{9.8}} \simeq 5,100 \text{ seconds} \simeq 85 \text{ minutes}.$$

7. If T is the period of a circular orbit of radius r then

$$vT = 2\pi r \Leftrightarrow T\sqrt{\frac{M}{r}} = 2\pi r \Leftrightarrow r = \left(\frac{MT^2}{4\pi^2}\right)^{\frac{1}{3}}.$$

For a geostationary orbit we have $M \simeq 4.5$ millimeters and $T \simeq 24$ hours. Converting to kilometers yields

$$r \simeq \left(\frac{4.5 \times 10^{-6} \times (24 \times 3,600 \times 300,000)^2}{4\pi^2}\right)^{\frac{1}{3}} \simeq 42,000 \text{ kilometers}.$$

For the GPS satellites' orbits $T \simeq 12$ hours, and hence

$$r \simeq \left(\frac{4.5 \times 10^{-6} \times (12 \times 3,600 \times 300,000)^2}{4\pi^2}\right)^{\frac{1}{3}} \simeq 27,000 \text{ kilometers}.$$

8. For the Moon's (circular) orbit we have $r \simeq 1.3$ light-seconds and $M \simeq 4.5$ millimeters. Converting to seconds, the period of the orbit is

$$T = \frac{2\pi r}{v} = 2\pi\sqrt{\frac{r^3}{M}} \simeq 2\pi\sqrt{\frac{1.3^3 \times 3 \times 10^5}{4.5 \times 10^{-6}}} \simeq 2.4 \times 10^6 \text{ seconds,}$$

that is, about 28 days.

9. From Figure 4.3 we see that the orbit of the star SO–20 is roughly circular, with radius approximately 0.2 arcseconds wide, corresponding to a distance

$$\frac{0.2}{3,600} \times \frac{\pi}{180} \times 26,000 \simeq 2.52 \times 10^{-2} \text{ light-years.}$$

This distance is about

$$\frac{2.52 \times 10^{-2} \times 365 \times 24 \times 60}{8.3} \simeq 1,600$$

times the distance from the Earth to the Sun. If T is the period of a circular orbit of radius r then

$$vT = 2\pi r \Leftrightarrow T\sqrt{\frac{M}{r}} = 2\pi r \Leftrightarrow M = \frac{4\pi^2 r^3}{T^2}.$$

Since the star SO–20 took 8 years to complete about a quarter of a revolution, its period is around 32 years. We conclude that the mass of Sagittarius A* is approximately

$$\frac{1,600^3}{32^2} \simeq 4 \times 10^6$$

times the mass of the Sun.

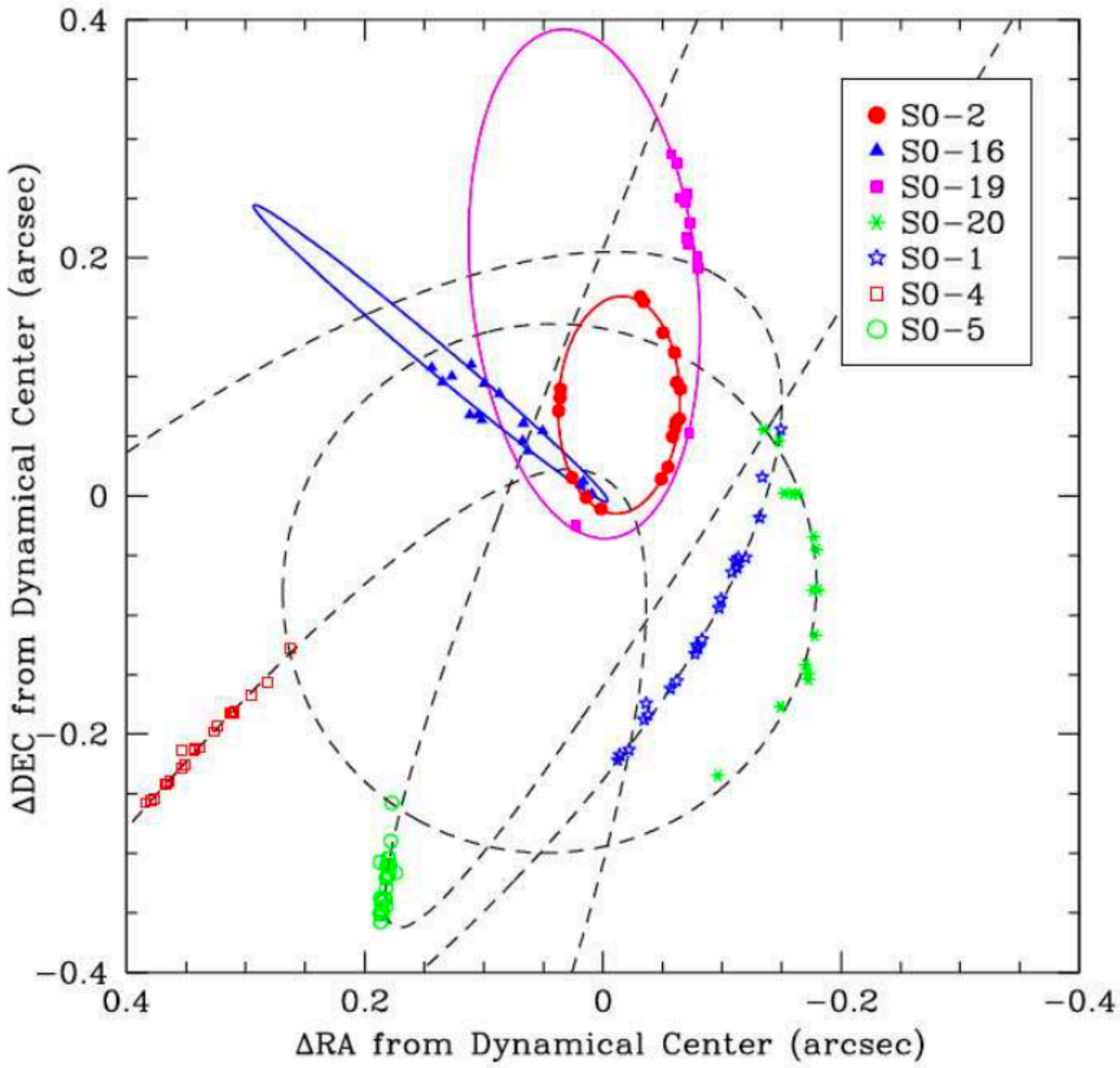

Figure 4.3: Stars orbiting Sagittarius A* (from "Stellar Orbits Around the Galactic Center Black Hole", by A. M. Ghez, S. Salim, S. D. Hornstein, A. Tanner, J. R. Lu, M. Morris, E. E. Becklin and G. Duchene, The Astrophysical Journal 620 (2005)).

1. The percentual variation was

$$\frac{T' - T}{T} = \Delta\phi = g\Delta z \simeq \frac{9.8 \times 23}{(3 \times 10^8)^2} \simeq 2.5 \times 10^{-15}.$$

2. Let E and T be the energy and the period of a photon on a given point P of the gravitational field, and let E', T' be the same quantities at another point P'. Let $\Delta\phi$ be the potential difference between P' and P. The Planck-Einstein relation implies

$$ET = E'T',$$

and if we accept the formula for the photon's energy loss we are force to conclude that

$$E' = E - \frac{E}{c^2}\Delta\phi = \left(1 - \frac{\Delta\phi}{c^2}\right) E.$$

Therefore

$$T = \left(1 - \frac{\Delta\phi}{c^2}\right) T',$$

or, using the fact that $\left|\frac{\Delta\phi}{c^2}\right| \ll 1$, we have the approximate formula

$$T' = \left(1 + \frac{\Delta\phi}{c^2}\right) T,$$

which is just the gravitational redshift formula ($c = 1$ in our units).

3. Since the airplane flew at an average altitude of 10 kilometers, the clocks on board the airplane registered a longer travel time than similar clocks on the surface by a fraction

$$\Delta\phi = g\Delta z \simeq \frac{9.8 \times 10^4}{(3 \times 10^8)^2} \simeq 1.1 \times 10^{-12}.$$

Since the travel took about 124,000 seconds, this corresponds to an additional time of about

$$1.1 \times 10^{-12} \times 1.24 \times 10^5 \simeq 1.4 \times 10^{-7} \text{ seconds},$$

that is, about 140 nanoseconds (independently of the flight direction). Therefore the clock which flew eastwards was only late about $170 - 140 = 30$ nanoseconds, whereas the clock which flew westwards was about $80 + 140 = 220$ nanoseconds early[7].

4. If an inertial observer "at infinity" (i.e. far away from the Earth) measures a time interval Δt, a satellite moving with speed v on a point at a distance r from the center of the Earth measures a time interval

$$\Delta t_{SAT} = \sqrt{1 - v^2}\left(1 - \frac{M}{r}\right)\Delta t \simeq \left(1 - \frac{v^2}{2}\right)\left(1 - \frac{M}{r}\right)\Delta t \simeq \left(1 - \frac{v^2}{2} - \frac{M}{r}\right)\Delta t.$$

In the same way, an observer on the Earth's surface measures a time interval

$$\Delta t_{EARTH} \simeq \left(1 - \frac{V^2}{2} - \frac{M}{R}\right)\Delta t,$$

where V is the Earth's rotation speed and R is the radius of the Earth. Therefore

$$\frac{\Delta t_{SAT}}{\Delta t_{EARTH}} = \frac{1-\frac{v^2}{2}-\frac{M}{r}}{1-\frac{V^2}{2}-\frac{M}{R}} \simeq \left(1-\frac{v^2}{2}-\frac{M}{r}\right)\left(1+\frac{V^2}{2}+\frac{M}{R}\right) \simeq 1-\frac{v^2}{2}-\frac{M}{r}+\frac{V^2}{2}+\frac{M}{R}.$$

We have already seen that $r \simeq 27,000$ kilometers, so that

$$\frac{M}{r} \simeq \frac{4.5 \times 10^{-6}}{27,000} \simeq 1.7 \times 10^{-10}.$$

On the other hand,

$$\frac{v^2}{2} = \frac{M}{2r} \simeq 0.8 \times 10^{-10}.$$

Analogously, since $R \simeq 6,400$ kilometers, we have

$$\frac{M}{R} \simeq \frac{4.5 \times 10^{-6}}{6,400} \simeq 7.0 \times 10^{-10}.$$

Finally, the Earth's rotation speed on the equator is about

$$\frac{2\pi \times 6,400}{24 \times 3,600} \simeq 0.47 \text{ kilometers per second},$$

so that

$$\frac{V^2}{2} \simeq 1.2 \times 10^{-12}.$$

We conclude that

$$\frac{\Delta t_{SAT}}{\Delta t_{EARTH}} \simeq 1 + 4.5 \times 10^{-10},$$

that is, the clock on the satellite is fast by about $4.5 \times 10^{-10} \times 24 \times 3,600 \simeq 4.0 \times 10^{-5}$ seconds per day, corresponding to about $4.0 \times 10^{-5} \times 3 \times 10^5 = 12$ light-kilometers.

5. If an inertial observer "at infinity" (i.e. far away from the Earth) measures a time interval Δt, an observer moving with speed v on a point where the Earth's gravitational potential is ϕ measures a time interval

$$\Delta t' = \sqrt{1 - v^2}\,(1 + \phi)\,\Delta t \simeq \left(1 - \frac{v^2}{2}\right)(1 + \phi)\,\Delta t \simeq \left(1 - \frac{v^2}{2} + \phi\right)\Delta t.$$

Since $\phi - \frac{v^2}{2}$ is equal for all points on the surface of the Earth, we conclude that all clocks on the surface of the Earth tick at the same rate. For instance, a clock on the equator is moving faster than a clock on the north pole (so it should tick slower), but it is further away from the center of the Earth (so it should tick faster); these two effects exactly cancel.

6. (a) We have already seen that Bob takes 6 years to complete the first leg of the journey. Since during this time Alice is moving at 80% of the speed of light with respect to him, he expects her to experience

$$6\sqrt{1-0.8^2} = 6 \times 0.6 = 3.6 \text{ years.}$$

The same is true for the return leg, and so he expects Alice to experience in total $3.6 + 3.6 = 7.2$ years.

(b) Let a be Bob's acceleration and Δt the (small) time interval he spends accelerating. Then we must have $a\Delta t = 0.8 + 0.8 = 1.6$. When Bob is accelerating he can imagine that he is in an uniform gravitational field, with Alice 8 light-years higher up, corresponding to a potential difference $\Delta\phi = 8a$. By the gravitational redshift formula, she must experience a time interval

$$\Delta t' = (1 + \Delta\phi)\Delta t = \Delta t + 8a\Delta t = \Delta t + 8 \times 1.6 \simeq 12.8 \text{ years.}$$

Notice that this is exactly the missing time: $12.8 + 7.2 = 20$. (The fact that this works out perfectly is however just a happy coincidence: the gravitational redshift formula can only be expected to hold for $|\Delta\phi| \ll 1$. Moreover, Alice is not at rest in the accelerated frame: she must be in free fall as she is an inertial observer).

Section 6

1. If $\frac{M}{r}, \frac{M}{r'} \ll 1$ then

$$\sqrt{\frac{1-\frac{2M}{r'}}{1-\frac{2M}{r}}} \simeq \sqrt{\left(1-\frac{2M}{r'}\right)\left(1+\frac{2M}{r}\right)} \simeq \sqrt{1-\frac{2M}{r'}+\frac{2M}{r}} \simeq 1-\frac{M}{r'}+\frac{M}{r} = 1+\phi'-\phi,$$

where ϕ and ϕ' are the gravitational potential at points with coordinates r and r'. Consequently,

$$T' = T\sqrt{\frac{1-\frac{2M}{r'}}{1-\frac{2M}{r}}} \simeq (1+\Delta\phi)T.$$

2. The period T of a circular orbit of radius r satisfies

$$\frac{2\pi r}{T} = \sqrt{\frac{M}{r}} \Leftrightarrow T = 2\pi\sqrt{\frac{r^3}{M}}.$$

Therefore the period of Mercury's orbit is about

$$\sqrt{\left(\frac{3.1}{8.3}\right)^3} \simeq 0.23 \text{ years}.$$

The precession of Mercury's perihelion is about

$$\frac{6\pi \times 1.5}{3.1 \times 60 \times 300,000}$$

radians per orbit, that is, about

$$\frac{6\pi \times 1.5}{3.1 \times 60 \times 300,000} \times \frac{180}{\pi} \times 3,600 \times \frac{100}{0.23} \simeq 45$$

arcseconds per century (the difference to the exact value of 43 arcseconds per century is due to our approximations).

The precession of Earth's perihelion is about

$$\frac{6\pi \times 1.5}{8.3 \times 60 \times 300,000}$$

radians per orbit, that is, about

$$\frac{6\pi \times 1.5}{8.3 \times 60 \times 300,000} \times \frac{180}{\pi} \times 3,600 \times 100 \simeq 4$$

arcseconds per century.

3. (a) Since circular orbits satisfy

$$\left(\frac{\Delta\theta}{\Delta t}\right)^2 = \frac{M}{r^3} \Leftrightarrow \Delta t = \pm\Delta\theta\sqrt{\frac{r^3}{M}},$$

we see that the period of the orbit for an observer at infinity (i.e. the value of Δt as $\Delta\theta = \pm 2\pi$) is

$$T_\infty = 2\pi\sqrt{\frac{r^3}{M}}.$$

(b) As seen by a stationary observer, the period is

$$T_S = T_\infty \sqrt{1 - \frac{2M}{r}} = 2\pi \sqrt{\frac{r^3}{M}} \sqrt{1 - \frac{2M}{r}}.$$

The orbital velocity measured by these observers is then

$$v = \frac{2\pi r}{T_S} = \frac{\sqrt{\frac{M}{r}}}{\sqrt{1 - \frac{2M}{r}}}.$$

As r approaches $3M$ this value tends to

$$\frac{\sqrt{\frac{1}{3}}}{\sqrt{1 - \frac{2}{3}}} = 1,$$

i.e. the speed of light. There are indeed null geodesics corresponding to light rays in circular orbits of radius $r = 3M$.

(c) An orbiting observer satisfies $\Delta r = 0$ and

$$\Delta\theta^2 = \frac{M}{r^3} \Delta t^2,$$

and so she measures a proper time interval $\Delta\tau$ given by and so she measures a proper time interval $\Delta\tau$ given by

$$\begin{aligned}\Delta\tau^2 &= \left(1-\frac{2M}{r}\right)\Delta t^2 - \left(1-\frac{2M}{r}\right)^{-1}\Delta r^2 - r^2\Delta\theta^2 \\ &= \left(1-\frac{2M}{r}\right)\Delta t^2 - \frac{M}{r}\Delta t^2 \\ &= \left(1-\frac{3M}{r}\right)\Delta t^2,\end{aligned}$$

that is,

$$\Delta\tau = \Delta t\sqrt{1-\frac{3M}{r}}.$$

Consequently, the period of the orbit for an orbiting observer is

$$T_O = T_\infty\sqrt{1-\frac{3M}{r}} = 2\pi\sqrt{\frac{r^3}{M}}\sqrt{1-\frac{3M}{r}}.$$

Note that T_O tends to zero as r approaches $3M$. This was to be expected, as the velocity of the orbiting observer (with respect to the stationary observers) tends to the speed of light as r approaches $3M$.

(d) Geometrically, the period measured by the orbiting observer is the length of the geodesic which represents a full orbit, whereas the period measured by the stationary observer is the length of a non-geodesic curve joining the same events (because the stationary observer is not free-falling). Therefore the geodesic corresponding to the circular orbit does not have maximum length. This phenomenon is due to the curvature of the Schwarzschild space-time. Analogously, there are always two geodesic segments connecting two non-antipodal points on the sphere, of which only one has minimum length (together they form the great circle defined by the two points). For the two events connected by the circular orbit, the maximizing geodesic is the one which describes a particle thrown upwards (i.e. in the radial direction) with the right speed so that it reaches its maximum altitude after half an orbit, returning to the initial point at the end of the orbit.

4. (a) The internal angles of the triangle are α, β and $\pi - \gamma$. Therefore we must have

$$\alpha + \beta + \pi - \gamma = \pi \Leftrightarrow \alpha + \beta = \gamma.$$

(b) With this interpretation, α is the velocity of OP with respect to OQ, β is the velocity of OQ with respect to PQ and γ is the velocity of OP with respect to PQ. Since the velocities are much smaller than the speed of light we have $\gamma = \alpha + \beta$.

(c) For $\frac{M}{r} \ll 1$, the speed of each of the circular orbits is approximately

$$v = \sqrt{\frac{M}{r}}.$$

Each of the polygon's angles is approximately $2v$, and the sum of the polygon's internal angles is then approximately $4v$. The area of the cylinder of radius r and height equal to half the orbit's period is approximately

$$A = 2\pi r \times \frac{\pi r}{v} = \frac{2\pi^2 r^2}{v}.$$

Therefore the curvature of the Schwarzschild solution for $\frac{M}{r} \ll 1$ is of the order

$$\frac{8v}{A} = \frac{8v^2}{2\pi^2 r^2} = \frac{4}{\pi^2} \frac{M}{r^3}.$$

5. (a) A particle thrown in the radial direction has angular momentum

$$L = r^2 \frac{\Delta\theta}{\Delta\tau} = 0,$$

and so the differential equations describing its motion are

$$\frac{\Delta r}{\Delta\tau} = \pm\sqrt{2E + \frac{2M}{r}};$$
$$\frac{\Delta t}{\Delta\tau} = \left(1 - \frac{2M}{r}\right)^{-1} \sqrt{1 + 2E}.$$

The first equation implies

$$2E + \frac{2M}{r} \geq 0 \Leftrightarrow -Er \leq M.$$

If $E < 0$ then the range of the r coordinate is limited. Consequently the escape velocity corresponds to $E = 0$. The above equations then imply

$$\frac{\Delta r}{\Delta t} = \sqrt{\frac{2M}{r}} \left(1 - \frac{2M}{r}\right).$$

The distance measured by a stationary observer is

$$\Delta s = \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} \Delta r,$$

and her proper time is

$$\Delta\tau = \Delta t\sqrt{1 - \frac{2M}{r}}.$$

Consequently the escape velocity measured by a stationary observer is

$$v = \frac{\Delta s}{\Delta\tau} = \left(1 - \frac{2M}{r}\right)^{-1} \frac{\Delta r}{\Delta t} = \sqrt{\frac{2M}{r}}$$

(which coincidentally is the Newtonian result). Note that when r approaches $2M$ the escape velocity approaches 1 (i.e. the speed of light).

(b) Recall from Exercise 3 that the orbital speed measured by a stationary observer is

$$v = \frac{\sqrt{\frac{M}{r}}}{\sqrt{1 - \frac{2M}{r}}}.$$

This speed is equal to the escape velocity for

$$\frac{\sqrt{\frac{M}{r}}}{\sqrt{1 - \frac{2M}{r}}} = \sqrt{\frac{2M}{r}} \Leftrightarrow 1 = 2\left(1 - \frac{2M}{r}\right) \Leftrightarrow r = 4M,$$

corresponding to a velocity $\frac{\sqrt{2}}{2} \simeq 71\%$ of the speed of light. The circular orbit of radius $r = 4M$, like all orbits with radius smaller than $6M$, is unstable, and can be reached by a particle dropped from infinity with the right angular momentum.

6. (a) The escape velocity for a stationary observer on the neutron star's surface is

$$v \simeq \sqrt{\frac{2 \times 1.4 \times 1.5}{10}} \simeq 0.65$$

(i.e. about 65% of the speed of light).

(b) Using kilometers as units, the gravitational field measured by the stationary observer on the neutron star's surface is

$$g = \frac{\frac{M}{r^2}}{\sqrt{1 - \frac{2M}{r}}} \simeq \frac{\frac{1.4 \times 1.5}{10^2}}{\sqrt{1 - \frac{2 \times 1.4 \times 1.5}{10}}} \simeq 2.8 \times 10^{-2}.$$

In the same units, the Earth's gravitational acceleration is

$$g = \frac{M}{r^2} \simeq \frac{4.5 \times 10^{-6}}{6,400^2} \simeq 1.1 \times 10^{-13}.$$

Therefore the gravitational field measured by the stationary observer on the neutron star's surface is about

$$\frac{2.8 \times 10^{-2}}{1.1 \times 10^{-13}} \simeq 2.5 \times 10^{11}$$

Earth gravities.

7. Using kilometers as units, we see that the deflection of a light ray just grazing the Sun's surface is about

$$\frac{4 \times 1.5}{2.3 \times 300,000} \times \frac{180}{\pi} \times 3,600 \simeq 1.8 \text{ arcseconds}$$

(the difference to the exact value of 1.75 arcseconds is due to our approximations).

8. The space-time diagram describing the observation of the Einstein Cross from Earth at a given time is depicted in Figure 6.12. There are four null geodesics connecting the history of the quasar to the event in which it is observed on Earth at the given time. Generically, the light rays corresponding to these geodesics were emitted at different times.