The following notes are taken from a very old textbook (1964) that I used in graduate school to learn this subject.

The textbook is

Relativistic Quantum Mechanics

by

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1-The Dirac Equation

1.1 Formulation of a Relativistic Quantum Theory

Since the principles of special relativity are generally accepted at this time, a correct quantum theory should satisfy the requirement of relativity: laws of motion valid in one inertial system must be true in all inertial systems.

Stated mathematically, relativistic quantum theory must be formulated in a Lorentz covariant form.

In making the transition from nonrelativistic to relativistic quantum mechanics, we shall endeavor to retain the principles under-lying the nonrelativistic theory.

We review them briefly:

1 - For a given physical system there exists a state function Φ that summarizes all that we can know about the system. In our initial development of the relativistic one-particle theory, we usually deal directly with a coordinate realization of the state function, the wave function $\psi(q_i \cdots, s_i \cdots, t)$.

$\psi(q, s, t)$ is a complex function of all the classical degrees of freedom, $q_1 \cdots q_n$, of the time t and of any additional degrees of freedom, such as spin s_i, which are intrinsically quantum-mechanical.

The wave function has no direct physical interpretation; however, $|\psi(q_1 \cdots q_n, s_1 \cdots s_n, t)|^2 \geq 0$ is interpreted as the probability of the system having values $(q_1 \cdots s_n)$ at time t.

Evidently this probability interpretation requires that the sum of positive contributions $|\psi|^2$ for all values of $q_1 \cdot \cdot \cdot s_n$ at time t be finite for all physically acceptable wave functions ψ.

2 - Every physical observable is represented by a linear hermitian operator.

In particular, for the canonical momentum p_i the operator correspondence in a coordinate realization is

$$p_i \rightarrow \frac{\hbar}{i}\frac{\partial}{\partial q_i}$$

3 - A physical system is in an eigenstate of the operator Ω if

$$\Omega\Phi_n = \omega_n\Phi_n \tag{1.1}$$

where Φ_n is the *n*th eigenstate corresponding to the eigenvalue ω_n.

For a hermitian operator, ω_n is real.

In a coordinate realization the equation corresponding to (1.1) is

$$\Omega(q,s,t)\psi_n(q,s,t) = \omega_n\psi_n(q,s,t)$$

the the

4 - The expansion postulate states that an arbitrary wave function, or state function, for a physical system can be expanded in a complete orthonormal set of eigenfunctions ψ_n of a complete set of commuting operators (Ω_n).

We write, then,

$$\psi = \sum_n a_n \psi_n$$

where the statement of orthonormality is

$$\sum_s \int (dq_1 \cdots) \psi_n^*(q_1 \cdots, s \cdots, t) \psi_m(q_1 \cdots, s \cdots, t) = \delta_{nm}$$

$|a_n|^2$ records the probability that the system is in the nth eigenstate.

5 - The result of a measurement of a physical observable is any one of its eigenvalues.

In particular, for a physical system described by the wave function $\psi = \sum_n a_n \psi_n$, with $\Omega\, \psi_n = \omega_n \psi_n$, measurement of a physical observable Ω results in the eigenvalue ω_n with a probability $|a_n|^2$.

The average of many measurements of the observable Ω on identically prepared systems is given by

$$\langle \Omega \rangle_\psi \equiv \sum_s \int \psi^*(q_1 \cdots, s \cdots, t) \Omega \psi(q_1 \cdots, s \cdots, t) (dq_1 \cdots)$$
$$= \sum_n |a_n|^2 \omega_n$$

6 - The time development of a physical system is expressed by the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi \tag{1.2}$$

where the hamiltonian H is a linear hermitian operator.

It has no explicit time dependence for a closed physical system, that is, $\partial H/\partial t = 0$, in which case its eigenvalues are the possible stationary states of the system.

A superposition principle follows from the linearity of H and a statement of conservation of probability from the hermitian property of H:

$$\begin{aligned} \frac{d}{dt} \sum_s \int \psi^* \psi (dq_1 \cdots) &= \frac{i}{\hbar} \sum_s \int (dq_1 \cdots)[(H\psi)^* \psi - \psi^* (H\psi)] \\ &= 0 \end{aligned} \tag{1.3}$$

We strive to maintain these familiar six principles as underpinnings of a relativistic quantum theory.

1.2 Early Attempts

The simplest physical system is that of an isolated free particle, for which the nonrelativistic hamiltonian is

$$H = \frac{p^2}{2m} \tag{1.4}$$

The transition to quantum mechanics is achieved with the transcription

$$H \rightarrow i\hbar \frac{\partial}{\partial t}$$

$$\mathbf{p} \rightarrow \frac{\hbar}{i} \nabla \tag{1.5}$$

which leads to the nonrelativistic Schrödinger equation

$$i\hbar \frac{\partial \psi(q,t)}{\partial t} = \frac{-\hbar^2 \nabla^2}{2m} \psi(q,t) \tag{1.6}$$

Equations (1.4) and (1.6) are noncovariant and therefore unsatisfactory.

The left- and right-hand sides transform differently under Lorentz transformations.

According to the theory of special relativity, the total energy E and momenta (p_x, p_y, p_z) transform as components of a contravariant four-vector

$$p^\mu = (p^0, p^1, p^2, p^3) = \left(\frac{E}{c}, p_x, p_y, p_z \right)$$

of invariant length

$$\sum_{\mu=0}^{3} p_\mu p^\mu \equiv p_\mu p^\mu = \frac{E^2}{c^2} - \mathbf{p} \cdot \mathbf{p} \equiv m^2 c^2 \tag{1.7}$$

m is the rest mass of the particle and c the velocity of light in vacuo.

The covariant notation used throughout this book is discussed in more detail in Appendix A.

Here we only note that the operator transcription (1.5) is Lorentz covariant, since it is a correspondence between two contravariant four-vectors $p^\mu \rightarrow \mathrm{i}\,\hbar\,\partial/\partial \mathrm{x}_\mu$, where we defined $x^\mu = (ct,\mathbf{x})$ and $\nabla^\mu = \partial/\partial x_\mu$.

Following this it is natural to take as the hamiltonian of a relativistic free particle

$$H = \sqrt{p^2c^2 + m^2c^4} \tag{1.8}$$

and to write for a relativistic quantum analogue of (1.6)

$$i\hbar \frac{\partial \psi}{\partial t} = \sqrt{-\hbar^2c^2\nabla^2 + m^2c^4}\,\psi \tag{1.9}$$

Immediately we are faced with the problem of interpreting the square-root operator on the right in Eq. (1.9).

If we expand it, we obtain an equation containing all powers of the derivative operator and thereby a nonlocal theory.

Such theories are very difficult to handle and present an unattractive version of the Schrödinger equation in which the space and time coordinates appear in unsymmetrical form.

In the interest of mathematical simplicity (though perhaps with a lack of complete physical cogency) we remove the square-root operator in (1.9), writing

$$H^2 = p^2c^2 + m^2c^4 \tag{1.10}$$

Equivalently, iterating (1.9) and using the fact that if $[A, B] = 0$, $A\psi = B\psi$ implies $A^2\psi = B^2\psi$, we have

$$-\hbar^2 \frac{\partial^2}{\partial t^2}\psi = (-\hbar^2\nabla^2c^2 + m^2c^4)\psi$$

This is recognized as the classical wave equation

where

$$\left[\Box + \left(\frac{mc}{\hbar}\right)^2\right]\psi = 0 \qquad \Box \equiv \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu} \tag{1.11}$$

Before looking further into (1.11), we note first that in squaring the energy relation we have introduced an extraneous negative-energy root

$$H = -\sqrt{p^2c^2 + m^2c^4}$$

In order to gain a simple equation, we have sacrificed positive definite energy and introduced the difficulty of "extra" negative-energy solutions.

This difficulty is eventually surmounted (as we shall study in Chap. 5), and the negative-energy solutions prove capable of physical interpretation.

In particular, they are associated with antiparticles. and the existence of antiparticles in nature lends strong experimental support for this procedure. So let us for a moment consider Eq. (1.10) and the inferred wave equation (1.11).

Our first task is to construct a conserved current, since (1.11) is a second-order wave equation and is altered from the Schrödinger form (1.2) upon which the probability interpretation in the nonrelativistic theory is based.

This we do in analogy with the Schrödinger equation, taking ψ^* times (1.11), ψ times the complex conjugate equation, and subtracting:

$$\psi^*\left[\Box + \left(\frac{mc}{\hbar}\right)^2\right]\psi - \psi\left[\Box + \left(\frac{mc}{\hbar}\right)^2\right]\psi^* = 0$$

$$\nabla^\mu(\psi^*\nabla_\mu\psi - \psi\nabla_\mu\psi^*) = 0$$

or

$$\frac{\partial}{\partial t}\left[\frac{i\hbar}{2mc^2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right)\right] + \operatorname{div}\frac{\hbar}{2im}[\psi^*(\nabla\psi) - \psi(\nabla\psi^*)] = 0 \tag{1.12}$$

We would like to interpret $(i\hbar/2mc^2)\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right)$ as a probability density ρ.

However, this is impossible, since it is not a positive definite expression.

For this reason we follow the path of history' and temporarily discard Eq. (1.11) in the hope of finding an equation of first order in the time derivative which admits a straightforward probability interpretation as in the Schrödinger case.

We shall return to (1.11), however.

Although we shall find a first-order equation, it still proves impossible to retain a positive definite probability density for a single particle while at the same time providing a physical interpretation of the negative-energy root of (1.10).

Therefore Eq. (1.11), also referred to frequently as the Klein-Gordon equation, remains an equally strong candidate for a relativistic quantum mechanics as the one which we now discuss.

1.3 The Dirac Equation

We follow the historic path taken in 1928 by Dirac in seeking a relativistically covariant equation of the form (1.2) with positive definite probability density.

Since such an equation is linear in the time derivative, it is natural to attempt to form a hamiltonian linear in the space derivatives as well.

Such an equation might assume a form

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{\hbar c}{i}\left(\alpha_1 \frac{\partial \psi}{\partial x^1} + \alpha_2 \frac{\partial \psi}{\partial x^2} + \alpha_3 \frac{\partial \psi}{\partial x^3}\right) + \beta mc^2 \psi \equiv H\psi \tag{1.13}$$

The coefficients α_i here cannot simply be numbers, since the equation would not be invariant even under a spatial rotation.

Also, if we wish to proceed at this point within the framework stated in Sec. 1.1, the wave function ψ cannot be a simple scalar.

In fact, the probability density $\rho = \psi^* \psi$ should be the time component of a conserved four-vector if its integral over all space, at fixed t, is to be an invariant.

To free (1.13) from these limitations, Dirac proposed that it be considered as a matrix equation.

The wave function ψ, in analogy with the spin wave functions of nonrelativistic quantum mechanics, is written as a column matrix with N components

$$\psi = \begin{bmatrix} \psi_1 \\ \cdot \\ \cdot \\ \cdot \\ \psi_N \end{bmatrix}$$

and the constant coefficients α_i, β are N x N matrices.

In effect then, Eq. (1.13) is replaced by N coupled first-order equations

$$i\hbar \frac{\partial \psi_\sigma}{\partial t} = \frac{\hbar c}{i} \sum_{\tau=1}^{N} \left(\alpha_1 \frac{\partial}{\partial x^1} + \alpha_2 \frac{\partial}{\partial x^2} + \alpha_3 \frac{\partial}{\partial x^3} \right)_{\sigma\tau} \psi_\tau + \sum_{\tau=1}^{N} \beta_{\sigma\tau} mc^2 \psi_\tau$$

$$= \sum_{\tau=1}^{N} H_{\sigma\tau} \psi_\tau \qquad (1.14)$$

Hereafter we adopt matrix notation and drop summation indices, in which case Eq. (1.14) appears as (1.13), to be now interpreted as a matrix equation.

If this equation is to serve as a satisfactory point of departure, first, it must give the correct energy-momentum relation

$$E^2 = p^2c^2 + m^2c^4$$

for a free particle, second, it must allow a continuity equation and a probability interpretation for the wave function ψ, and third, it must be Lorentz covariant.

We now discuss the first two of these requirements.

In order that the correct energy-momentum relation emerge from Eq. (1.13), each component ψ_σ of ψ must satisfy the Klein-Gordon second-order equation, or

$$-\hbar^2 \frac{\partial^2 \psi_\sigma}{\partial t^2} = (-\hbar^2 c^2 \nabla^2 + m^2 c^4) \psi_\sigma \qquad (1.15)$$

Iterating Eq. (1.13), we find

$$-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = -\hbar^2 c^2 \sum_{i,j=1}^{3} \frac{\alpha_j \alpha_i + \alpha_i \alpha_j}{2} \frac{\partial^2 \psi}{\partial x^i \, \partial x^j} + \frac{\hbar m c^3}{i} \sum_{i=1}^{3} (\alpha_i \beta + \beta \alpha_i) \frac{\partial \psi}{\partial x^i} + \beta^2 m^2 c^4 \psi$$

We may resurrect (1.15) if the four matrices α_i, β obey the algebra:

$$\begin{aligned} \alpha_i \alpha_k + \alpha_k \alpha_i &= 2\delta_{ik} \\ \alpha_i \beta + \beta \alpha_i &= 0 \\ \alpha_i^2 = \beta^2 &= 1 \end{aligned} \tag{1.16}$$

What other properties do we require of these four matrices α_i, β, and can we explicitly construct them?

The α_i and β must be hermitian matrices in order that the hamiltonian $H_{\sigma\tau}$ in (1.14) be a hermitian operator as desired according to the postulates of Sec. 1.1.

Since, by (1.16), $\alpha_i^2 = \beta^2 = 1$, the eigenvalues of α_i and β are ± 1.

Also, it follows from their anticommutation properties that the trace, that is, the sum of the diagonal elements, of each α_i and β is zero.

For example,

$$\alpha_i = -\beta\alpha_i\beta$$

and by the cyclic property of the trace

$$\mathrm{Tr}\ AB = \mathrm{Tr}\ BA$$

one has

$$\mathrm{Tr}\ \alpha_i = +\ \mathrm{Tr}\ \beta^2\alpha_i = +\ \mathrm{Tr}\ \beta\alpha_i\beta = -\ \mathrm{Tr}\ \alpha_i = 0$$

Since the trace is just the sum of eigenvalues, the number of positive and negative eigenvalues ±1 must be equal, and the α_i and β must therefore be even-dimensional matrices.

The smallest even dimension, N = 2, is ruled out, since it can accommodate only the three mutually anticommuting Pauli matrices σ_i plus a unit matrix.

The smallest dimension in which the α_i and β can be realized is N= 4, and that is the case we shall study.

In a particular explicit representation the matrices are

$$\alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix} \qquad \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1.17}$$

where the σ_i are the familiar 2 x 2 Pauli matrices and the unit entries in β stand for 2 x 2 unit matrices.

To construct the differential law of current conservation, we first introduce the hermitian conjugate wave functions $\psi^\dagger = (\psi_1^* \cdots \psi_4^*)$ and left-multiply (1.13) by $\psi^\dagger$:

$$i\hbar\psi^\dagger \frac{\partial\psi}{\partial t} = \frac{\hbar c}{i} \sum_{k=1}^{3} \psi^\dagger \alpha_k \frac{\partial}{\partial x^k} \psi + mc^2\psi^\dagger\beta\psi \tag{1.18}$$

Next we form the hermitian conjugate of (1.13) and right-multiply by ψ:

$$-i\hbar \frac{\partial\psi^\dagger}{\partial t} \psi = -\frac{\hbar c}{i} \sum_{k=1}^{3} \frac{\partial\psi^\dagger}{\partial x^k} \alpha_k\psi + mc^2\psi^\dagger\beta\psi \tag{1.19}$$

where $\alpha_i^\dagger = \alpha_i, \beta^\dagger = \beta$.

Subtracting (1.19) from (1.18), we find

$$i\hbar \frac{\partial}{\partial t} \psi^\dagger\psi = \sum_{k=1}^{3} \frac{\hbar c}{i} \frac{\partial}{\partial x^k} (\psi^\dagger \alpha^k \psi)$$

or

$$\frac{\partial}{\partial t} \rho + \operatorname{div} \mathbf{j} = 0 \tag{1.20}$$

where we make the identification of probability density

$$\rho = \psi^\dagger \psi = \sum_{\sigma=1}^{4} \psi_\sigma^* \psi_\sigma \tag{1.21}$$

and of a probability current with three components

$$j^k = c\psi^\dagger \alpha^k \psi \tag{1.22}$$

Integrating (1.20) over all space and using Green's theorem, we find

$$\frac{\partial}{\partial t} \int d^3x \psi^\dagger \psi = 0 \tag{1.23}$$

which encourages the tentative interpretation of $\rho = {}_{\psi}\dagger\, \psi$ as a positive definite probability density.

The notation (1.20) anticipates that the probability current j forms a vector if (1.22) is to be invariant under three-dimensional space rotations.

We must actually show much more than this.

The density and current in (1.20) must form a four-vector under Lorentz transformations in order to ensure the covariance of the continuity equation and of the probability interpretation.

Also, the Dirac equation (1.13) must be shown to be Lorentz covariant before we may regard it as satisfactory.

Before delving into the problem of establishing Lorentz invariance of the Dirac theory, it is perhaps more urgent to see first that the equation makes sense physically.

We may start simply by considering a free electron and counting the number of solutions corresponding to an electron at rest.

Equation (1.13) then reduces to

$$i\hbar \frac{\partial \psi}{\partial t} = \beta mc^2 \psi$$

since the de Broglie wavelength is infinitely large and the wave function is uniform over all space.

In the specific representation of Eq. (1.17) for β, we can write down by inspection four solutions:

$$\psi^1 = e^{-(imc^2/\hbar)t} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \psi^2 = e^{-(imc^2/\hbar)t} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\psi^3 = e^{+(imc^2/\hbar)t} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \psi^4 = e^{+(imc^2/\hbar)t} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{1.24}$$

the first two of which correspond to positive energy, and the second two to negative energy.

The extraneous negative-energy solutions which result from the quadratic form of $H^2 = p^2c^2 + m^2c^4$ are a major difficulty, but one for which the resolution leads to an important triumph in the form of antiparticles.

We come to this point in Chap. 5.

Here we confine ourselves to the "acceptable" positive-energy solutions.

In particular, we wish to show that they have a sensible nonrelativistic reduction to the two-component Pauli spin theory.

To this end we introduce an interaction with an external electromagnetic field described by a four-potential

$$A^\mu : (\Phi, \mathbf{A})$$

The coupling is most simply introduced by means of the gauge-invariant substitution

$$p^\mu \longrightarrow p^\mu - \frac{e}{c} A^\mu \tag{1.25}$$

made in classical relativistic mechanics to describe the interaction of a point charge e with an applied field.

In the present case

$$p^\mu \longrightarrow i\hbar\, \partial/\partial x_\mu \equiv \mathrm{p}^\mu$$

according to (1.5), and (1.25) takes the Dirac equation (1.13) to

$$i\hbar \frac{\partial \psi}{\partial t} = \left(c\boldsymbol{\alpha} \cdot \left(\mathbf{p} - \frac{e}{c}\mathbf{A} \right) + \beta mc^2 + e\Phi \right) \psi \tag{1.26}$$

Equation (1.26) expresses the "minimal" interaction of a Dirac particle, considered to be a point charge, with an applied electromagnetic field.

To emphasize its classical parallel, we write in (1.26) $H = H_0 + H'$, with $H' = -e\boldsymbol{\alpha} \cdot \mathbf{A} + e\Phi$.

The matrix $c\boldsymbol{\alpha}$ appears here as the operator transcription of the velocity operator in the classical expression for the interaction energy of a point charge:

$$H'_{\text{classical}} = -\frac{e}{c}\mathbf{v} \cdot \mathbf{A} + e\Phi$$

This operator correspondence $\mathbf{v}_{\text{op}} = c\boldsymbol{\alpha}$ is again evident in Eq. (1.22) for the probability current.

It also follows if we make the relativistic extension of the Ehrenfest relations:

$$\frac{d}{dt}\mathbf{r} = \frac{i}{\hbar}[H, \mathbf{r}] = c\boldsymbol{\alpha} \equiv \mathbf{v}_{\text{op}}$$

and

$$\frac{d}{dt}(\boldsymbol{\pi}) = \frac{i}{\hbar}[H,\boldsymbol{\pi}] - \frac{e}{c}\frac{\partial}{\partial t}\mathbf{A}$$

$$\frac{d}{dt}(\boldsymbol{\pi}) = e\left[\mathbf{E} + \frac{1}{c}\mathbf{v}_{\mathrm{op}} \times \mathbf{B}\right] \qquad (1.27)$$

with = $\boldsymbol{\pi} \equiv \mathbf{p} - (e/c)\mathbf{A}$ the operator corresponding to the kinetic momentum and

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi \qquad \text{and} \qquad \mathbf{B} = \operatorname{curl}\mathbf{A}$$

the field strengths.

Equation (1.27) is the operator equation of motion for a point charge e.

More general couplings in (1.26) would lead to specific dipole and higher multipole terms in analogy with the classical development.

In taking the nonrelativistic limit of Eq. (1.26), it is convenient to work in the specific representation of Eq. (1.17) and to express the wave function in terms of two-component column matrices $\tilde{\varphi}$ and $\tilde{\chi}$:

$$\psi = \begin{bmatrix} \tilde{\varphi} \\ \tilde{\chi} \end{bmatrix} \qquad (1.28)$$

We then obtain for (1.26)

$$i\hbar \frac{\partial}{\partial t}\begin{bmatrix}\tilde{\varphi}\\ \tilde{\chi}\end{bmatrix} = c\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\begin{bmatrix}\tilde{\chi}\\ \tilde{\varphi}\end{bmatrix} + e\Phi\begin{bmatrix}\tilde{\varphi}\\ \tilde{\chi}\end{bmatrix} + mc^2\begin{bmatrix}\tilde{\varphi}\\ -\tilde{\chi}\end{bmatrix}$$

In the nonrelativistic limit the rest energy mc^2 is the largest energy in the problem and we write

$$\begin{bmatrix}\tilde{\varphi}\\ \tilde{\chi}\end{bmatrix} = e^{-(imc^2/\hbar)t}\begin{bmatrix}\varphi\\ \chi\end{bmatrix} \tag{1.29}$$

where now φ and χ are relatively slowly varying functions of time which are solutions of the coupled equations

$$i\hbar \frac{\partial}{\partial t}\begin{bmatrix}\varphi\\ \chi\end{bmatrix} = c\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\begin{bmatrix}\chi\\ \varphi\end{bmatrix} + e\Phi\begin{bmatrix}\varphi\\ \chi\end{bmatrix} - 2mc^2\begin{bmatrix}0\\ \chi\end{bmatrix} \tag{1.30}$$

The second of Eqs. (1.30) may be approximated, for kinetic energies and field interaction energies small in comparison with mc^2, to

$$\chi = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\pi}}{2mc}\varphi \tag{1.31}$$

Equation (1.31) reveals χ as the "small" components of the wave function ψ in comparison with the "large" components φ.

Relative to φ, χ is reduced by $\sim v/c \ll 1$ in the nonrelativistic approximation.

Inserting (1.31) into the first of Eqs. (1.30), we obtain a two-component spinor equation

$$i\hbar \frac{\partial \varphi}{\partial t} = \left(\frac{\boldsymbol{\sigma} \cdot \boldsymbol{\pi}\, \boldsymbol{\sigma} \cdot \boldsymbol{\pi}}{2m} + e\Phi \right) \varphi \tag{1.32}$$

This is further reduced by the identity for Pauli spin matrices

or, here,

$$\begin{aligned} \boldsymbol{\sigma} \cdot \mathbf{a}\, \boldsymbol{\sigma} \cdot \mathbf{b} &= \mathbf{a} \cdot \mathbf{b} + i\boldsymbol{\sigma} \cdot \mathbf{a} \times \mathbf{b} \\ \boldsymbol{\sigma} \cdot \boldsymbol{\pi}\, \boldsymbol{\sigma} \cdot \boldsymbol{\pi} &= \boldsymbol{\pi}^2 + i\boldsymbol{\sigma} \cdot \boldsymbol{\pi} \times \boldsymbol{\pi} \\ &= \boldsymbol{\pi}^2 - \frac{e\hbar}{c} \boldsymbol{\sigma} \cdot \mathbf{B} \end{aligned} \tag{1.33}$$

Then we have

$$i\hbar \frac{\partial \varphi}{\partial t} = \left[\frac{(\mathbf{p} - (e/c)\mathbf{A})^2}{2m} - \frac{e\hbar}{2mc} \boldsymbol{\sigma} \cdot \mathbf{B} + e\Phi \right] \varphi \tag{1.34}$$

which is recognized as the Pauli equation.

Equation (1.34) gives us confidence that we are on the right track in accepting Eqs. (1.13) and (1.26) as a starting point in constructing a relativistic electron theory.

The two components of φ suffice to accommodate the two spin degrees of freedom of a spin one-half electron; and the correct magnetic moment of the electron, corresponding to the gyromagnetic ratio $g = 2$, automatically emerges.

To see this explicitly, we reduce (1.34) further, keeping only first-order terms in the interaction with a weak uniform magnetic field **B** = curl**A**; **A** = ½**B** x **r**:

$$i\hbar \frac{\partial \varphi}{\partial t} = \left[\frac{\mathbf{p}^2}{2m} - \frac{e}{2mc} (\mathbf{L} + 2\mathbf{S}) \cdot \mathbf{B} \right] \varphi \tag{1.35}$$

Here **L** = **r** x **p** is the orbital angular momentum, **S** = $\frac{1}{2}\hbar\boldsymbol{\sigma}$ is the electron spin, with eigenvalues $\pm\hbar/2$, and the coefficient of the interaction of the spin with **B** field gives the correct magnetic moment of the electron corresponding to a g value of 2.

Fortified by this successful nonrelativistic reduction of the Dirac equation, we go on and establish the Lorentz covariance of the Dirac theory, as required by special relativity.

Next we must investigate further physical consequences of this theory; especially we must interpret those "negative-energy" solutions.

2 Lorentz Covariance of the Dirac Equation

2.1 Covariant Form of the Dirac Equation

It is necessary that the Dirac equation and the continuity equation upon which its physical interpretation rests be covariant under Lorentz transformations.

Let us first review what is meant by a Lorentz transformation.

Two observers O and O' who are in different inertial reference frames will describe the same physical event with the different space-time coordinates.

The rule which relates the coordinates x^μ with which observer O describes the event to the coordinates $(x^\mu)'$ used by observer O' to describe the same event is given by the Lorentz transformation between the two sets of coordinates:

$$(x^\nu)' = \sum_{\mu=0}^{3} a^\nu{}_\mu x^\mu \equiv a^\nu{}_\mu x^\mu \tag{2.1}$$

It is a linear homogeneous transformation, and the coefficients $a^\upsilon{}_\mu$ depend only upon the relative velocities and spatial orientations of the two reference frames of 0 and O'.

The basic invariant of the Lorentz transformation is the proper time interval

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = dx^\mu\, dx_\mu \tag{2.2}$$

This is derived from the physical observation that the velocity of light in vacuo is the same in all Lorentz frames.

Equations (2.1) and (2.2) lead to the relation on the transformation coefficients

$$a_\mu{}^\nu a^\mu{}_\sigma = \delta^\nu{}_\sigma \tag{2.3}$$

Equations (2.1) and (2.3) serve as defining relations for both proper and improper Lorentz transformations.

In the former case the determinant of the transformation coefficients satisfies the relation

$$\det |a| = +1$$

Proper Lorentz transformations can be built up by an infinite succession of infinitesimal transformations.

They include transformations to coordinates in relative motion along any spatial direction as well as ordinary three-dimensional rotations.

The improper Lorentz transformations are the discrete transformations of space inversion and of time inversion.

They cannot be built up from a succession of infinitesimal ones.

Their transformation coefficients satisfy the relation

$$\det |a| = -1$$

in both cases.

Our task is to construct a correspondence relating a given set of observations of a Dirac particle made by observers O and O' in their respective reference frames.

In other words, we seek a transformation law relating the wave functions $\psi(x)$ and $\psi'(x')$ used by observers O and O', respectively.

This transformation law is a rule which allows O' to compute $\psi'(x')$ if given $\psi(x)$.

According to the requirement of Lorentz covariance, this transformation law must lead to wave functions which are solutions of Dirac equations of the same form in the primed as well as unprimed reference frame.

This form invariance of the Dirac equation expresses the Lorentz invariance of the underlying energy-momentum connection

$$p_\mu p^\mu = m^2c^2$$

upon which the considerations of Chap. 1 were based.

In discussing covariance it is desirable to express the Dirac equation in a four-dimensional notation which preserves the symmetry between ct and x^i.

To this end we multiply (1.13) by β/c and introduce the notation

$$\gamma^0 = \beta \qquad \gamma^i = \beta\alpha_i \qquad i = 1, 2, 3$$

This gives

$$i\hbar\left(\gamma^0\frac{\partial}{\partial x^0}+\gamma^1\frac{\partial}{\partial x^1}+\gamma^2\frac{\partial}{\partial x^2}+\gamma^3\frac{\partial}{\partial x^3}\right)\psi-mc\psi=0 \tag{2.4}$$

The new matrices γ^μ provide an elegant restatement of the commutation relations (1.16)

$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}1 \tag{2.5}$$

where 1 is the 4 x 4 unit matrix and hereafter will not be explicitly indicated.

It is clear from their definition that the γ^i are anti-hermitian, with $(\gamma^i)^2 = -1$, and that γ^0 is hermitian.

In the representation (1.17) they have the form

$$\gamma^i=\begin{bmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{bmatrix}\qquad \gamma^0=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \tag{2.6}$$

It is convenient to introduce the Feynman dagger, or slash, notation:

$$\not{A}=\gamma^\mu A_\mu=g_{\mu\nu}\gamma^\mu A^\nu=\gamma^0A^0-\boldsymbol{\gamma}\cdot\mathbf{A}$$

and in particular

$$\not{\nabla}=\gamma^\mu\frac{\partial}{\partial x^\mu}=\frac{\gamma^0}{c}\frac{\partial}{\partial t}+\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}$$

Equation (2.4) then abbreviates to

$$(i\hbar\nabla - mc)\psi = 0 \tag{2.7}$$

or, with $p^\mu = i\hbar \frac{\partial}{\partial x_\mu}$,

$$(\not{p} - mc)\psi = 0 \tag{2.8}$$

Addition of the electromagnetic interaction according to the “minimal" substitution (1.25) gives

$$\left(\not{p} - \frac{e\not{A}}{c} - mc\right)\psi = 0$$

This in no way influences considerations of covariance, because both p^μ and A^μ, and hence their difference, are four-vectors.

2.2 Proof of Covariance

In order to establish Lorentz covariance of the Dirac equation, we must satisfy two requirements.

The first is that there must be an explicit prescription which allows observer O', given the $\psi(x)$ of observer O, to compute the $\psi'(x)$ which describes to O' the same physical state.

Second, according to the relativity principle, $\psi'(x)$ will be a solution of an equation which takes the form of (2.7) in the primed system

$$\left(i\hbar\tilde{\gamma}^{\mu}\frac{\partial}{\partial x^{\mu\prime}} - mc\right)\psi'(x') = 0$$

The $\tilde{\gamma}^{\mu}$ satisfy the anticommutation relations (2.5); therefore $\tilde{\gamma}^{0\dagger} = \tilde{\gamma}^{0}$ and $\tilde{\gamma}^{i\dagger} = -\tilde{\gamma}^{i}$ as required for a hermitian hamiltonian.

As may be shown by a lengthy algebraic proof, all such 4 x 4 matrices $\tilde{\gamma}^{\mu}$ are equivalent up to a unitary transformation U:

$$\tilde{\gamma}_{\mu} = U^{\dagger}\gamma_{\mu}U \qquad U^{\dagger} = U^{-1}$$

and so we drop the distinction between $\tilde{\gamma}^{\mu}$ and γ^{μ} and write

with

$$(\not{p}' - mc)\psi'(x') = 0 \qquad (2.9)$$
$$\not{p}' = i\hbar\gamma^{\mu}\frac{\partial}{\partial x^{\mu\prime}}$$

We ask that the transformation between ψ and ψ' be linear, since both the Dirac equation and the Lorentz transformation (2.1) of the coordinates are themselves linear.

We introduce it in the form

$$\psi'(x') = \psi'(ax) = S(a)\psi(x) = S(a)\psi(a^{-1}x') \qquad (2.10)$$

where $S(a)$ is a 4 x 4 matrix which operates upon the four-component column vector $\psi(x)$.

It depends upon the relative velocities and spatial orientations of O and O'.

S must have an inverse, so that if O knows $\psi'(x')$ which O' uses to describe his observations of a given physical state, he may construct his own wave function $\psi(x)$

$$\psi(x) = S^{-1}(a)\psi'(x') = S^{-1}(a)\psi'(ax) \tag{2.11}$$

We could equally well write, using (2.10),

$$\psi(x) = S(a^{-1})\psi'(ax)$$

which provides the identification

$$S(a^{-1}) = S^{-1}(a)$$

The main problem is to find S.

It must satisfy (2.10) and (2.11).

If S exists, observer O', given $\psi(x)$ by O, may construct $\psi'(x')$ using (2.10).

By reexpressing the Dirac equation (2.7) of O in terms of $\psi'(x')$ with the aid of (2.11), O' could then check whether $\psi'(x')$ satisfies his own equation (2.9).

He would find after left-multiplication by S(a)

$$\left[i\hbar S(a)\gamma^{\mu}S^{-1}(a)\,\frac{\partial}{\partial x^{\mu}} - mc \right]\psi'(x') = 0$$

Using (2.1) to write

$$\frac{\partial}{\partial x^{\mu}} = \frac{\partial x'^{\nu}}{\partial x^{\mu}}\,\frac{\partial}{\partial x'^{\nu}} = a^{\nu}{}_{\mu}\,\frac{\partial}{\partial x'^{\nu}}$$

the primed equation is found to be

$$\left[i\hbar S(a)\gamma^{\mu}S^{-1}(a)a^{\nu}{}_{\mu}\,\frac{\partial}{\partial x^{\nu\prime}} - mc \right]\psi'(x') = 0$$

This is form-invariant, that is, identical with (2.9), provided an S can be found which has the property

$$S(a)\gamma^{\mu}S^{-1}(a)a^{\nu}{}_{\mu} = \gamma^{\nu}$$

or equivalently

$$a^{\nu}{}_{\mu}\gamma^{\mu} = S^{-1}(a)\gamma^{\nu}S(a) \qquad (2.12)$$

Equation (2.12) is the fundamental relation determining S.

In seeking S we are seeking a solution to (2.12).

Once we show that (2.12) has a solution and find it, the covariance of the Dirac equation is established.

By way of terminology, a wave function transforming according to (2.10) and (2.12) is a four-component Lorentz spinor.

We anticipate that S will present novel features not found in tensor calculus, since bilinear forms in ψ such as the probability current (1.20) are expected to form four-vectors.

We first construct S for an infinitesimal proper Lorentz transformation

$$a^{\nu}{}_{\mu} = g^{\nu}{}_{\mu} + \Delta\omega^{\nu}{}_{\mu} \tag{2.13a}$$

with

$$\Delta\omega^{\nu\mu} = -\Delta\omega^{\mu\nu} \tag{2.13b}$$

according to Eq. (2.3) for an invariant proper time interval.

Each of the six independent nonvanishing $\Delta\omega^{\mu\nu}$ generates an infinitesimal Lorentz transformation,

$$\Delta\omega^{01} = \Delta\beta$$

for a transformation to a coordinate system moving with a velocity c $\Delta\beta$ along the x direction,

$$\Delta\omega^{1}{}_{2} = -\Delta\omega^{12} = \Delta\varphi$$

for a rotation through an angle $\Delta\varphi$ about the z axis, and so forth.

Expanding S in powers of $\Delta\omega^{\mu\nu}$ and keeping only the linear term in the infinitesimal generators, we write

$$S = 1 - \frac{i}{4}\sigma_{\mu\nu}\,\Delta\omega^{\mu\nu} \qquad \text{and} \qquad S^{-1} = 1 + \frac{i}{4}\sigma_{\mu\nu}\,\Delta\omega^{\mu\nu} \tag{2.14}$$

with

$$\sigma_{\mu\nu} = -\sigma_{\nu\mu}$$

by (2.13b).

Each of the six coefficients $\sigma_{\mu\nu}$ is a 4 x 4 matrix, as are the transformation S and the unit matrix 1.

Inserting (2.13) and (2.14) into (2.12) and keeping first-order terms in $\Delta\omega^{\mu\nu}$, we find

$$\Delta\omega^{\nu}{}_{\mu}\gamma^{\mu} = -\frac{i}{4}(\Delta\omega)^{\alpha\beta}(\gamma^{\nu}\sigma_{\alpha\beta} - \sigma_{\alpha\beta}\gamma^{\nu})$$

From the antisymmetry of the generators $\Delta\omega^{\mu\nu}$ there follows

$$2i[g^{\nu}{}_{\alpha}\gamma_{\beta} - g^{\nu}{}_{\beta}\gamma_{\alpha}] = [\gamma^{\nu},\sigma_{\alpha\beta}] \tag{2.15}$$

The problem of establishing proper Lorentz covariance of the Dirac equation is now reduced to that of finding six matrices $\sigma_{\alpha\beta}$ which satisfy (2.15).

The simplest guess to make is an antisymmetric product of two matrices, and directly we find, using (2.5), that

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] \tag{2.16}$$

tis the desired matrix.

According to (2.14), S for an infinitesimal Lorentz transformation is given by

$$S = 1 + \frac{1}{8}[\gamma_\mu, \gamma_\nu]\,\Delta\omega^{\mu\nu} = 1 - \frac{i}{4}\sigma_{\mu\nu}\,\Delta\omega^{\mu\nu} \tag{2.17}$$

We now complete our task by constructing the finite proper transformations by a succession of infinitesimal ones.

First, to build up (2.1) from (2.13), we write

$$\Delta\omega^{\nu}{}_{\mu} = \Delta\omega(I_n)^{\nu}{}_{\mu} \tag{2.18}$$

where $\Delta\omega$ is the infinitesimal parameter, or "angle of rotation" about an axis in the direction labeled n, and I_n is the 4 x 4 (in space-time) matrix of coefficients for a unit Lorentz rotation about this axis.

υ and μ label row and column respectively.

Thus for a transformation to a primed system in motion along the x axis with an infinitesimal velocity
c $\Delta\omega$ = c $\Delta\beta$

$$I^{\nu}{}_{\mu} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.19}$$

so that

$$I^{0}{}_{1} = I^{1}{}_{0} = -I^{01} = +I^{10} = -1$$

Using the algebraic property of $I^{\mu}{}_{\upsilon}$, that

$$I^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad I^3 = +I$$

we can write the finite transformation for uniform relative x-axis motion as

$$\begin{aligned} x^{\nu'} &= \lim_{N\to\infty} \left(g + \frac{\omega}{N} I\right)^{\nu}{}_{\alpha_1} \left(g + \frac{\omega}{N} I\right)^{\alpha_1}{}_{\alpha_2} \cdots x^{\alpha_N} \\ &= (e^{\omega I})^{\nu}{}_{\mu} x^{\mu} \\ &= (\cosh \omega I + \sinh \omega I)^{\nu}{}_{\mu} x^{\mu} \\ &= (1 - I^2 + I^2 \cosh \omega + I \sinh \omega)^{\nu}{}_{\mu} x^{\mu} \end{aligned}$$

For the individual components this gives

$$\begin{bmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{bmatrix} = \begin{bmatrix} \cosh\omega & -\sinh\omega & 0 & 0 \\ -\sinh\omega & \cosh\omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \tag{2.20}$$

or

$$\begin{aligned} x^{0'} &= (\cosh\omega)(x^0 - \tanh\omega\, x^1) \\ x^{1'} &= (\cosh\omega)(x^1 - \tanh\omega\, x^0) \\ x^{2'} &= x^2 \\ x^{3'} &= x^3 \end{aligned} \tag{2.21}$$

where

$$\tanh\omega = \beta \qquad \text{and} \qquad \cosh\omega = \frac{1}{\sqrt{1-\beta^2}}$$

relate the Lorentz rotation angle ω with the relative velocity $c\beta$.

This result can be generalized to include motion along any direction or spatial rotation about any axis.

The six matrices I^{ν}_{μ} generating the six independent Lorentz rotations are the four-dimensional generalizations of the three-dimensional space rotations familiar in the nonrelativistic theory.

Turning now to the construction of a finite spinor transformation S, we have from (2.14) and (2.18)

$$\psi'(x') = S\psi(x) = \lim_{N\to\infty}\left(1 - \frac{i}{4}\frac{\omega}{N}\sigma_{\mu\nu}I_n^{\mu\nu}\right)^N \psi(x)$$
$$= \exp\left(-\frac{i}{4}\omega\sigma_{\mu\nu}I_n^{\mu\nu}\right)\psi(x) \qquad (2.22)$$

Specializing again to the transformation (2.19) we have

$$\psi'(x') = e^{-(i/2)\omega\sigma_{01}}\psi(x) \qquad (2.23)$$

where x'and x are related by (2.21).

Similarly, for a rotation through an angle φ about the z axis, $I^{12} = -I^{21} = -1$ and

$$\psi'(x') = e^{(i/2)\varphi\sigma^{12}}\psi(x) \qquad (2.24)$$

where

$$\sigma^{12} = \begin{bmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{bmatrix}$$

in the representation (1.17), with

$$\sigma_3 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

the Pauli 2 x 2 matrix.

We recognize the similarity of (2.24) with the form of rotation of a two-component Pauli spinor

$$\varphi'(x') = e^{(i/2)\boldsymbol{\omega}\cdot\boldsymbol{\sigma}}\varphi(x) \tag{2.25}$$

The covariant "angle" variables $\omega^{\mu\nu}$ in (2.18) are associated with the Lorentz transformation in the same sense that the rotation angle and direction in ω are for the three-dimensional rotation.

The appearance of half-angles in (2.24), as in (2.25), is an expression of the double-valuedness of the spinor law of rotation; it takes a rotation of 4π radians to return ψ(x) to its original value.

Because of this, physical observables in the Dirac theory must be bilinear, or an even power in ψ(x).

For spatial rotations, $S = S_R$ is unitary, since the σ_{ij} are hermitian, and

$$S_R^\dagger = e^{-(i/4)\sigma^{\dagger ij}\omega_{ij}} = e^{-(i/4)\sigma^{ij}\omega_{ij}} = S_R^{-1}$$

This is not true for transformations to a moving coordinate system $S = S_L$.

For instance, for the transformation (2.23)

$$S_L = e^{-(i/2)\omega\sigma_{01}} = e^{-(\omega/2)\alpha_1} = S_L^\dagger \neq S_L^{-1}$$

However, S_L does have the property

$$S_L^{-1} = \gamma_0 S_L^\dagger \gamma_0$$

found by expanding S_L, in a power series.

Since $[\gamma_0,\sigma^{ij}] = 0$, this can be generalized to include rotations

$$S^{-1} = \gamma_0 S^\dagger \gamma_0 \tag{2.26}$$

The continuity equation is also covariant.

The probability current (1.21) and (1.22), in the notation of (2.4), is

$$j^\mu(x) = c\psi^\dagger(x)\gamma^0\gamma^\mu\psi(x)$$

and under (2.1) transforms to

$$\begin{aligned} j^{\mu'}(x') &= c\psi'^\dagger(x')\gamma^0\gamma^\mu\psi'(x') \\ &= c\psi^\dagger(x)S^\dagger\gamma_0\gamma^\mu S\psi(x) \\ &= c\psi^\dagger(x)\gamma_0 S^{-1}\gamma^\mu S\psi(x) \\ &= ca^\mu{}_\nu\psi^\dagger(x)\gamma_0\gamma^\nu\psi(x) \\ &= a^\mu{}_\nu j^\nu(x) \end{aligned} \tag{2.27}$$

Evidently $j^\mu(x)$ is a Lorentz four-vector and the continuity equation

$$\frac{\partial j^\mu(x)}{\partial x^\mu} = 0$$

is invariant.

Also, the probability density $j^0(x) = c\rho(x)$ transforms as the time component of a conserved four-vector.

This is the desired result noted in Sec. 1.3 for an invariant probability.

Because the combination $\psi^\dagger\gamma_0$ in (2.27) occurs so often, it is dignified by a new notation

$$\bar{\psi}(x) = \psi^\dagger\gamma_0 \tag{2.28}$$

where $\bar{\psi}(x)$ is known as the adjoint spinor.

Its Lorentz transformation property is given by

$$\bar{\psi}'(x') = \bar{\psi}(x)S^{-1} \tag{2.29}$$

2.3 Space Reflection

We now expand our outlook to take into account the existence of the improper Lorentz transformation of space reflection

$$\mathbf{x}' = -\mathbf{x} \qquad t' = t$$

Again covariance requires a solution of (2.12), but in this case we cannot build it up from the infinitesimal transformations.

However, it is easy enough to solve (2.12) directly.

The transformation matrix is

$$a^{\nu}{}_{\mu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = g^{\nu\mu} \tag{2.30}$$

Denoting $S = P$ for the coordinate reflection, (2.12) becomes

$$P^{-1}\gamma^{\nu}P = g^{\nu\nu}\gamma^{\nu} \tag{2.31}$$

which is satisfied by

$$P = e^{i\varphi}\gamma_0 \tag{2.32}$$

The phase factor is of no physical interest here and may be narrowed down to the four choices $\pm 1, \pm i$ if we require that four reflections return the spinor to itself in analogy with a rotation through 4π radians.

P in (2.32) evidently is unitary, $P^{-1} = P^{\dagger}$, and satisfies (2.26) as well.

Equation (2.32) tells us that

$$\psi'(x') = \psi'(-\mathbf{x},t) = e^{i\varphi}\gamma_0\psi(\mathbf{x},t) \tag{2.33}$$

In the nonrelativistic limit ψ approaches an eigenstate of P, and by (1.24) and (2.6) the positive- and negative-energy states at rest have opposite eigenvalues, or *intrinsic parities*.

The discussion of the other improper transformations, such as time reversal, is more involved; it is given in Chap. 5.

2.4 Bilinear Covariants

By forming products of the γ matrices it is possible to construct 16 linearly independent 4 x 4 matrices $\Gamma^{n}{}_{\alpha\beta}$ as which appear often in applications of the Dirac theory.

These are

$$\Gamma^S = 1 \qquad \Gamma^V_\mu = \gamma_\mu \qquad \Gamma^T_{\mu\nu} = \sigma_{\mu\nu}$$
$$\Gamma^P = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_5 \equiv \gamma^5 \qquad \Gamma^A_\mu = \gamma_5\gamma_\mu \tag{2.34}$$

By using the anticommutation relations (2.5) the Γ^n are readily established to be linearly independent by the following argument:

1. For each Γ^n, $(\Gamma^n)^2 = \pm 1$.
2. For each Γ^n except Γ^S, there exists a Γ^m such that

$$\Gamma^n\Gamma^m = -\Gamma^m\Gamma^n$$

From this it follows that the trace of Γ^n vanishes:

$$\pm \operatorname{Tr} \Gamma^n = \operatorname{Tr} \Gamma^n(\Gamma^m)^2 = -\operatorname{Tr} \Gamma^m\Gamma^n\Gamma^m = -\operatorname{Tr} \Gamma^n(\Gamma^m)^2 = 0$$

3. Given Γ^a and Γ^b, $a \neq b$, there exists a $\Gamma^n \neq \Gamma^S$ such that

$$\Gamma^a\Gamma^b = \Gamma^n$$

This follows by direct inspection of the Γ's.

4. Suppose there exist numbers a_n such that

$$\sum_n a_n\Gamma^n = 0$$

Then multiply by $\Gamma^m \neq \Gamma^S$ and take the trace; using (3), we find $a_m = 0$.

If $\Gamma^m = \Gamma^S$, we find $a_s = 0$, and all coefficients vanish.

This establishes the linear independence of the Γ^n.

It follows that any 4 x 4 matrix can be written in terms of the Γ^n.

We may now write down the Lorentz transformation properties of the bilinear forms $\bar{\psi}(x)\Gamma^n\psi(x)$ constructed from the 16 Γ^n.

We need only the observation that

$$\gamma^\mu\gamma_5 + \gamma_5\gamma^\mu = 0 \tag{2.35}$$

and therefore

$$[\gamma_5,\sigma_{\mu\nu}] = 0$$

or

$$[S,\gamma_5] = 0 \tag{2.36}$$

for all proper Lorentz transformations.

As a special case of (2.35)

$$P\gamma_5 = -\gamma_5 P \tag{2.37}$$

Carrying out calculations similar to (2.27) we find:

$$\bar{\psi}'(x')\psi'(x') = \bar{\psi}(x)\psi(x)$$
a scalar

$$\bar{\psi}'(x')\gamma_5\psi'(x') = \bar{\psi}(x)S^{-1}\gamma_5 S\psi(x) = \det|a|\bar{\psi}(x)\gamma_5\psi(x)$$
a pseudoscalar

$$\bar{\psi}'(x')\gamma^\nu\psi'(x') = a^\nu{}_\mu\bar{\psi}(x)\gamma^\mu\psi(x)$$
a vector

$$\bar{\psi}'(x')\gamma_5\gamma^\nu\psi'(x') = \det|a|a^\nu{}_\mu\bar{\psi}(x)\gamma_5\gamma^\mu\psi(x)$$
a pseudovector

$$\bar{\psi}'(x')\sigma^{\mu\nu}\psi'(x') = a^\mu{}_\alpha a^\nu{}_\beta\bar{\psi}(x)\sigma^{\alpha\beta}\psi(x)$$
a second-rank tensor

(2.38)

the the the the the

3 Solutions to the Dirac Equation for a Free Particle

3.1 Plane-wave Solutions

We have seen that the Dirac theory meets the requirements of Lorentz covariance and that the positive-energy solutions to the Dirac equation have a sensible nonrelativistic correspondence.

Further insight into the nature and interpretation of solutions of the Dirac equation may be gained by considering the free-particle equation.

The four solutions corresponding to a free particle at rest were given in (1.24) and are written in the combined form

$$\psi^r(x) = w^r(0)e^{-(i\epsilon_r mc^2/\hbar)t} \qquad r = 1, 2, 3, 4 \tag{3.1}$$

with

$$\epsilon_r = \begin{cases} +1 & r = 1, 2 \\ -1 & r = 3, 4 \end{cases}$$

The spinors are

$$w^1(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad w^2(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad w^3(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad w^4(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{3.2}$$

in this representation, Eq. (1.17), of the Dirac matrices.

The first two solutions describe the two spin degrees of freedom of a Schrödinger-Pauli electron.

The "negative-energy" solutions, r = 3 and 4, remain to be interpreted.

They are all eigenfunctions of $\sigma_z = \sigma_{12}$ with eigenvalues +1 and -1.

The Lorentz transformation (2.10) may be used to build the free-particle solutions for an arbitrary velocity.

By transforming to a coordinate system moving with velocity -**v** relative to that of the solutions at rest, we construct free-particle wave functions for an electron with the observed velocity +**v**.

In order to exhibit the general space-time coordinate variation, we need only express the exponent in (3.1) in invariant form:

$$\exp\left(-i\epsilon_r \frac{mc^2}{\hbar} t\right) = \exp\left(-i\epsilon_r \frac{p_\mu{}^{(0)} x^\mu}{\hbar}\right) = \exp\left(-i\epsilon_r \frac{p_\mu x^{\mu'}}{\hbar}\right) \tag{3.3}$$

where $x^{\mu'} = a^\mu{}_\nu x^\nu$ and $p^\mu = a^\mu{}_\nu p^{\nu(0)} = a^\mu{}_0 mc$; our notation throughout is such that $p^0 \equiv E/c = +\sqrt{\mathbf{p}^2 + m^2c^2} > 0.$

The positive- and negative-energy solutions transform among themselves separately and do not mix with each other under proper Lorentz transformations, as well as under spatial inversions.

This is seen to follow from (3.3), since the four-momentum of a free particle is time-like, $p^\mu p_\mu = m^2c^2 > 0$.

Therefore, p_μ is within the light cones in p space.

Under the transformations mentioned above, the future and past light cones, and hence the positive- and negative-energy solutions, remain distinct.

We transform the spinors with

$$S = e^{-(i/2)\omega\sigma_{01}} \tag{3.4}$$

according to (2.23), where for simplicity we have specified the velocity to lie along the x axis.

The Lorentz angle ω in (3.4) is given by $\omega = \tanh^{-1}(-v/c) = -\tanh^{-1}(v/c)$ and differs by a minus sign from (2.21), since we are transforming to a system moving in the x direction with velocity -**v**.

Applying the transformation (3.4) to the spinors (3.2), we find

$$\begin{aligned} w^r(\mathbf{p}) &= e^{-(i\omega/2)\sigma_{01}} w^r(0) = \left(\cosh\frac{\omega}{2} - \alpha_1 \sinh\frac{\omega}{2}\right) w^r(0) \\ &= \cosh\frac{\omega}{2} \begin{bmatrix} 1 & 0 & 0 & -\tanh\frac{\omega}{2} \\ 0 & 1 & -\tanh\frac{\omega}{2} & 0 \\ 0 & -\tanh\frac{\omega}{2} & 1 & 0 \\ -\tanh\frac{\omega}{2} & 0 & 0 & 1 \end{bmatrix} w^r(0) \end{aligned} \tag{3.5}$$

From the form (3.2) for $\omega^r(0)$, it is clear that the rth column of this transformation matrix is identically the column spinor corresponding to $\omega^r(\mathbf{p})$.

We may reexpress it in terms of the energy and momentum of the particle by using the trigonometric identities,

$$-\tanh\frac{\omega}{2}=\frac{-\tanh\omega}{1+\sqrt{1-\tanh^2\omega}}=\frac{v/c}{1+\sqrt{1-(v^2/c^2)}}=\frac{pc}{E+mc^2}$$

and

$$\cosh\frac{\omega}{2}=\sqrt{\frac{E+mc^2}{2mc^2}} \tag{3.6}$$

Also, we may generalize (3.5) to the case of arbitrary direction of the velocity **v**.

In this case the matrix I in (2.19) is replaced by

$$I^{\mu}{}_{\nu}=\begin{bmatrix} 0 & -\cos\alpha & -\cos\beta & -\cos\gamma \\ -\cos\alpha & 0 & 0 & 0 \\ -\cos\beta & 0 & 0 & 0 \\ -\cos\gamma & 0 & 0 & 0 \end{bmatrix}$$

where $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are the direction cosines of the velocity **v**, and in the transformation matrix

$$\sigma_{\mu\nu}I_n^{\mu\nu}=2(\sigma_{01}\cos\alpha+\sigma_{02}\cos\beta+\sigma_{03}\cos\gamma)=-2i\frac{\boldsymbol{\alpha}\cdot\mathbf{v}}{|\mathbf{v}|}$$

This gives, with the aid of (3.6),

$$S = \exp\left(-\frac{\omega}{2}\frac{\boldsymbol{\alpha}\cdot\mathbf{v}}{|\mathbf{v}|}\right)$$

$$= \sqrt{\frac{E+mc^2}{2mc^2}}\begin{bmatrix} 1 & 0 & \dfrac{p_z c}{E+mc^2} & \dfrac{p_- c}{E+mc^2} \\ 0 & 1 & \dfrac{p_+ c}{E+mc^2} & \dfrac{-p_z c}{E+mc^2} \\ \dfrac{p_z c}{E+mc^2} & \dfrac{p_- c}{E+mc^2} & 1 & 0 \\ \dfrac{p_+ c}{E+mc^2} & \dfrac{-p_z c}{E+mc^2} & 0 & 1 \end{bmatrix} \tag{3.7}$$

where $p_\pm = p_x \pm i p_y$.

The general form of a free-particle solution is

$$\psi^r(x) = w^r(\mathbf{p}) e^{-i\epsilon_r (p_\mu x^\mu/\hbar)} \tag{3.8}$$

where the rth column of (3.7) gives the corresponding spinor $\omega^r(\mathbf{p})$ in the representation of the γ matrices given by Eg. (1.17).

The $\omega^r(\mathbf{p})$ satisfy the following useful relations:

$$(\not{p} - \epsilon_r mc) w^r(\mathbf{p}) = 0 \qquad \bar{w}_r(\mathbf{p})(\not{p} - \epsilon_r mc) = 0 \tag{3.9a}$$

$$\bar{w}^r(\mathbf{p}) w^{r'}(\mathbf{p}) = \delta_{rr'}\epsilon_r \tag{3.9b}$$

$$\sum_{r=1}^{4} \epsilon_r w^r_\alpha(\mathbf{p}) \bar{w}^r_\beta(\mathbf{p}) = \delta_{\alpha\beta} \tag{3.9c}$$

Equation (3.9a), obtained by applying the Dirac operator (i∇ - m) to (3.8), states the Dirac equation for a free particle in momentum space.

For r = 1 or 2, $\epsilon_r = +1$ and $(\not{p} - mc)w^r(\mathbf{p}) = 0$.

This is the equation for the two positive-energy solutions given by the first two columns of (3.7). In this representation their third and fourth components are the "small components" in a nonrelativistic approximation, and they reduce to Eqs. (1.29) and (1.31) in the absence of external fields.

For the negative-energy solutions the "large" and "small" components are interchanged in (3.7).

We also introduce the adjoint spinor according to the definition in (2.28): $\bar{w}^r(\mathbf{p}) \equiv w^{r\dagger}(\mathbf{p})\gamma_0$.

It satisfies the adjoint wave equation

$$\bar{w}^r(\mathbf{p})(\not{p} - \epsilon_r mc) = 0 \tag{3.10}$$

which is obtained by taking the hermitian conjugate of (3.9a) and multiplying from the right by γ^0 with the aid of the identities $(\gamma^0)^2 = +1$ and $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$.

Equation (3.96) is a covariant normalization statement.

The bilinear form $\bar{w}^r(\mathbf{p})w^{r'}(\mathbf{p})$ is a Lorentz scalar as discussed in the preceding chapter see Eq. (2.38)), and so we verify (3.96) simply by returning to the rest solutions (3.2).

The probability density $w^{r\dagger}(\mathbf{p})w^{r}(\mathbf{p})$ will not be an invariant but transforms as the fourth component of a vector according to (2.27).

Calculating from the columns of (3.7) we find

$$w^{r\dagger}(\epsilon_r\mathbf{p})w^{r'}(\epsilon_{r'}\mathbf{p}) = \frac{E}{mc^2}\,\delta_{rr'} \tag{3.11}$$

This shows that the probability density acquires the correct factor E/mc² to compensate the Lorentz contraction of the volume element along the direction of motion and to preserve thereby the normalization of the invariant probability.

Notice that (3.9b) is an orthogonality statement between a spinor and its adjoint of the same momentum ***p***, whereas in (3.11) the positive-energy spinor is orthogonal to its hermitian conjugate spinor of negative energy and reversed momentum.

Thus two plane-wave solutions of the same spatial momentum ***p*** but of opposite energy are orthogonal in the sense that $\psi^{r\dagger}(x)\psi^{r'}(x) = 0$ if $r = 1, 2$ and $r' = 3, 4$, or vice versa.

Equation (3.9c) is a completeness statement applying to the four Dirac spinors for a given momentum.

It is clearly true for a free particle at rest.

To prove it for an arbitrary momentum, we can make an appropriate Lorentz transformation to the rest system and then use (3.2) to find

$$\sum_{r=1}^{4} \epsilon_r w_\alpha^r(\mathbf{p}) \bar{w}_\beta^r(\mathbf{p}) = \sum_{r=1}^{4} \epsilon_r S_{\alpha\gamma}\left(-\frac{\mathbf{p}}{E}\right) w_\gamma^r(0) \bar{w}_\lambda^r(0) S_{\lambda\beta}^{-1}\left(-\frac{\mathbf{p}}{E}\right)$$

$$= S_{\alpha\gamma}\delta_{\gamma\lambda}S_{\lambda\beta}^{-1} = \delta_{\alpha\beta}$$

That $\bar{w}$ and not $w^\dagger$ appears in the completeness relation is due to the relation $S^\dagger = \gamma^0 S^{-1}\gamma^0$ derived in (2.26) and again reflects the fact that the Lorentz transformation is not unitary.

By using the rotation operators

$$S = e^{(i/2)\varphi \mathbf{d}\cdot\mathbf{s}}$$

upon the solutions (3.2) for the electron at rest and polarized in the z direction, it is possible to form states which are polarized in any arbitrary direction **s**.

In particular, the defining relation for such states is

$$\mathbf{d}\cdot\mathbf{s}w = w$$

if the spinor ω corresponds to a particle polarized along direction of the unit vectors.

The specific form of these solutions is similar to that of the two-component Pauli theory owing to the structure of $\mathbf{d}$ in (2.24).

In this description it is convenient to introduce a different notation.

Let $u(p, s)$ denote the spinor which is a positive-energy solution of the Dirac equation with momentum p^μ and spin s^μ.

Thus $u(p, s)$ satisfies the equation

$$(\not{p} - mc)_{\alpha\beta} u_\beta(p,s) = 0 \tag{3.12}$$

The spin vector s^μ is defined in terms of the polarization vector $\check{s}$ in the rest frame by $s^\mu = a^\mu{}_\nu \check{s}^\nu$, where $\check{s}^\nu = (0,\check{\mathbf{s}})$ and $a^\mu{}_\nu$ are the transformation coefficients to the rest frame, that is, $p^\mu = a^\mu{}_\nu \check{p}^\nu$, where $\check{p}^\nu = (m,\mathbf{0})$.

Notice that $s_\mu s^\mu = -1$ and that $\check{p}^\mu \check{s}_\mu = 0$ and therefore $p^\mu s_\mu = 0$.

In the rest frame u satisfies

$$\boldsymbol{\sigma} \cdot \check{\mathbf{s}} u(\check{p},\check{s}) = u(\check{p},\check{s}) \tag{3.13}$$

Similarly let $v(p, s)$ denote a negative-energy solution

$$(\not{p} + mc) v(p,s) = 0 \tag{3.14}$$

with polarization $-\check{\mathbf{s}}$ in the rest frame, that is,

$$\boldsymbol{\sigma} \cdot \check{\mathbf{s}} v(\check{p},\check{s}) = -v(\check{p},\check{s}) \tag{3.15}$$

The $u(p, s)$ and $v(p, s)$ are related to the $\omega^r(p)$ by

$$\begin{aligned} w^1(\mathbf{p}) &= u(p,u_z) \\ w^2(\mathbf{p}) &= u(p,-u_z) \\ w^3(\mathbf{p}) &= v(p,-u_z) \\ w^4(\mathbf{p}) &= v(p,u_z) \end{aligned} \tag{3.16}$$

with u_z^μ a four-vector, which in the rest frame takes the form

$$\check{u}_z^\mu = (0,\check{\mathbf{u}}_z) = (0,0,0,1)$$

An arbitrary spinor is thus specified by the momentum p_μ, the sign of the energy, and the polarization in the rest frame $\check{s}_\mu$.

3.2 Projection Operators for Energy and Spin

In practical calculations, it is often convenient to have operators which project out a spinor of given sign of energy and polarization.

These projection operators are the four-dimensional analogues of the nonrelativistic two-component operators

$$P_\pm = \frac{1 \pm \sigma_z}{2}$$

which project out of an arbitrary state the spin-up or spin-down amplitude.

For the Dirac equation, we search for four operators which project from a given plane-wave solution of momentum **p** the four independent solutions corresponding to positive and negative energy and to spin up and spin down along a given direction.

We would like these operators in a covariant form so that we may transform with ease among different Lorentz systems, as will prove useful in practical calculations.

The four projection operators are denoted by $P_r(\mathbf{p}) = P(p_\mu, u_z, \epsilon)$ and are defined to satisfy the following properties:

$$P_r(\mathbf{p})w^{r'}(\mathbf{p}) = \delta_{rr'}w^{r'}(\mathbf{p})$$

or equivalently (3.17)

$$P_r(\mathbf{p})P_{r'}(\mathbf{p}) = \delta_{rr'}P_r(\mathbf{p})$$

An operator which projects out positive- or negative-energy eigenstates for a given **p** may be found directly from (3.9a), already in covariant form.

We denote it by

$$\Lambda_r(p) = \frac{\epsilon_r \not{p} + mc}{2mc}$$

or, alternatively, (3.18)

$$\Lambda_\pm(p) = \frac{\pm \not{p} + mc}{2mc}$$

By direct calculation, using $\not{p}\not{p} = p^2 = m^2c^2$, we verify that

$$\Lambda_r(p)\Lambda_{r'}(p) = \frac{m^2c^2(1 + \epsilon_r\epsilon_{r'}) + mc\not{p}(\epsilon_r + \epsilon_{r'})}{4m^2c^2} = \left(\frac{1 + \epsilon_r\epsilon_{r'}}{2}\right)\Lambda_r(p)$$

that is,

$$\Lambda_+^2(p) = \Lambda_+(p)$$
$$\Lambda_+(p)\Lambda_-(p) = 0$$

Also notice that

$$\Lambda_+(p) + \Lambda_-(p) = 1$$

To exhibit the analogous operator for the spin, we go to the rest frame, where the spin is most easily described, and try to find a projection operator which may be cast into covariant form.

The natural candidate for a spin-up particle is $(1 + \sigma_z)/2$.

In the same way as the two-component nonrelativistic spin projection operator is liberated from explicit dependence upon the z direction by rewriting $(1 + \sigma_z)/2$ as a scalar,

$$\frac{1 + \sigma \cdot \check{u}_z}{2}$$

we try to write the Dirac spin projection operator in scalar form by using the four-vector $\check{u}_z^\mu$, that is

$$\frac{1+\sigma_z}{2} = \frac{1+\gamma_5\gamma_3\check{u}_z^3\gamma_0}{2} = \frac{1+\gamma_5\check{\not{u}}_z\gamma_0}{2}$$

This may now be cast into covariant form by eliminating the γ_0.

Because we are in the rest frame, γ_0 acting upon the Dirac spinor becomes ±1.

With the convention established in (3.14) and (3.15), the covariant Dirac spin projection operator is finally

$$\Sigma(u_z) = \frac{1+\gamma_5\not{u}_z}{2}$$

or for a general spin vector s^μ, with $s^\mu p_\mu = 0$,

$$\Sigma(s) = \frac{1+\gamma_5\not{s}}{2} \tag{3.19}$$

Thus in the rest frame

$$\Sigma(\check{u}_z)w^1(0) = \frac{1+\gamma_5\check{\not{u}}_z}{2}w^1(0) = \frac{1+\sigma_z}{2}w^1(0) = w^1(0) \tag{3.20}$$

and

$$\Sigma(-\check{u}_z)w^2(0) = w^2(0)$$

Similarly, for the negative-energy spinors

$$\Sigma(-\check{u}_z)w^3(0) = \frac{1-\gamma_5\check{\not{u}}_z}{2}w^3(0) = \frac{1+\gamma_5\check{\not{u}}_z\gamma_0}{2}w^3(0)$$
$$= \frac{1+\sigma_z}{2}w^3(0) = w^3(0) \tag{3.21}$$
$$\Sigma(\check{u}_z)w^4(0) = w^4(0)$$

and

In terms of the definitions (3.16) of the spinors *u* and v, these are

$$\Sigma(u_z)u(p,u_z) = u(p,u_z)$$
$$\Sigma(u_z)v(p,u_z) = v(p,u_z)$$
$$\Sigma(-u_z)u(p,u_z) = \Sigma(-u_z)v(p,u_z) = 0$$

Because of the covariant form of the projection operator $\sum$, we may write for any polarization vector $s^\mu (s^\mu p_\mu = 0)$ that

$$\Sigma(s)u(p,s) = u(p,s)$$
$$\Sigma(s)v(p,s) = v(p,s) \tag{3.25}$$
$$\Sigma(-s)u(p,s) = \Sigma(-s)v(p,s) = 0$$

With the four projection operators $\Lambda_\pm$(p) and $\sum(\pm s)$ we can now completely specify free-particle motion in terms of four-momentum p_μ, sign of energy ϵ, and polarization s^μ with $s^\mu\, p_\mu = 0$.

In particular, we construct from (3.18) and (3.19) the four projection operators

$$
\begin{aligned}
P_1(\mathbf{p}) &= \Lambda_+(p)\Sigma(u_z) \\
P_2(\mathbf{p}) &= \Lambda_+(p)\Sigma(-u_z) \\
P_3(\mathbf{p}) &= \Lambda_-(p)\Sigma(-u_z) \\
P_4(\mathbf{p}) &= \Lambda_-(p)\Sigma(u_z)
\end{aligned}
$$

Notice that $[\Sigma(s), \Lambda_\pm(p)] = 0$ for all vectors satisfying $s^\mu p_\mu = 0$, since $\not{p}$ anticommutes with both γ_5 and $\not{s}$.

From this it follows that these $P_r(\mathbf{p})$ satisfy the defining relations (3.17).

We shall rely upon these projection operators very frequently in developing rapid and efficient calculational techniques.

They permit us to use closure methods, thus avoiding the necessity of writing out matrices and spinor solutions component by component.

In order to achieve an invariant formulation, we have introduced negative-energy solutions of momentum **p** which, according to (3.8), are eigenfunctions of the momentum operator ***p*** with eigenvalue -**p**.

Similarly, according to (3.19) and (3.21), the negative-energy solutions representing spin-up and spin-down states reduce in their rest frames to eigenfunctions of σ_z, with eigenvalues -1 and +1, respectively.

The physical motivation for this apparently backward association of eigenvalues for the negative-energy solutions will appear when we come to the hole theory in Chap. 5.

3.3 Physical Interpretation of Free-particle Solutions and Packets

We may now superpose the plane-wave solutions at our disposal to construct localized packets.

These packets are still solutions of the the Dirac equation as required by the superposition principle, since the Dirac equation is linear. We study them to gain further insight into the interpretation of the free-particle solutions.

To begin, we form a packet by superposing positive-energy solutions only:

$$\psi^{(+)}(\mathbf{x},t) = \int \frac{d^3p}{(2\pi\hbar)^{3/2}} \sqrt{\frac{mc^2}{E}} \sum_{\pm s} b(p,s)u(p,s)e^{-ip_\mu x^\mu/\hbar} \tag{3.23}$$

To normalize the expansion coefficients $b(p, s)$ to unit probability, we call on the spinor orthogonality relations (3.11) and find

$$\int \psi^{(+)\dagger}(\mathbf{x},t)\psi^{(+)}(\mathbf{x},t)\, d^3x = \int d^3p \frac{mc^2}{E} \sum_{\pm s, \pm s'} b^*(p,s')b(p,s)u^\dagger(p,s')u(p,s)$$

$$= \int d^3p \sum_{\pm s} |b(p,s)|^2 = 1 \tag{3.24}$$

The average current for such a packet is given by the expectation value of the velocity operator

$$\mathbf{J}^{(+)} = \int \psi^{(+)\dagger} c\boldsymbol{\alpha} \psi^{(+)}\, d^3x \tag{3.25}$$

In evaluating this we use the following important relation between the three four-vectors that can be formed from free-particle solutions:

For $\psi_1(x)$ and $\psi_2(x)$ any two solutions to the Dirac equation, $(\not{p} - mc)\psi(x) = 0,$

$$c\bar{\psi}_2 \gamma^\mu \psi_1 = \frac{1}{2m}\left[\bar{\psi}_2 p^\mu \psi_1 - (p^\mu \bar{\psi}_2)\psi_1\right] - \frac{i}{2m} p_\nu (\bar{\psi}_2 \sigma^{\mu\nu} \psi_1) \tag{3.26}$$

To prove (3.26), we observe that if a^μ and b^μ are two arbitrary four-vectors

$$\begin{aligned} \not{a}\not{b} &= a_\mu b_\nu [\tfrac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) + \tfrac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)] \\ &= a^\mu b_\mu - i a^\mu b^\nu \sigma_{\mu\nu} \end{aligned} \tag{3.27}$$

Then with the Dirac equation we construct

$$\begin{aligned} 0 &= \bar{\psi}_2(-\overleftarrow{\not{p}} - mc)\not{a}\psi_1 + \bar{\psi}_2 \not{a}(\overrightarrow{\not{p}} - mc)\psi_1 \\ &= -2mc\bar{\psi}_2 \not{a}\psi_1 + \bar{\psi}_2[a^\mu \overrightarrow{p}_\mu - i a^\mu \overrightarrow{p}^\nu \sigma_{\mu\nu} - \overleftarrow{p}^\mu a_\mu + i\overleftarrow{p}^\mu a^\nu \sigma_{\mu\nu}]\psi_1 \end{aligned}$$

$(\bar{\psi}\overleftarrow{\not{p}} = p_\mu \bar{\psi}\gamma^\mu)$

and (3.26) emerges as the coefficient of an arbitrary vector a^μ.

This identity is known as the Gordon decomposition.

It expresses the Dirac current as the sum of a convection current similar to the nonrelativistic one, and a spin current.

With the help of (3.26) for the special case $\psi_2 = \psi_1 = \psi$ and (3.23), we now find for the current (3.25)

$$J_i^{(+)} = \int d^3x \int \frac{d^3p\, d^3p'}{(2\pi\hbar)^3} \frac{mc^2}{\sqrt{EE'}} \sum_{\pm s, \pm s'} b^*(p',s')b(p,s)e^{i(p'-p)_\mu x_\mu/\hbar}$$

$$\times \frac{1}{2m} \bar{u}(p',s')[(p_i' + p_i) + i\sigma_i{}^\nu(p_\nu' - p_\nu)]u(p,s)$$

$$= \int d^3p \frac{p_i c^2}{E} \sum_{\pm s} |b(p,s)|^2 \qquad (3.28)$$

According to the normalization (3.24), the current can be written

$$\mathbf{J}^{(+)} = \langle c\boldsymbol{\alpha} \rangle_+ = \left\langle \frac{c^2\mathbf{p}}{E} \right\rangle_+ = \langle \mathbf{v}_{gp} \rangle_+ \qquad (3.29)$$

where $< >_+$ denotes expectation value with respect to a positive-energy packet.

Thus the average current for an arbitrary packet formed of positive-energy solutions is just the classical group velocity.

The corresponding statement is familiar in the nonrelativistic Schrödinger theory.

Now we come to an important difference in the relativistic theory.

In the Schrödinger theory the velocity operator appearing in current is just $\boldsymbol{p}/m$ and is a constant of the motion for free particles.

The current is not, however, proportional to the momentum in the Dirac theory, and whereas the Ehrenfest relation (1.27) has shown that $d\mathbf{p}/dt = 0$ for free-particle motion, the velocity operator $c\alpha$ is not constant, since $[\alpha,H] \neq 0$.

Indeed in constructing eigenfunctions of $c\alpha$ we have to include both positive- and negative-energy solutions, since the eigenvalues of $c\alpha$ are $\pm c$ whereas $|\langle c\alpha^i \rangle + | < c$, according to (3.29).

Let us now enlarge our considerations to include the negative-as well as positive-energy solutions informing a packet from complete set of free-particle solutions.

We generalize (3.23) to

$$\psi(\mathbf{x},t) = \int \frac{d^3p}{(2\pi\hbar)^{3/2}} \sqrt{\frac{mc^2}{E}} \sum_{\pm s} [b(p,s)u(p,s)e^{-ip^\mu x_\mu/\hbar} + d^*(p,s)v(p,s)e^{+ip^\mu x_\mu/\hbar}] \tag{3.30}$$

again normalized to unit probability.

A short calculation gives for the probability

$$\int d^3x\, \psi^\dagger(\mathbf{x},t)\psi(\mathbf{x},t) = \int d^3p \sum_{\pm s} [|b(p,s)|^2 + |d(p,s)|^2] = 1$$

and for the current for such a packet

$$J^k = \int d^3p \left\{ \sum_{\pm s} [|b(p,s)|^2 + |d(p,s)|^2] \frac{p^k c^2}{E} + i \sum_{\pm s, \pm s'} b^*(-p,s') d^*(p,s) e^{2ix_0 p_0/\hbar} \bar{u}(-p,s') \sigma^{k0} v(p,s) - i \sum_{\pm s, \pm s'} b(-p,s') d(p,s) e^{-2ix_0 p_0/\hbar} \bar{v}(p,s') \sigma^{k0} u(-p,s) \right\} \tag{3.31}$$

NOTE: Despite a certain inconsistency, we denote hereafter

$$u(\sqrt{\mathbf{p}^2 + m^2}, -\mathbf{p}, s) \equiv u(-p,s)$$

with similar conventions for expansion coefficients b, d^*, etc.

In addition to the time-independent group velocity there now appear cross terms between the positive- and negative-energy solutions which oscillate rapidly in time with frequencies

$$\frac{2p_0 c}{\hbar} > \frac{2mc^2}{\hbar} = 2 \times 10^{21}\ \text{sec}^{-1}$$

This rapid oscillation, or *zitterbewegung*, is proportional to the amplitude of the negative-energy solutions in the packet.

We have as yet no physical interpretation of these solutions, but we may ask when to expect them to be present in the packet with appreciable amplitude.

The general form of a free-particle solution (3.30) shows explicitly by the time independence of $b(p,s)$ that a packet initially formed with positive-energy solutions only does not develop negative-energy components in the absence of forces.

However, a packet formed to represent an electron somehow localized initially in a region of finite extent generally includes solutions of both signs of energy.

Consider, for example, the solution

$$\psi(\mathbf{r},0,s) = (\pi d^2)^{-3/4} e^{-\frac{1}{2}r^2/d^2} w^1(0) \tag{3.32}$$

which corresponds to a Gaussian density distribution of half-width ~d about the origin at time t = 0.

At a later time t it can be expressed as a packet (3.30) with the coefficients b and d^* fixed by the initial conditions, viz., at t = 0

$$\int \frac{d^3p}{(2\pi\hbar)^{3/2}} \sqrt{\frac{\overline{mc^2}}{E}} \sum_{\pm s} [b(p,s)u(p,s)e^{i\mathbf{p}\cdot\mathbf{r}/\hbar} + d^*(p,s)v(p,s)e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar}]$$

$$= (\pi d^2)^{-3/4} e^{-\frac{1}{2}r^2/d^2} w^1(0)$$

Taking the Fourier transform and using

$$\int_{-\infty}^{\infty} d^3r\, e^{-r^2/2d^2} e^{i\mathbf{l}\cdot\mathbf{r}/\hbar} = (2\pi d^2)^{3/2} e^{-\frac{1}{2}l^2d^2/\hbar^2}$$

we find

$$\sqrt{\frac{mc^2}{E}} \sum_{\pm s'} [b(p,s')u(p,s') + d^*(-p,s')v(-p,s')] = \left(\frac{d^2}{\pi\hbar^2}\right)^{3/4} e^{-\frac{1}{2}p^2d^2/\hbar^2} w^1(0)$$

The orthogonality relation (3.11) gives

$$b(p,s) = \sqrt{\frac{mc^2}{E}} \left(\frac{d^2}{\pi\hbar^2}\right)^{3/4} e^{-p^2d^2/2\hbar^2} u^\dagger(p,s) w^1(0)$$

$$d^*(-p,s) = \sqrt{\frac{mc^2}{E}} \left(\frac{d^2}{\pi\hbar^2}\right)^{3/4} e^{-p^2d^2/2\hbar^2} v^\dagger(-p,s) w^1(0) \qquad (3.33)$$

Thus the amplitude d^* of the negative-energy solutions in the packet (3.32) is nonzero.

Relative to the positive-energy components b it is reduced by the ratio of the upper, or small, components of v to the upper, or large, components of u, that is, by $\sim pc/(E + mc^2)$.

This shows that the negative-energy amplitudes are appreciable for momenta $\sim mc$.

We also see in (3.33), however, that the packet is composed predominantly of momenta $p \leq \hbar/d$.

Therefore, packet must be localized in a region of space comparable with the electron Compton wavelength, that is, with $d \sim \hbar/mc$, before the negative-energy solutions enter appreciably.!

This result can be equally well inferred on dimensional groups using $\Delta p\, \Delta x \sim \hbar$ without reference to the particular gaussian shape.

In discussing problems and interactions in which the electron is "spread out" over distances large compared with its Compton wave-length, we may simply ignore the existence of the uninterpreted negative-energy solutions and hope to obtain physically sensible and accurate results.

This will not work, however, in situations which find electrons localized to distances comparable with $\hbar$/mc.

The negative-frequency amplitudes will then be appreciable, the zitterbewegung terms in the current important, and indeed we shall find ourselves beset by paradoxes and dilemmas which defy interpretation within the framework so far developed by the Dirac theory of an electron.

A celebrated example of these difficulties is the Klein paradox, illustrated by the following example.

In order to localize electrons, we must introduce strong external forces confining them to the desired region.

Suppose, for example, we want to confine a free electron of energy E to region I to the left of the origin z = 0 in the one-dimensional potential diagram of Fig. 3.1.

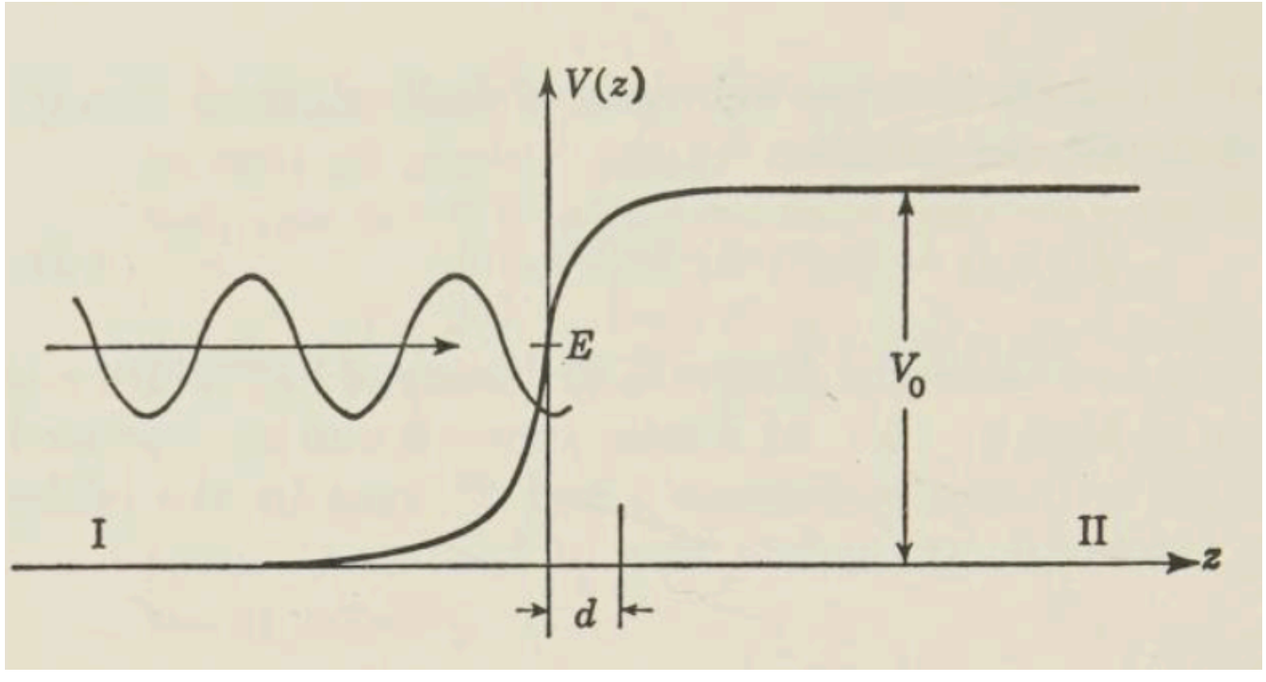

Fig. 3-1 Potential barrier confining electron of energy *E* in region *I* to the left.

If the electron is not to be found more than a distance d to the right of $z = 0$, in region II, then V must rise sharply within an interval $z \leq$ d to a height $V_0 > E$ so that the solution in II falls off with a characteristic width $\leq$ d.

This is as in the Schrödinger theory, until the confining length d shrinks to ~k/mc and V_0 - E increases beyond mc.

To see what happens, let us consider an electrostatic potential with a sharp boundary as in Fig. 3.2 and calculate the reflected and transmitted current for an electron of wave number k incident from the left with spin up along the z direction.

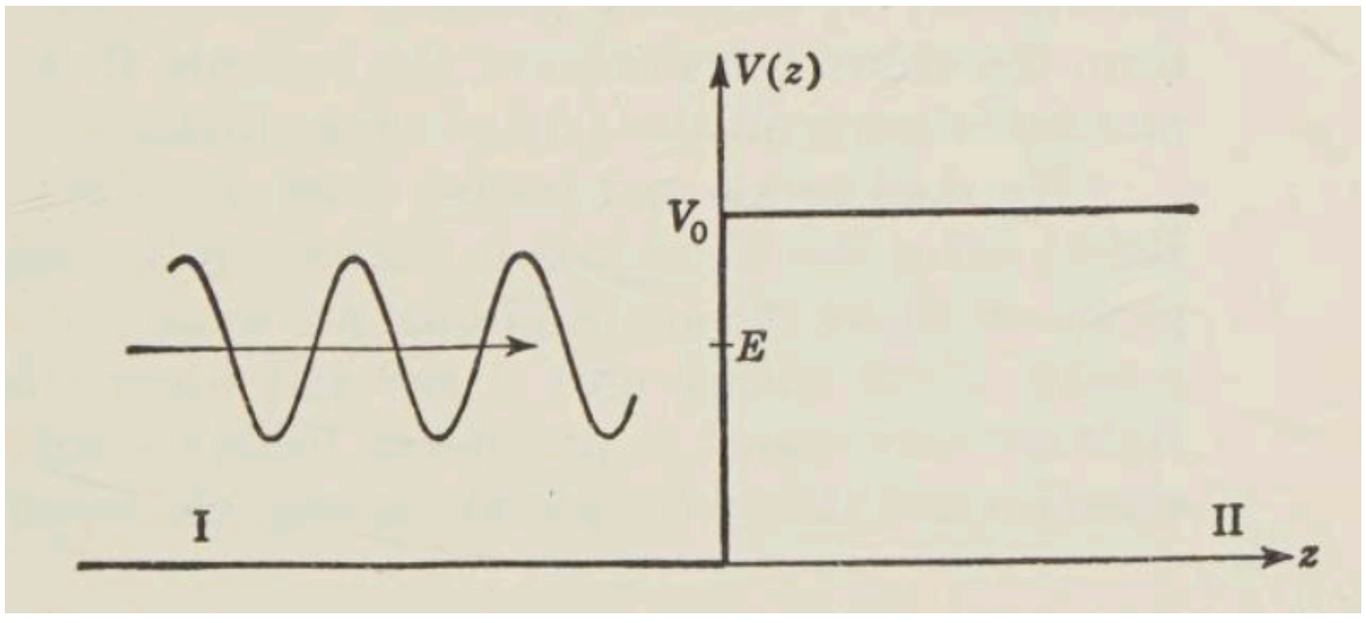

Fig. 3-2 Electrostatic potential idealized with a sharp boundary, with an incident free electron wave of energy E moving to the right in region I. For $V_0 > E + mc$ the reflected current from the potential exceeds the incident one; this is an example of the Klein paradox.

The positive-energy solutions for the incident and reflected waves in region I may be written

$$\psi_{\text{inc}} = ae^{ik_1 z}\begin{bmatrix} 1 \\ 0 \\ \dfrac{ck_1\hbar}{E + mc^2} \\ 0 \end{bmatrix}$$

$$\psi_{\text{ref}} = be^{-ik_1 z}\begin{bmatrix} 1 \\ 0 \\ -\dfrac{ck_1\hbar}{E + mc^2} \\ 0 \end{bmatrix} + b'e^{-ik_1 z}\begin{bmatrix} 0 \\ 1 \\ 0 \\ \dfrac{ck_1\hbar}{E + mc^2} \end{bmatrix} \qquad (3.34)$$

For the transmitted wave we need the solutions of the Dirac equation in the presence of a constant external potential e$\Phi = V_0$.

These differ from the free-particle solutions only by the substitution $p_0 = (1/c)(E - V_0)$, so that in region II

$$\hbar^2 k_2^2 c^2 = (E - V_0)^2 - m^2c^4 = (E - mc^2 - V_0)(E + mc^2 - V_0)$$

We therefore write the transmitted wave of positive energy E > 0 as

$$\psi_{\text{trans}} = de^{ik_2 z}\begin{bmatrix} 1 \\ 0 \\ \dfrac{c\hbar k_2}{E - V_0 + mc^2} \\ 0 \end{bmatrix} + d'e^{ik_2 z}\begin{bmatrix} 0 \\ 1 \\ 0 \\ \dfrac{-c\hbar k_2}{E - V_0 + mc^2} \end{bmatrix} \qquad (3.35)$$

The amplitudes d and d' are fixed by continuity of the solution at the potential boundary as required by current conservation:

$$a + b = d$$

$$a - b = \frac{k_2}{k_1} \frac{E + mc^2}{E - V_0 + mc^2} d \equiv rd \tag{3.36}$$

$$b' = d' = 0 \qquad \text{(there is no spin flip)}$$

If $V_0 > 0$ and $|E - V_0| < mc^2$, the wave number is imaginary, $k_2 = +i\,|\,k_z\,|$, and the solution in region II is a decaying exponential corresponding to damping in a distance $d > \hbar/mc$.

However, as we increase the height of the barrier beyond $V_0 = E + mc^2$ in order to further confine the electron, the transmitted wave becomes oscillatory.

The transmitted and reflected currents may be computed, and we find

$$\frac{j_{\text{trans}}}{j_{\text{inc}}} = \frac{4r}{(1+r)^2} \qquad \frac{j_{\text{ref}}}{j_{\text{inc}}} = \frac{(1-r)^2}{(1+r)^2} = 1 - \frac{j_{\text{trans}}}{j_{\text{inc}}} \tag{3.37}$$

Whereas the form of these results reminds us of the analogous predictions of the Schrödinger theory, we must now observe that, (3.36) and the above condition $V_0 > E + mc^2$, $r < 0$.

So we find in (3.37) a result contradicting our ordinary reasoning by indicating a *negative* transmitted current and a reflected current *exceeding* the *incident* one.

What is the source of a current in region II moving left in Fig. 3.2 into region I in this case of $V_0 > E + mc^2$?

We increased the potential height V_0 beyond $E + mc^2$ in attempting to localize the solution within one Compton wavelength $\hbar$/mc, but ended up with undamped oscillatory solutions instead.

How do we understand this?

Only by understanding and interpreting the negative-energy solutions.

It is clear from the packet discussion that they enter prominently in solutions localized within $\hbar$/mc.

It is equally clear from the above calculation of the currents that our physical picture of what is going on also fails at these distances.

We shall tackle and resolve these questions starting in Chap. 5.

Before doing this let us look in the vast, if limited, domain of physical problems where the applied forces are weak and smoothly varying on a scale whose energy unit is me and whose distance unit is $\hbar$/mc.

Here we may expect to find fertile fields for application of the Dirac equation and theory for positive-energy electrons.

4 The Foldy-Wouthuysen Transformation

4.1 Introduction

Aside from the negative-energy problem, the Dirac equation appears to provide a suitable description of the electron.

It has a sensible nonrelativistic limit, and it automatically yields the correct magnetic moment.

We now investigate the interaction of the Dirac electron with prescribed external potentials.

In particular, we shall be primarily interested in low-energy properties, avoiding the difficulties associated with the as yet uninterpreted negative-energy solutions, which are an essentially relativistic feature.

We anticipate from our discussions of the packet in the preceding chapter that in practice they play a very minor role in a problem such as the hydrogen atom, which finds the electron localized in Bohr orbits of radius' $1/\alpha m >> 1/m$.

We shall see, in fact, that the stationary energy levels deduced from the Dirac equation for the hydrogen atom are in exceedingly close agreement with the observed eigenvalues.

However, before indicating the solution to the eigenvalue problem in the Coulomb potential, it is instructive to cast the Dirac theory in a form which displays the different interaction terms between the electron and an applied field in a nonrelativistic and easily interpretable form.

We consider, then, a systematic procedure developed by Foldy and Wouthuysen, namely, a canonical transformation which decouples the Dirac equation into two two-component equations: one reduces to the Pauli description in the nonrelativistic limit; the other describes the negative-energy states.

4.2 Free-particle Transformation

As a first illustration of the Foldy-Wouthuysen transformation we consider the Dirac equation for a free particle, most conveniently-for this purpose—written in hamiltonian form and with the a matrices in the representation introduced in Eq. (1.17).

We search for a unitary transformation U_F which will remove from the equation all operators such as α which couple the large to the small components.

We call any such operator "odd"; operators which do not couple large and small components are "even"; thus α, γ, γ_5, etc., are odd, and $1, \beta, \sigma$, etc., are even.

Writing $U_F = e^{+iS}$ with S hermitian and not explicitly time-dependent, the unitary transformation is

$$\psi' = e^{+iS}\psi$$

and

$$i\frac{\partial \psi'}{\partial t} = e^{+iS}H\psi = e^{+iS}He^{-iS}\psi' = H'\psi'$$

H' is to contain no odd operators by construction.

Since $H = \alpha \cdot \mathbf{p} + \beta m$ with $\{\alpha,\beta\} = 0$, our problem is quite analogous to that of attempting to find a unitary transformation which changes a two-component spin hamiltonian $\mathcal{H} = \sigma_x B_x + \sigma_z B_z$ into a form which contains only even operators (that is, 1 and σ_z).

Such a transformation is simply a rotation about the y axis and the operator is $e^{+(i/2)\sigma_y\theta_0} = e^{+\frac{1}{2}\sigma_z\sigma_x\theta_0}$, with $\tan\theta_0 = B_x/B_z$.

This suggests that a good operator to try in our case would be

$$e^{iS} = e^{\beta\alpha\cdot\mathbf{p}\theta(\mathbf{p})} = \cos|\mathbf{p}|\theta + \frac{\beta\alpha\cdot\mathbf{p}}{|\mathbf{p}|}\sin|\mathbf{p}|\theta$$

where the right-hand side is established by expansion of the exponential in powers of θ.

With this choice H' becomes:

$$H' = \left(\cos|\mathbf{p}|\theta(\mathbf{p}) + \frac{\beta\alpha\cdot\mathbf{p}}{|\mathbf{p}|}\sin|\mathbf{p}|\theta(\mathbf{p})\right)(\alpha\cdot\mathbf{p} + \beta m)\left(\cos|\mathbf{p}|\theta - \frac{\beta\alpha\cdot\mathbf{p}}{|\mathbf{p}|}\sin|\mathbf{p}|\theta\right)$$

$$= (\alpha\cdot\mathbf{p} + \beta m)\left(\cos|\mathbf{p}|\,\theta - \frac{\beta\alpha\cdot\mathbf{p}}{|\mathbf{p}|}\sin|\mathbf{p}|\theta\right)^2$$

$$= (\alpha\cdot\mathbf{p} + \beta m)\exp(-2\beta\alpha\cdot\mathbf{p}\theta)$$

$$= \alpha\cdot\mathbf{p}(\cos 2|\mathbf{p}|\theta - \frac{m}{|\mathbf{p}|}\sin 2|\mathbf{p}|\theta) + \beta(m\cos 2|\mathbf{p}|\theta + |\mathbf{p}|\sin 2|\mathbf{p}|\theta)$$

In order to eliminate the odd operator, we choose

$$\tan 2|\mathbf{p}|\theta = \frac{|\mathbf{p}|}{m}$$

and the transformed hamiltonian is

$$H' = \beta \sqrt{m^2 + \mathbf{p}^2} \tag{4.1}$$

as may be verified with the aid of the triangle construction of Fig. 4.1.

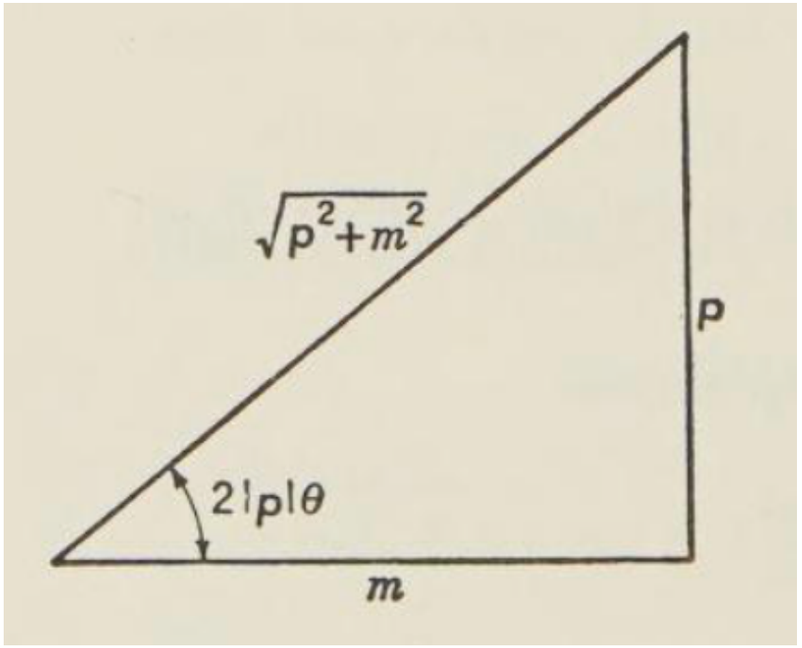

Fig. 4-1 Foldy-Wouthrysen triangle construction.

The new hamiltonian is just the one rejected in Chap. 1, with the important change that now the negative energies are also accepted.

The negative energies and four-component wave functions are the price we must pay in order to have a factorization of H' in (4.1) into a linear Dirac equation.

4.3 The General Transformation

We turn now to the more general case of an electron in a prescribed external electromagnetic field and search for the corresponding transformation S.

The hamiltonian is

$$H = \boldsymbol{\alpha} \cdot (\mathbf{p} - e\mathbf{A}) + \beta m + e\Phi = \beta m + \mathcal{O} + \mathcal{E} \tag{4.2}$$

with $\mathcal{O} = \boldsymbol{\alpha} \cdot (\mathbf{p} - e\mathbf{A})$ and $\mathcal{E} = e\Phi$; as before, $\beta\mathcal{O} = -\mathcal{O}\beta$ and

$$\beta\mathcal{E} = +\mathcal{E}\beta$$

The fields appearing in (4.2) and hence the hamiltonian itself may be time-dependent.

In the general case the transformation S is also time-dependent and it is not possible to construct an S which removes the odd operators from H' to all orders, as was achieved in (4.1).

Therefore, we content ourselves with a nonrelativistic expansion of the transformed hamiltonian in a power series in $1/m$, keeping terms only through order (kinetic energy/m)$_3$ and (kinetic energy) (field energy)/m^2.

Again we introduce the transformation by

$$\psi' = e^{iS}\psi$$

finding

$$i\frac{\partial}{\partial t}e^{-iS}\psi' = H\psi = He^{-iS}\psi' = e^{-iS}\left(i\frac{\partial\psi'}{\partial t}\right) + \left(i\frac{\partial}{\partial t}e^{-iS}\right)\psi'$$

Thus

$$i\frac{\partial\psi'}{\partial t} = \left[e^{iS}\left(H - i\frac{\partial}{\partial t}\right)e^{-iS}\right]\psi' = H'\psi'$$

Since S is expanded in powers of $1/m$ and is therefore "small" in the nonrelativistic limit, we expand the quantity in brackets in a series of multiple commutators, using the relation

$$e^{+iS}He^{-iS} = H + i[S,H] + \frac{(i)^2}{2!}[S,[S,H]] + \cdots + \frac{(i)^n}{n!}[S,[S,\ \cdots\ ,[S,H]\cdots]] + \cdots$$

Since $S = \mathrm{O}(1/m)$, to the desired order of accuracy we have

$$H' = H + i[S,H] - \frac{1}{2}[S,[S,H]] - \frac{i}{6}[S,[S,[S,H]]] + \frac{1}{24}[S,[S,[S,[S,\beta m]]]] - \dot{S} - \frac{i}{2}[S,\dot{S}] + \frac{1}{6}[S,[S,\dot{S}]]$$

To start constructing S, we consider just the terms through order unity:

$$H' = \beta m + \mathcal{E} + \mathcal{O} + i[S,\beta]m \tag{4.3}$$

We require that the odd term in (4.3) vanish; and taking our cue from the behavior in the free-particle case, we choose $S = -i\beta\mathcal{O}/2m$.

We then find, to the order of accuracy desired,

$$i[S,H] = -\mathcal{O} + \frac{\beta}{2m}[\mathcal{O},\mathcal{E}] + \frac{1}{m}\beta\mathcal{O}^2$$

$$\frac{i^2}{2}[S,[S,H]] = -\frac{\beta\mathcal{O}^2}{2m} - \frac{1}{8m^2}[\mathcal{O},[\mathcal{O},\mathcal{E}]] - \frac{1}{2m^2}\mathcal{O}^3$$

$$\frac{i^3}{3!}[S,[S,[S,H]]] = \frac{\mathcal{O}^3}{6m^2} - \frac{1}{6m^3}\beta\mathcal{O}^4$$

$$\frac{i^4}{4!}[S,[S,[S,[S,H]]]] = \frac{\beta\mathcal{O}^4}{24m^3}$$

$$-\dot{S} = +\frac{i\beta\dot{\mathcal{O}}}{2m}$$

$$-\frac{i}{2}[S,\dot{S}] = -\frac{i}{8m^2}[\mathcal{O},\dot{\mathcal{O}}]$$

Collecting everything together,

$$H' = \beta\left(m + \frac{\mathcal{O}^2}{2m} - \frac{\mathcal{O}^4}{8m^3}\right) + \mathcal{E} - \frac{1}{8m^2}[\mathcal{O},[\mathcal{O},\mathcal{E}]] - \frac{i}{8m^2}[\mathcal{O},\dot{\mathcal{O}}]$$

$$+ \frac{\beta}{2m}[\mathcal{O},\mathcal{E}] - \frac{\mathcal{O}^3}{3m^2} + \frac{i\beta\dot{\mathcal{O}}}{2m} = \beta m + \mathcal{E}' + \mathcal{O}' \qquad (4.4)$$

The odd terms now appear in (4.4) only in order 1/m.

To reduce them further, we apply a second Foldy-Wouthuysen transformation using the same prescription:

$$S' = \frac{-i\beta}{2m}\mathcal{O}' = \frac{-i\beta}{2m}\left(\frac{\beta}{2m}[\mathcal{O},\mathcal{E}] - \frac{\mathcal{O}^3}{3m^2} + \frac{i\beta\dot{\mathcal{O}}}{2m}\right)$$

Under this transformation we find

$$H'' = e^{iS'}\left(H' - i\frac{\partial}{\partial t}\right)e^{-iS'} = \beta m + \mathcal{E}' + \frac{\beta}{2m}[\mathcal{O}',\mathcal{E}'] + \frac{i\beta\dot{\mathcal{O}}'}{2m}$$
$$= \beta m + \mathcal{E}' + \mathcal{O}''$$

where $\mathcal{O}''$ is now $O(1/m^2)$.

Finally, by a third canonical transformation

$$S'' = \frac{-i\beta\mathcal{O}''}{2m}$$

this term may also be eliminated in the same way, the end result being

$$H''' = e^{iS''}\left(H'' - i\frac{\partial}{\partial t}\right)e^{-iS''} = \beta m + \mathcal{E}'$$
$$= \beta\left(m + \frac{\mathcal{O}^2}{2m} - \frac{\mathcal{O}^4}{8m^3}\right) + \mathcal{E} - \frac{1}{8m^2}[\mathcal{O},[\mathcal{O},\mathcal{E}]] - \frac{i}{8m^2}[\mathcal{O},\dot{\mathcal{O}}]$$

Evaluating the operator products to the desired order of accuracy, we find

$$\frac{\mathcal{O}^2}{2m} = \frac{(\boldsymbol{\alpha}\cdot(\mathbf{p}-e\mathbf{A}))^2}{2m} = \frac{(\mathbf{p}-e\mathbf{A})^2}{2m} - \frac{e}{2m}\,\boldsymbol{\sigma}\cdot\mathbf{B}$$

$$\frac{1}{8m^2}\left([\mathcal{O},\varepsilon]+i\dot{\mathcal{O}}\right) = \frac{e}{8m^2}\left(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\Phi - i\boldsymbol{\alpha}\cdot\dot{\mathbf{A}}\right) = \frac{ie}{8m^2}\,\boldsymbol{\alpha}\cdot\mathbf{E}$$

$$\left[\mathcal{O}, \frac{ie}{8m^2}\,\boldsymbol{\alpha}\cdot\mathbf{E}\right] = \frac{ie}{8m^2}\,[\boldsymbol{\alpha}\cdot\mathbf{p},\boldsymbol{\alpha}\cdot\mathbf{E}]$$

$$= \frac{ie}{8m^2}\sum_{ij}\alpha_i\alpha_j\left(-i\,\frac{\partial E^j}{\partial x^i}\right) + \frac{e}{4m^2}\,\boldsymbol{\sigma}\cdot\mathbf{E}\times\mathbf{p}$$

$$= \frac{e}{8m^2}\,\mathrm{div}\,\mathbf{E} + \frac{ie}{8m^2}\,\boldsymbol{\sigma}\cdot\mathrm{curl}\,\mathbf{E} + \frac{e}{4m^2}\,\boldsymbol{\sigma}\cdot\mathbf{E}\times\mathbf{p}$$

and thus the reduced hamiltonian is to this order

$$H''' = \beta\left(m + \frac{(\mathbf{p}-e\mathbf{A})^2}{2m} - \frac{\mathbf{p}^4}{8m^3}\right) + e\Phi - e\,\frac{1}{2m}\,\beta\boldsymbol{\sigma}\cdot\mathbf{B}$$
$$- \frac{ie}{8m^2}\,\boldsymbol{\sigma}\cdot\mathrm{curl}\,\mathbf{E} - \frac{e}{4m^2}\,\boldsymbol{\sigma}\cdot\mathbf{E}\times\mathbf{p} - \frac{e}{8m^2}\,\mathrm{div}\,\mathbf{E} \qquad (4.5)$$

The individual terms in (4.5) have a direct physical interpretation.

The terms in the first bracket give the expansion of

$$\sqrt{(\mathbf{p}-e\mathbf{A})^2 + m^2}$$

to the desired order, showing the relativistic mass increase.

The second and third terms are the electrostatic and magnetic dipole. energies.

The next pair of terms which taken together are hermitian comprise the spin-orbit energy, and they have a very familiar form in a spherically symmetric static potential. In this case curl **E** = 0,

$$\boldsymbol{\sigma} \cdot \mathbf{E} \times \mathbf{p} = -\frac{1}{r}\frac{\partial V}{\partial r}\boldsymbol{\sigma} \cdot \mathbf{r} \times \mathbf{p} = -\frac{1}{r}\frac{\partial V}{\partial r}\boldsymbol{\sigma} \cdot \mathbf{L}$$

and this term reduces to

$$H_{\text{spin-orbit}} = \frac{e}{4m^2}\frac{1}{r}\frac{\partial V}{\partial r}\boldsymbol{\sigma} \cdot \mathbf{L} \tag{4.6}$$

Equation (4.6) is in agreement with the classical result obtained by considering the magnetic field $\mathbf{B}' = -\mathbf{v} \times \mathbf{E}$ experienced by the moving electron.

The interaction energy one would expect is thus

$$-\frac{e}{2m}\boldsymbol{\sigma} \cdot \mathbf{B}' = \frac{e}{2m^2}\boldsymbol{\sigma} \cdot (\mathbf{p} \times \mathbf{E})$$

However, this is reduced by a factor of 2 owing to the Thomas precession effect and indicates that the orbital moment of the electron has the standard gyromagnetic ratio of $g_e = 1$.

The last term-known as the Darwin term-may be attributed to the zitterbewegung.

Because the electron coordinate fluctuates over distances or ~1/m, it sees a somewhat smeared out Coulomb potential; the correction is

$$\langle \delta V \rangle = \langle V(\mathbf{r} + \delta\mathbf{r}) \rangle - \langle V(\mathbf{r}) \rangle = \left\langle \delta r \frac{\partial V}{\partial r} + \frac{1}{2} \sum_{ij} \delta r_i \, \delta r_j \frac{\partial^2 V}{\partial r_i \, \partial r_j} \right\rangle$$
$$\cong \frac{1}{6} \delta r^2 \, \nabla^2 V \simeq \frac{1}{6m^2} \nabla^2 V \tag{4.7}$$

in qualitative accord with the sign, form, and magnitude of the Darwin term.

4.4 The Hydrogen Atom

We turn to a discussion of the bound-state solutions of the Dirac equation, considering in particular the energy levels of the electron in a Coulomb field.

For this problem the Dirac equation is

$$H\psi = [\alpha \cdot \mathbf{p} + \beta m + V(r)]\psi = E\psi \tag{4.8}$$

with $V = -Z\alpha/r$.

In order to separate variables, we take advantage of the fact that the angular momentum of a particle in a central field is conserved.

Evidently $\mathbf{J} = \mathbf{L} + \mathbf{S} = \mathbf{r} \times \mathbf{p} + \frac{1}{2}\sigma$ commutes with the hamiltonian (4.8) and therefore we may construct simultaneous eigenfunctions of H, J^2, and J_z.

To do this, we call on experience with the Pauli matrices, observing that in the representation of Chap. w

$$\mathfrak{d} = \begin{bmatrix} \mathfrak{d} & 0 \\ 0 & \mathfrak{d} \end{bmatrix}$$

is diagonal in terms of 2 x 2 Pauli spin matrices.

Therefore, if we construct in terms of two-component spinors

$$\psi = \begin{bmatrix} \varphi \\ \chi \end{bmatrix}$$

the angular separation for the solutions of φ and x is precisely that of the Pauli two-component theory.

The two-component angular solutions are eigenfunctions of J^2, J_z, L^2, and S^2 and are of two types:

For $j = l + \frac{1}{2}$

$$\varphi^{(+)}_{j,m} = \begin{bmatrix} \sqrt{\dfrac{l + \frac{1}{2} + m}{2l+1}}\, Y_l^{m-\frac{1}{2}} \\ \sqrt{\dfrac{l + \frac{1}{2} - m}{2l+1}}\, Y_l^{m+\frac{1}{2}} \end{bmatrix} \qquad (4.9a)$$

For j = l - ½

$$\varphi^{(-)}_{j,m} = \begin{bmatrix} \sqrt{\frac{l+\frac{1}{2}-m}{2l+1}}\, Y_l^{m-\frac{1}{2}} \\ -\sqrt{\frac{l+\frac{1}{2}+m}{2l+1}}\, Y_l^{m+\frac{1}{2}} \end{bmatrix} \tag{4.9b}$$

The spherical harmonics here are written with the convention $Y^*_{lm} = (-)^m Y_{l,-m}$, and the solution $\varphi^{(-)}$ exists only for $l > 0$.

The two solutions above satisfy the eigenvalue equations

and

$$J^2 \varphi^{(\pm)}_{jm} = j(j+1)\varphi_{jm}$$

$$\mathbf{L}\cdot\boldsymbol{\sigma}\varphi^{(\pm)}_{jm} = (J^2 - L^2 - \tfrac{3}{4})\varphi^{(\pm)}_{jm} = -(1+\kappa)\varphi^{(\pm)}_{jm}$$

with

$$\kappa = \begin{cases} -(l+1) = -(j+\frac{1}{2}) & j = l+\frac{1}{2} \\ +l = +(j+\frac{1}{2}) & j = l-\frac{1}{2} \end{cases}$$

For a given j they are of opposite parity, since their l values differ by 1, and can be formed from each other by a scalar operator of odd parity.

This operator will be a linear combination of $Y^m{}_1(\theta,\varphi)$ since it must change the l value by 1, and is therefore proportional to **r**.

Dotting with σ, the only pseudovector at our disposal, we form the pseudoscalar $\sigma\cdot$ **r**/r and find with the above sign convention

$$\varphi_{jm}^{(+)} = \frac{\boldsymbol{\sigma} \cdot \mathbf{r}}{r} \varphi_{jm}^{(-)} \tag{4.10}$$

tThe general solution to the central field problem for a given *jm* is

$$\psi_{jm} = \begin{pmatrix} \dfrac{iG_j^+}{r} \varphi_{jm}^{(+)} + \dfrac{iG_j^-}{r} \varphi_{jm}^{(-)} \\ \dfrac{F_j^+}{r} \varphi_{jm}^{(-)} + \dfrac{F_j^-}{r} \varphi_{jm}^{(+)} \end{pmatrix}$$

We may finally break this down into two solutions each of definite parity.

Since $V(r)$ is invariant under reflection of coordinates, we know that the energy eigenfunctions can be classified into parity eigenstates along with (j , m); and therefore we form the even and odd solutions, which have the property under the transformation $\mathbf{x}' = -\mathbf{x}$

$$\psi'(x') = \beta\psi(x) = \pm\psi(x') \tag{4.11}$$

These are given by

$$\psi_{jm}^{l} = \begin{bmatrix} \dfrac{iG_{lj}}{r} \varphi_{jm}^{l} \\ \dfrac{F_{lj}}{r} \dfrac{\boldsymbol{\sigma} \cdot \mathbf{r}}{r} \varphi_{jm}^{l} \end{bmatrix} \tag{4.12}$$

where as a common notation we have introduced

$$G_{lj} = \begin{cases} G_j^+ & j = l + \tfrac{1}{2} \\ G_j^- & j = l - \tfrac{1}{2} \end{cases} \qquad F_{lj} = \begin{cases} F_j^+ & j = l + \tfrac{1}{2} \\ F_j^- & j = l - \tfrac{1}{2} \end{cases}$$

$$\varphi_{jm}^l = \begin{cases} \varphi_{jm}^+ & j = l + \tfrac{1}{2} \\ \varphi_{jm}^- & j = l - \tfrac{1}{2} \end{cases}$$

and have made use of (4.10).

The parity of these solutions is $(-)^l$ by the convention (4.11).

With the aid of the following identities we can now find the radial equations following from (4.8):

$$\boldsymbol{\sigma} \cdot \mathbf{p} \frac{f(r)}{r} \varphi_{jm}^l = \frac{\boldsymbol{\sigma} \cdot \mathbf{r}}{r^2} (\boldsymbol{\sigma} \cdot \mathbf{r}\, \boldsymbol{\sigma} \cdot \mathbf{p}) \frac{f(r)}{r} \varphi_{jm}^l$$

$$= \frac{\boldsymbol{\sigma} \cdot \mathbf{r}}{r^2} \left(\frac{1}{i} r \frac{\partial}{\partial r} + i\, \boldsymbol{\sigma} \cdot \mathbf{L} \right) \frac{f(r)}{r} \varphi_{jm}^l$$

$$= \left[\frac{1}{i} \frac{d}{dr} \frac{f(r)}{r} - i(1 + \kappa) \frac{f(r)}{r^2} \right] \left(\frac{\boldsymbol{\sigma} \cdot \mathbf{r}}{r} \right) \varphi_{jm}^l$$

The radial equations are then

$$\left(E - m + \frac{Z\alpha}{r} \right) G_{lj}(r) = - \frac{dF_{lj}(r)}{dr} + \frac{\kappa}{r} F_{lj}(r)$$

$$\left(E + m + \frac{Z\alpha}{r} \right) F_{lj}(r) = + \frac{dG_{lj}(r)}{dr} + \frac{\kappa}{r} G_{lj}(r) \qquad (4.13)$$

The bound-state solutions of these equations may be found by standard methods; we quote only some of the results.

The energy eigenvalues are

$$E_n = m\left[1 + \left(\frac{Z\alpha}{n - (j + \tfrac{1}{2}) + \sqrt{(j + \tfrac{1}{2})^2 - Z^2\alpha^2}}\right)^2\right]^{-\frac{1}{2}} \tag{4.14}$$

where the quantum number $n = 1, 2, \ldots, \infty$ is a positive integer and the angular-momentum eigenvalues range from 0 to $j + \tfrac{1}{2} \leq n$, with the restriction $0 \leq l \leq n - 1$.

Expanding (4.14) in powers of $(Z\alpha)^2$, we see that n corresponds to the principal quantum number of the nonrelativistic theory

$$E_n \approx m\left\{1 - \frac{1}{2}\frac{Z^2\alpha^2}{n^2}\left[1 + \frac{(Z\alpha)^2}{n}\left(\frac{1}{j + \tfrac{1}{2}} - \frac{3}{4n}\right)\right] + 0((Z\alpha)^6)\right\} \tag{4.15}$$

The ground-state energy is, with $n = 1, j = \tfrac{1}{2}$,

$$E_g = m\sqrt{1 - Z^2\alpha^2} \cong m - \tfrac{1}{2}Z^2\alpha^2 m - \tfrac{1}{8}Z^4\alpha^4 m + \cdots$$

The corresponding spin-up and spin-down normalized eigenfunctions are

$$\psi_{n=1,j=1/2,\uparrow}(r,\theta,\varphi)$$

$$= \frac{(2mZ\alpha)^{3/2}}{\sqrt{4\pi}} \sqrt{\frac{1+\gamma}{2\Gamma(1+2\gamma)}} (2mZ\alpha r)^{\gamma-1} e^{-mZ\alpha r} \begin{bmatrix} 1 \\ 0 \\ \dfrac{i(1-\gamma)}{Z\alpha} \cos\theta \\ \dfrac{i(1-\gamma)}{Z\alpha} \sin\theta e^{i\varphi} \end{bmatrix}$$

$$\psi_{n=1,j=1/2,\downarrow}(r,\theta,\varphi)$$

$$= \frac{(2mZ\alpha)^{3/2}}{\sqrt{4\pi}} \sqrt{\frac{1+\gamma}{2\Gamma(1+2\gamma)}} (2mZ\alpha r)^{\gamma-1} e^{-mZ\alpha r} \begin{bmatrix} 0 \\ 1 \\ \dfrac{i(1-\gamma)}{Z\alpha} \sin\theta e^{-i\varphi} \\ \dfrac{-i(1-\gamma)}{Z\alpha} \cos\theta \end{bmatrix}$$

with $\gamma = \sqrt{1 - Z^2\alpha^2}$.

In the nonrelativistic limit $\gamma \rightarrow 1$ and $(1 - \gamma)/Z\alpha \rightarrow 0$, and they reduce to the Schrödinger wave functions multiplied by two-component Pauli spinors.

In the relativistic case we see that as $r \rightarrow 0$, ψ exhibits a mild singularity of order $(2mZ\alpha r)^{-(Z^2\alpha^2)/2}$ which becomes important only at distances

$$r \sim \frac{1}{2mZ\alpha} e^{-2/Z^2\alpha^2}$$

For $Z\alpha \geq 1$, γ is imaginary and the solutions exhibit an oscillatory behavior reminiscent of that found in the Klein paradox.

In this case there is no longer a gap between the positive- and negative-energy spectra, and again we lack a physical interpretation of the solution.

In classifying the energy levels (4.14) it is customary to denote them by their nonrelativistic labels, that is, by the orbital angular momentum l appearing in ψ^l_{jm} and by the total j.

In the following table we list a few of the first terms:

	n	l	j	E_{nj}
$1S_{1/2}$	1	0	$\frac{1}{2}$	$m\sqrt{1-Z^2\alpha^2}$
$2S_{1/2}$	2	0	$\frac{1}{2}$	$m\sqrt{\frac{1+\sqrt{1-Z^2\alpha^2}}{2}}$
$2P_{1/2}$	2	1	$\frac{1}{2}$	$m\sqrt{\frac{1+\sqrt{1-Z^2\alpha^2}}{2}}$
$2P_{3/2}$	2	1	$\frac{3}{2}$	$\frac{m}{2}\sqrt{4-Z^2\alpha^2}$

The $2S_{1/2}$ and $2P_{3/2}$ states are degenerate, being the two eigenstates of opposite parity corresponding to the same n and j.

The $2P_{1/2}$ state is higher in energy than the $2P_{1/2}$ state; the energy difference, $[m(Z\alpha)^4/32](1 + 0(Z\alpha)^2 + \cdots)$, is the fine-structure splitting due to the spin-orbit interaction, (4.6).

In general, the state of larger j, for a given n, lies higher in energy according to (4.15).

How do these predictions agree with observations for the H atom?

Prior to 1947 the agreement was completely satisfactory after the above predictions were modified to take into account the hyperfine splitting of each level due to coupling between the electron and proton spins.

In 1947 the Lamb-Retherford measurements of the H-atom fine structure confirmed an earlier suspicion of a shift of the 2S, levels upward relative to the $2P_{1/2}$ lines.

This "Lamb shift," breaking the degeneracy of levels with the same n and j but differing l, arises from the interaction of the electrons with the fluctuations of the quantized radiation field.

Both the hyperfine structure splitting and the Lamb shift have been measured and calculated to a very high precision with good agreement?

The hyperfine structure results from the interaction of the proton with the electron magnetic moment.

This has the effect of splitting all lines into doublets corresponding to the two possible states of total angular momentum compounded from the j of the electron system and the half-integer spin of the proton.

Let us compute the magnitude of this effect for s states.

For our purpose a nonrelativistic description of the electron suffices.

The interaction is of the form

$$H' = + \frac{|e|}{2m} \boldsymbol{\sigma} \cdot \mathbf{B}$$

and

$$\mathbf{B} = \frac{g_p e}{2M_p} \int d^3r' \, \rho(r') \nabla \times (\mathbf{I} \times \nabla) \frac{1}{4\pi |\mathbf{r} - \mathbf{r}'|}$$

Here **I** is the proton spin operator ($I_z = \pm ½$) and $\rho(r')$ is the magnetic moment density of the proton, owing to the fact it is not a and taking the angular average for spherically symmetric wave functions so that

$$\nabla_i \nabla_j \Rightarrow \tfrac{1}{3} \delta_{ij} \nabla^2$$

we find

$$\mathbf{B} = \frac{2}{3} g_p \frac{e}{2M_p} \int d^3r' \, \rho(r') \mathbf{I} \nabla^2 \left(\frac{1}{4\pi |\mathbf{r} - \mathbf{r}'|} \right) = \frac{2}{3} g_p \frac{e}{2M_p} \mathbf{I} \rho(r)$$

The energy shift is then given, in nonrelativistic theory, by

$$\Delta E_n = (\psi_n H' \psi_n) = \frac{2}{3} \frac{g_p e^2}{4mM_p} \boldsymbol{\sigma} \cdot \mathbf{I} \int d^3r \, \psi_n^*(r) \rho(r) \psi_n(r)$$

$$\approx \frac{1}{6} \frac{g_p e^2}{mM_p} \boldsymbol{\sigma} \cdot \mathbf{I} |\psi_n(0)|^2$$

$$= \frac{1}{2} m\alpha^2 \left[\frac{4}{3} g_p \frac{Z^3 \alpha^2}{n^3} \left(\frac{m}{M_p} \right) \boldsymbol{\sigma} \cdot \mathbf{I} \right]$$

with

$$\boldsymbol{\sigma}\cdot\mathbf{I} = \begin{cases} +\tfrac{1}{2} & \text{triplet states} \\ -\tfrac{3}{2} & \text{singlet state} \end{cases}$$

The splitting in of the nth s-state level is thus

$$\delta_n = \frac{1}{2} m\alpha^2 \left[\frac{8}{3} g_p \frac{Z^3\alpha^2}{n^3} \left(\frac{m}{M_p} \right) \right]$$

and is reduced by the mass ratio m/M, relative to the fine structure - see Fig. 4.2.

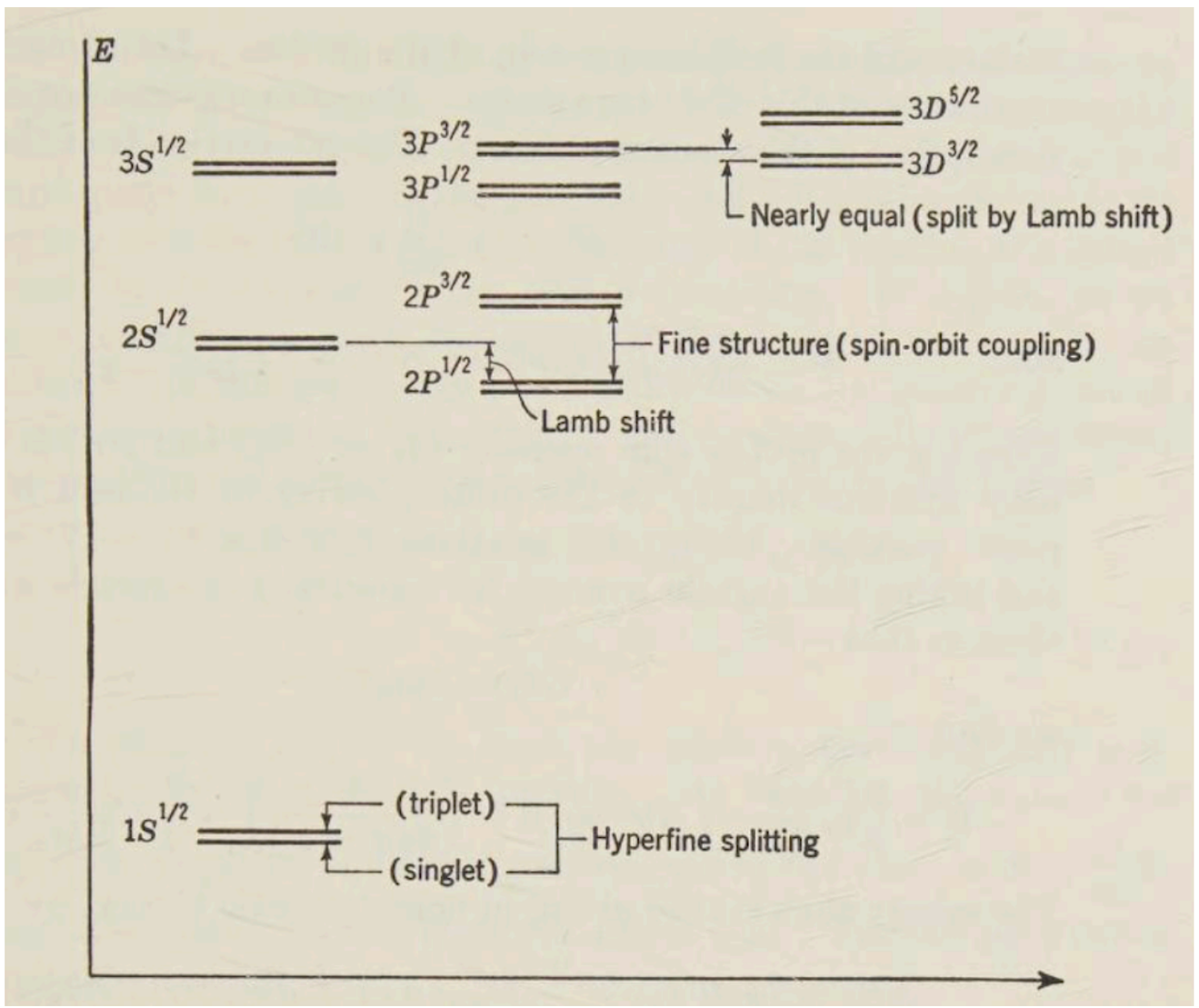

Fig. 4-2 Low-lying energy levels of atomic hydrogen. The diagram is not drawn to scale.

Welton has given a simple qualitative description of the Lamb shift by considering the interaction of an electron, treated nonrelativistically, with the vacuum fluctuations of the electromagnetic field.

Since the dynamics of a normal mode of the electromagnetic field is equivalent to that of a harmonic oscillator, each mode upon quantization acquires a zero-point energy of $\omega/2$.As a result of this quantum

As a result of this quantum effect there are now fluctuating electromagnetic fields even when no external fields are applied.

Although the average field strengths are zero, their mean-square values are nonvanishing, and this leads to a mean-square fluctuation in the electron's position coordinate due to its coupling with the field.

It is the amplitude of this jiggling of a bound electron in the hydrogen atom that we estimate.

It implies, as we saw in our discussion of the origin of the Darwin term, (4.7), an additional interaction energy $1/6\langle(\delta r)^2\rangle\nabla^2 V$ from the smearing out of the Coulomb potential $V(r)$ seen by the electron.

To lowest order, the change in the energy level for the electron due to this is then

$$\begin{aligned}\Delta E_n(\text{Lamb}) &= \tfrac{1}{6}\langle(\delta r)^2\rangle\int\psi_n^*\nabla^2 V(r)\psi_n\, d^3r \\ &= \frac{2\pi}{3} Z\alpha\langle(\delta r)^2\rangle|\psi_n(0)|^2\end{aligned} \tag{4.16}$$

To estimate $\langle(\delta r)^2\rangle$, we treat the electron classically and non-relativistically as a charged particle.

Its equation of motion for oscillation about its equilibrium coordinate in the atom is $\delta\ddot{\mathbf{r}} = \frac{e}{m}\mathbf{E}$, where $\mathbf{E}$ is the fluctuating electromagnetic field.

For the ωth Fourier amplitude we have

$$\delta r_\omega = -\frac{eE_\omega}{m\omega^2}$$

and hence for its mean-square amplitude

$$\langle (\delta r_\omega)^2 \rangle = \frac{e^2 \langle (E_\omega)^2 \rangle}{m^2 \omega^4}$$

and

$$\langle (\delta r)^2 \rangle = \frac{e^2}{m^2} \int \frac{d\omega}{\omega^4} \langle (E_\omega)^2 \rangle$$

To calculate the mean-square field strength, we consider the total vacuum field energy

$$\frac{1}{2} \int d^3x \, (E^2 + B^2) = \sum_{\lambda=1}^{2} \sum_{k} \frac{1}{2} \omega$$

where the two values of λ refer to the two states of transverse polarization and the sum extends over all modes in a large box of volume

$$L^3 = \int d^3x \qquad \sum_{k} \rightarrow \frac{L^3}{(2\pi)^3} \int d^3k$$

Since $\int d^3x\, E^2 = \int d^3x\, B^2$ and $\omega = |\mathbf{k}|$ for free electromagnetic waves, the mean-square field strength in vacuo is

$$\langle E^2 \rangle = \frac{1}{L^3} \int E^2 \, d^3x = 2 \frac{1}{(2\pi)^3} \int d^3k \, \frac{\omega}{2} = \frac{1}{2\pi^2} \int \omega^3 \, d\omega = \int d\omega \, \langle E_\omega^2 \rangle$$

Inserting in (4.17), we find

$$\langle (\delta r^2) \rangle = \frac{e^2}{2\pi^2 m^2} \int \frac{d\omega}{\omega} \tag{4.18}$$

where the frequency integral extends from 0 to ∞.

Because of the crudity of our approximate treatment of the electron, the integral diverges at both ends.

This is not the case for an accurate relativistic treatment of the electron localized in a hydrogen atom.

Wavelengths larger than the Bohr radius $\sim(Z\alpha m)^{-1}$ will not be effective, since there must be a minimum frequency for the induced oscillations corresponding to this typical atomic size; therefore, $\omega_{min} \sim mZ\alpha$.

There is also a high-frequency cutoff at distances ~ the electron Compton wavelength 1/m coming from the relativistic structure of the electron.

This structure corresponding to the zitterbewegung amplitude suggests that frequencies higher than $\omega_{max} \sim m$ will not be effective in jiggling the electron.

Hence we approximate $\int d\omega/\omega \sim \ln(1/Z\alpha)$ and find for the mean-square amplitude of the oscillations in the vacuum field, by (4.18),

$$\langle (\delta r)^2 \rangle = \left(\frac{2\alpha}{\pi} \ln \frac{1}{Z\alpha}\right)\left(\frac{1}{m}\right)^2 \tag{4.19}$$

The resulting energy shift is by (4.16)

$$\begin{aligned}\Delta E_n &= \frac{4Z\alpha^2}{3}\left(\frac{1}{m}\right)^2 \ln \frac{1}{Z\alpha}\,|\psi_n(0)|^2 \\ &= \left[\frac{8}{3\pi}\frac{Z^4\alpha^3}{n^3}\left(\ln \frac{1}{Z\alpha}\right)\right](\tfrac{1}{2}\alpha^2 m)\,\delta_{l0} \\ &\cong 1{,}000 \text{ mc/sec for } n = 2,\ Z = 1,\ l = 0\end{aligned}$$

This accounts for most of the measured shift of the $2S_{1/2}$ level in the hydrogen atom; for the p and higher l states the shifts are not precisely zero but are much smaller because the wave functions at the origin are zero.

By way of comparison with the ordinary fine structure we see by looking back at the hamiltonian (4.5) that the ratio of the Lamb term to the Darwin term is $(8\alpha/3\pi)[\ln(1/Z\alpha)]$ corresponding to the ratio of the mean-square fluctuation amplitude (4.19) to the zitterbewegung structure $\simeq(1/m)^2$.

5 Hole Theory

5.1 The Problem of Negative-energy Solutions

The negative-energy solutions of the Dirac equation have been touched on in some of our earlier discussions, and their presence in the construction of a localized packet, for example, has been computed.

However, we have so far managed to avoid coming to grips with the problems of interpreting them and of understanding their implications.

Let us now face up to these questions.

By their very existence the negative-energy solutions require a massive reinterpretation of the Dirac theory in order to prevent atomic electrons from making radiative transitions into negative-energy states and cascading down to oblivion.

This is no problem if we completely neglect interaction of the electrons with the radiation field.

We may then calculate stationary solutions as in the preceding chapter and find energy eigenvalues and transition amplitudes which agree in general very well with experiment.

However, the problem of keeping the electron from tumbling into a negative-energy state exists in principle, as well as in practice, if we wish to calculate atomic properties to such great accuracy as requires inclusion of the radiation interaction.

The transition rate for an electron in the ground state of a hydrogen atom to fall into a negative-energy state may be calculated by applying semiclassical radiation theory and using the wave functions found in Chap. 4.

The rate for the electron to make a transition into the energy interval $-mc^2$ to $-2mc^2$ is

$$\sim \frac{2\alpha^6}{\pi}\frac{mc^2}{\hbar} \simeq 10^8 \text{ sec}^{-1}$$

and it blows up if all the negative-energy states are included.

This is clearly nonsense!

We must find some treatment of the negative-energy states other than that suggested by the one-particle Schrödinger theory if the Dirac equation is to survive.

Dirac did this for us in 1930.

He formulated the "hole theory," which resolves the dilemma posed by the negative-energy solutions simply by filling up the negative-energy levels with electrons, in accord with the Pauli exclusion principle.

The vacuum state is then one with all negative-energy electron levels filled and all positive-energy levels empty.

The stability of the hydrogen-atom ground state, for example, is now assured, since no more electrons can be accommodated in the negative-energy sea by the Pauli principle.

There are many consequences of this new assumption of a filled sea of negative-energy electrons.

It is possible for a negative-energy electron to absorb radiation and be excited into a positive-energy state, as shown schematically in Fig. 5.1.

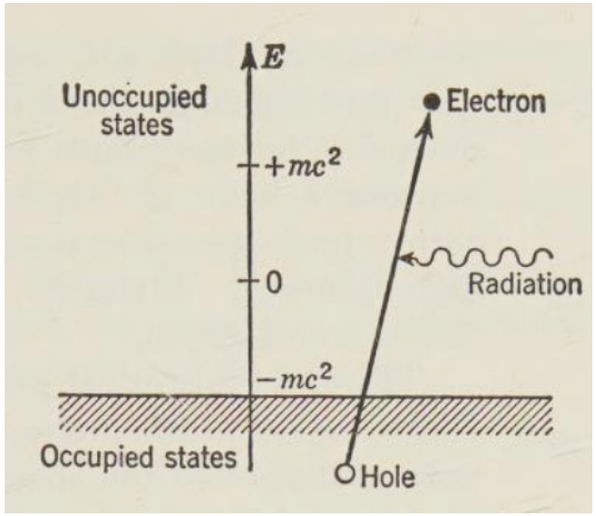

Fig. 5-1 Pair production: a negative-energy electron is excited to a positive-energy state by radiation.

If this occurs, we observe an electron of charge - | e | and energy +*E* and in addition a hole in the negative-energy sea.

The hole registers the absence of an electron of charge — | e | and energy -*E* and would be interpreted by an observer relative to the vacuum as the presence of a particle of charge + | e | and energy +*E*; that is, the positron.

This is the basis of the hole-theory interpretation of pair production.

Correspondingly, a hole in the negative-energy sea, or a positron, is a trap for a positive-energy electron and leads to electron-position pair annihilation with emission of radiation, as shown in Fig. 5.2.

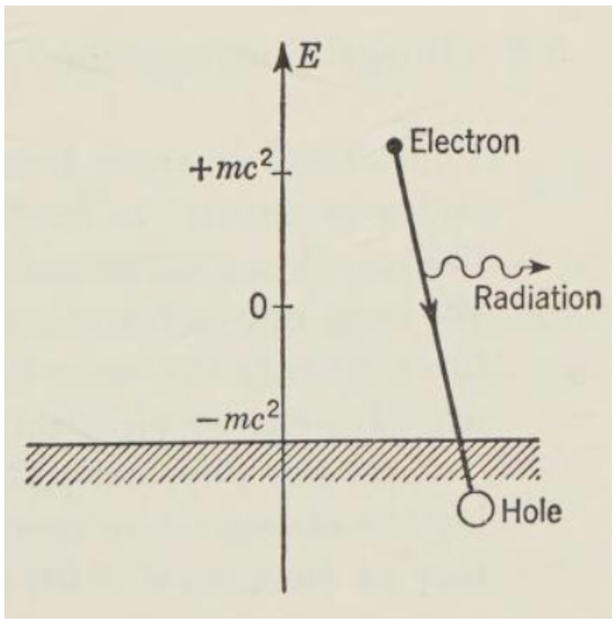

Fig. 5-2 Pair annihilation: a positive-energy electron falls into a negative-energy hole emitting radiation.

We recognize that with the hole theory we transit to a many-particle theory describing particles of both signs of charge.

No longer does the wave function have the simple probability interpretation of the one-particle theory, since it must now also record the production or annihilation of electron-positron pairs.

Recall, however, that the Klein-Gordon equation was discarded and the development of the Dirac equation was motivated by the desire to establish a one-particle theory.

Therefore, we may ask, why not abandon the Dirac equation too?

We are reluctant to discard it for the simple reason that by now we have uncovered an impressive body of “truth" in the Dirac equation - it predicts the correct hydrogen-atom energy spectrum and g value of the electron to high accuracy.

Moreover, positrons as first predicted by the theory have been observed.

Thus the historical path of reasoning mapped out originally by Dirac has led to the desired equation for an electron, though we have now reinterpreted the theory and thereby renounced the motivation that started the development.

The history of physics has numerous other examples of this pattern of progress.

Therefore, we shall retain the Dirac equation and the hole-theory interpretation and reject instead the one-particle probability interpretation which we originally set out to achieve.

We note here that it should also be possible to return to the second-order Klein-Gordon equation and rescue it by a suitable reinterpretation of the wave function there too.

The advantage of the Dirac over the Klein-Gordon equation is that it correctly describes electrons of spin ½ with g = 2.

The Klein-Gordon equation applies for spinless particles such as pions, as will be discussed in Chap. 9.

For both equations we have the invariant, quadratic energy-momentum relation for free particles $p_\mu p^\mu = m^2$.

In both cases we must reinterpret the negative-energy solutions in order to secure stable ground states, and this leads unavoidably to the existence of antiparticles as well as particles.

The particles are described by positive-energy solutions—for the Dirac equation, electrons of mass m and charge -| e |; the antiparticles are described by the reinterpreted negative-energy solutions and, in the present instance, are positrons of mass m and charge + | e |.

5.2 Charge Conjugation

There thus emerges from the hole theory a fundamental new symmetry in nature: to each particle there is an antiparticle and, in particular, the existence of electrons implies the existence of positrons.

We seek now a formal expression of this symmetry which we use to form directly the wave function of a positron from that of the missing negative-energy electron to which it corresponds.

By our physical picture a hole in the negative-energy sea recording the *absence* of an energy $-E(E>0)$, and the absence of a charge e (for an electron $e < 0$), is equivalent to the *presence* of a positron of positive energy $+E$ and charge $-e$.

We thus have a one-to-one correspondence between the negative-energy solutions of the Dirac equation

$$(i\nabla - e\not{A} - m)\psi = 0 \tag{5.1}$$

and the positron eigenfunctions.

Since by their interpretation positrons appear as positively charged electrons, the positron wave function ψ_e will be a positive-energy solution of the equation

$$(i\nabla + e\not{A} - m)\psi_c = 0 \tag{5.2}$$

Conversely and with historical hindsight we could equally well start with the Dirac equation (5.2) for positrons.

Nowhere in our considerations has the sign of the charge e played an essential role.

Electrons will then emerge from the hole theory reinterpretation as the absence of the negative-energy solutions of (5.2).

We have thereby a one-to-one correspondence between solutions of (5.1) and (5.2) for both signs of charge and are led to construct an operator transforming the two equations into each other.

First we observe that it is necessary to change the relative sign between the two terms iT and A in transforming from (5.1) to (5.2).

We accomplish this most readily simply by taking the complex conjugate: $i\partial/\partial x^\mu = -(i\partial/\partial x^\mu)^*$ and $A_\mu = +A_\mu^*$.

Upon doing this, we find that (5.1) becomes

$$\left[\left(i\frac{\partial}{\partial x^\mu}+eA_\mu\right)\gamma^{\mu^*}+m\right]\psi^*=0 \tag{5.3}$$

If we can now find a nonsingular matrix, denoted $C\gamma^0$, with the algebra

$$(C\gamma^0)\gamma^{\mu^*}(C\gamma^0)^{-1}=-\gamma^\mu$$

we shall have the desired form

$$(i\nabla\!\!\!/+eA\!\!\!/-m)(C\gamma^0\psi^*)=0$$

with

$$C\gamma^0\psi^*=C\bar{\psi}^T=\psi_c \tag{5.5}$$

the positron wave function.

That there exists such a matrix C may be verified by explicit construction.

Let us exhibit it in our representation of (2.6), according to which $\gamma^0\gamma^{\mu^*}\gamma^0=\gamma^{\mu T}$ so that (5.4) becomes $C\gamma^{\mu T}C^{-1}=-\gamma^\mu$, or

$$C^{-1}\gamma^\mu C=-\gamma^{\mu T}$$

In this representation C must commute with γ^1 and γ^3 and anticommute with γ^0 and γ^2, and a suitable choice is

$$C = i\gamma^2\gamma^0 = -C^{-1} = -C^\dagger = -C^T \tag{5.6}$$

It suffices to be able to construct a matrix C in any given representation; the unitary transformation to any other one when applied to this C will give a matrix appropriate to the new representation.

We note also that there is a phase arbitrariness in our definition of C in (5.6); the similar circumstance for the parity transformation was discussed earlier. In the present considerations the phase of a wave function is of no physical interest and we do not pursue this question.

Let us examine in detail what the transformation $\psi_c = C\bar{\psi}^T = i\gamma^2\psi^*$ does to a negative-energy free-particle eigenfunction.

For a negative-energy electron at rest with spin down we have the wave function

$$\psi^4 = \frac{1}{(2\pi)^{3/2}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} e^{+imt}$$

The corresponding positron solution is then

$$\begin{aligned} i\gamma^2\psi^{4*} &= i \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \frac{1}{(2\pi)^{3/2}} e^{-imt} \\ &= \frac{1}{(2\pi)^{3/2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{-imt} = \psi^1 \end{aligned} \tag{5.7}$$

That is, the absence of a spin-down negative-energy electron at rest is equivalent to the presence of a spin-up positive-energy positron at rest.

In the field-free case there is no difference between an electron and positron, and we see by (5.7) that the transformation (5.5) has formed just another electron solution.

Applying the same transformation to an arbitrary spin-momentum eigenstate, we find, using $[C,\gamma_5] = 0$ and $\gamma_5^T = \gamma_5 = \gamma_5^*$,

$$\begin{aligned}
\psi_c &= C\bar{\psi}^T = C\gamma_0\psi^* = C\gamma_0\left(\frac{\epsilon\not{p} + m}{2m}\right)^*\left(\frac{1 + \gamma_5\not{s}}{2}\right)^*\psi^* \\
&= C\left(\frac{\epsilon\not{p}^T + m}{2m}\right)\left(\frac{1 - \gamma_5\not{s}^T}{2}\right)\gamma_0\psi^* \\
&= \left(\frac{-\epsilon\not{p} + m}{2m}\right)\left(\frac{1 + \gamma_5\not{s}}{2}\right)\psi_c
\end{aligned} \tag{5.8}$$

Again we see that the operation of (take the complex conjugate) X (matrix multiplication with C γ_5 = i γ^2 has yielded from a negative-energy solution described by four-momentum p_μ and polarization s_μ a positive-energy solution described by the same p_μ and s_μ .

In terms of free-particle spinors, (5.5) reads

$$\begin{aligned}
e^{i\phi(p,s)}v(p,s) &= C\bar{u}^T(p,s) \\
e^{i\phi(p,s)}u(p,s) &= C\bar{v}^T(p,s)
\end{aligned}$$

showing that $v(p,s)$ and $u(p,s)$ are charge-conjugate spinors, within a phase factor $\phi(p,s)$.

Recall that the solutions were constructed such that

$$p_0 = +\sqrt{p^2 + m^2} = E > 0$$

Also notice that s does not change sign under charge conjugation but the spin does reverse as we saw in (5.7).

As discussed in Sec. 3.2, this difference lies in the fact that the spin-projection operator has the form $(1 + \gamma_0 \sigma \cdot \mathbf{s}/2)$ in the rest system where $s^\mu = (0, \mathbf{s})$, and the sign change comes from the yo matrix.

The operator in (5.5) explicitly constructs the wave function of a positron.

We may develop from it an invariance operation for the Dirac equation by defining the additional operator which changes the sign of the electromagnetic field.

Then the sequence of instructions (1) take complex conjugate, (2) multiply by C γ_0, and (3) replace all A_μ by $-A_\mu$ is a formal symmetry operation of the Dirac theory.

It transforms Eq. (5.1) [(5.2)] for the electron [positron] into the same equation for the positron [electron] and is called the charge conjugation transformation, denoted by $\mathcal{C}$.

The physical content of the transformation of charge conjugation is that for each physically realizable state containing an electron in a potential $A_\mu(x)$ there corresponds a physically realizable state of a positron in the potential $-A_\mu(x)$.

Thus $\mathcal{C}$ changes spin-up electrons of positive energy to spin-up positrons of positive energy by transforming a positive-energy solution of (5.1) to a negative-energy solution of the same equation, that is, to a positron according to the hole theory.

That the dynamics of a positron in a field $-A_\mu$, is exactly the same as that of an electron in a field $+A_\mu$ is not at all surprising to us from classical considerations.

The surprising and new result to which we have been led by the hole theory is that if there exist electrons of mass *m* and charge e, there *necessarily must* also exist positrons of the same mass *m* but of opposite charge -e.

It is indeed one of the strongest votes of confidence in at least the partial validity of relativistic quantum theory that electrons of both signs of charge and of the same mass are observed in nature.

5.3 Vacuum Polarization

The hole theory, while removing the negative-energy difficulty, leads to new fundamental barriers to be surmounted and new physical predictions to be verified by experiment.

For example, consider the influence of the vacuum on the definition of the charge of the electron and upon the interaction between two charges.

A positive-energy electron electrostatically repels the electrons in the negative-energy sea.

It thereby polarizes the vacuum in its vicinity, and the charge density of the electron, $\rho_0(\mathbf{r})$, plus polarized vacuum, $\rho_p(\mathbf{r})$, measured relative to the vacuum, is schematically shown in Fig. 5.3.

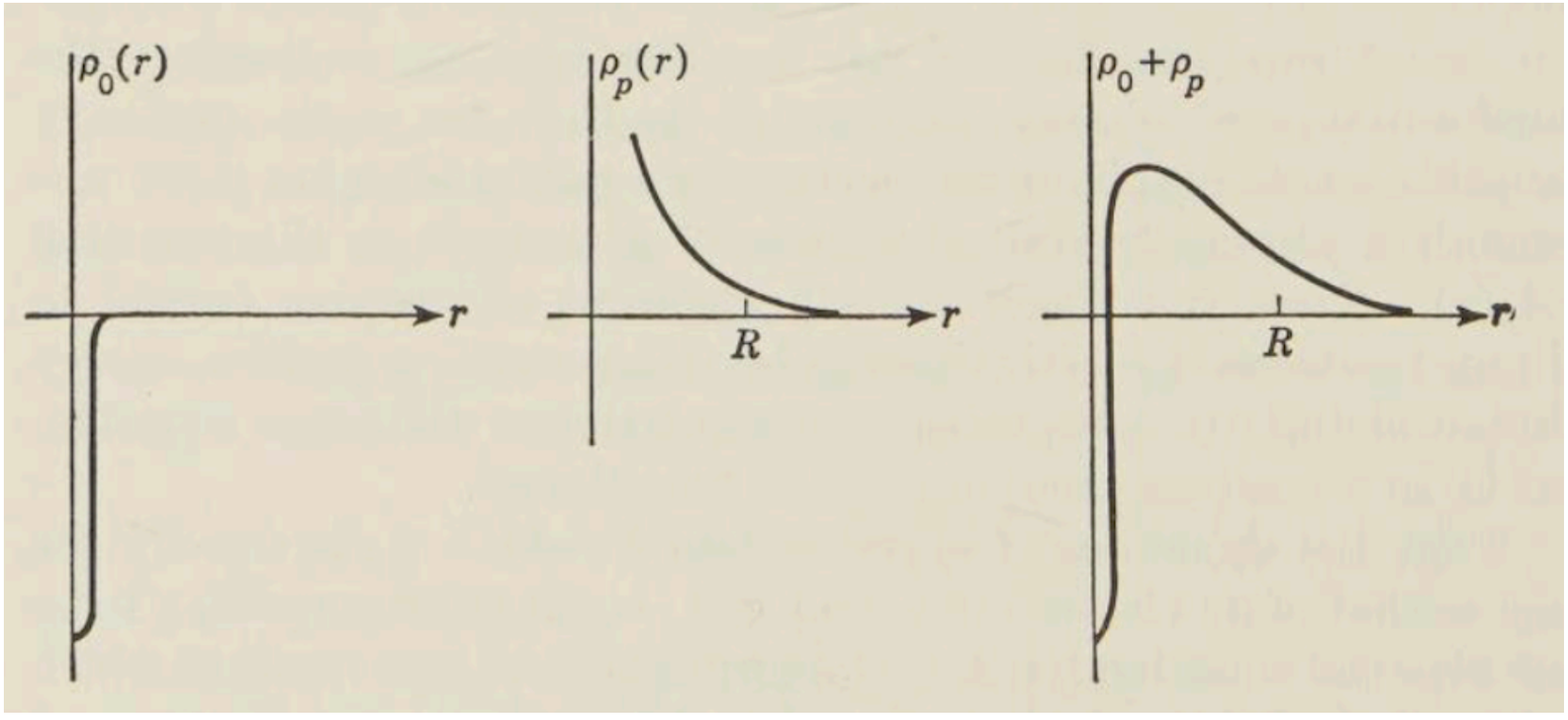

Fig. 5-3 Effect of vacuum polarization on the electron's charge density. ρ_0 is the charge density of the "bare" electron and ρ_p that of the induced polarization "cloud" of virtual electron-positron pairs.

The charge of the electron as seen by a macroscopic applied field, or by a test charge at a large distance, is $\int d^3r\,[\rho_0(\mathbf{r})+\rho_p(\mathbf{r})] = e$, the "physical" charge.

However, for a test charge probing at distances $r_0 < R$, the apparent charge is more negative until, as $r_0 \to 0$, the charge becomes $\int d^3r\,\rho_0(\mathbf{r}) = e_0$, the "bare" charge, with $|\,e_0\,| > |\,e\,|$.

This phenomenon is observed in the hydrogen-atom spectrum.

The electronic s levels are lowered relative to those with angular momentum $l \neq 0$, since the $l = 0$ wave functions bring electrons close to the protons.

This effect of vacuum polarization, calculated in Chap. 8, reduces the Lamb shift slightly.

We also take up there the question of how to connect the "bare" charge of an isolated electron with its observed value at large distances.

Another question arises from the hole theory: What is the meaning of the infinite negative total charge in the vacuum as we have defined it?

For the present we sidestep this question, remarking that there is no preferred direction in which an electric field from such a distribution could point.

Only inhomogeneities in this distribution due to vacuum polarization are observable.

5.4 Time Reversal and Other Symmetries

Let us turn now to the parity and the time-reversal transformations.

These are symmetry operations which are not included in the discussion of proper Lorentz invariance of the theory.

The additional symmetry of electromagnetic gauge invariance is evident from the form of the coupling, p_μ - eA_μ, as remarked in Chap. 1.

It is verified in just the same way as in the Schrödinger theory.

Recall that the parity, or space reflection, transformation was found in Sec. 2.3 to be expressed by

$$P\psi(\mathbf{x},t) = \psi'(\mathbf{x}',t) = e^{i\varphi}\gamma^0\psi(\mathbf{x},t) \qquad \text{for } \mathbf{x}' = -\mathbf{x} \tag{5.9}$$

$\psi'(\mathbf{x}',t)$ is readily interpreted as the space-reflected solution.

For plane-wave solutions the parity transformation (5.9) inverts the momenta and leaves the spins unchanged as we classically expect.

This transformation on the wave function coupled with the familiar one for the vector potentials, expressing their scalar and vector nature

$$\begin{aligned} P\Phi(\mathbf{x},t) &= \Phi'(\mathbf{x}',t) = \Phi(\mathbf{x},t) \\ P\mathbf{A}(\mathbf{x},t) &= \mathbf{A}'(\mathbf{x}',t) = -\mathbf{A}(\mathbf{x},t) \qquad \text{for } \mathbf{x}' = -\mathbf{x} \end{aligned} \tag{5.10}$$

leaves the Dirac equation and all physical observables unchanged.

The physical content of the parity invariance of the Dirac theory may be expressed simply in terms of a set of observations on a state described by a wave function $\psi(\mathbf{x},t)$.

We record these observations on motion-picture film, aiming our camera at a plane mirror forming an image of the experimental setup.

We say that the dynamics underlying our observations is invariant under parity if the movie we make of the mirror image describes a sequence of physically realizable observations, that is, if we cannot tell from the sequence of events observed in the film whether we are looking at a mirror image or not.

For this purpose a mirror image is all that need he considered, although it is not identical with spatial reflection.

A mirror inverts only the coordinates normal to its plane; this must be followed by a rotation through π about the normal for the parity transformation.

Such a rotation is already included, however, in our discussion of proper Lorentz invariance.

Turning next to time-reversal invariance, its physical content may be illustrated again in terms of the motion-picture film which we use to record a set of observations on a state described by $\psi(x)$.

Let us now run the movie backward.

We say that the dynamics underlying the set of observations is invariant under time reversal if the backward-run movie describes a set of physically realizable observations.

This invariance will be guaranteed if we may change t to t' = -t and carry out a transformation which reproduces the form of the Dirac equation with the same rules for its interpretation.

The transformed wave function will describe the original electron running backward in time and will be physically realizable, since it satisfies the Dirac equation.

To construct the desired time-reversal transformation, we write the Dirac equation in hamiltonian form

$$i\frac{\partial\psi(\mathbf{x},t)}{\partial t} = H\psi = [\boldsymbol{\alpha}\cdot(-i\boldsymbol{\nabla} - e\mathbf{A}) + \beta m + e\Phi]\psi(\mathbf{x},t) \tag{5.11}$$

and define the transformation $\mathcal{T}$ such that if t' = -t, $\psi'(t') = \mathcal{T}\psi(t)$.

Then (5.11) becomes

$$\frac{\partial}{\partial t'}(\mathcal{T}i\mathcal{T}^{-1})\psi'(t') = -\mathcal{T}H\mathcal{T}^{-1}\psi'(t') \tag{5.12}$$

and time-reversal invariance requires that either $\mathcal{T}H(t)\mathcal{T}^{-1} = -H(t')$, or

$$\mathcal{T}i\mathcal{T}^{-1} = -i$$

To investigate the behavior of H under $\mathcal{T}$, we must specify the behavior of the electromagnetic potentials A_μ when we let t' = -t.

Since **A** is generated by currents which reverse sign when the sense of time is reversed, we require

Similarly,

$$\mathbf{A}'(t') = -\mathbf{A}(t)$$

$$\Phi'(t') = +\Phi(t) \qquad (5.13)$$

since it is generated by charges; also $\nabla' = +\nabla$, since $\mathbf{x}' = +\mathbf{x}$.

To restore (5.12) to the original form, it is clear then that the transformations with $\mathcal{T} \cdots \mathcal{T}^{-1}$ must change i to -i; hence, $\mathcal{T}$ may be written in the form (take complex conjugate) X (multiply afterward by a 4 x 4 constant matrix T):

$$\psi'(t') = T\psi^*(t) \qquad (5.14)$$

This gives

$$i\frac{\partial\psi'(t')}{\partial t'} = \{(-T\boldsymbol{\alpha}^*T^{-1})\cdot[(-i\boldsymbol{\nabla}' - e\mathbf{A}'(t'))] + (T\beta^*T^{-1})m + e\Phi'(t')\}\psi'(t')$$

In our usual representation (1.17), this means T must commute with α_2 and β and anticommute with α_1 and α_3; thus

$$T = -i\alpha_1\alpha_3 = +i\gamma^1\gamma^3 \tag{5.15}$$

is satisfactory; the phase factor is again arbitrary.

To show that the transformation $\mathcal{T}$ corresponds to what we mean classically by time reversal, we apply (5.14) and (5.15) to a plane-wave solution for a free particle of positive energy:

$$\begin{aligned}
\mathcal{T}\left(\frac{\not{p}+m}{2m}\right)\left(\frac{1+\gamma_5\not{s}}{2}\right)\psi(\mathbf{x},t) \\
&= T\left(\frac{\not{p}^*+m}{2m}\right)T^{-1}T\left(\frac{1+\gamma_5\not{s}^*}{2}\right)T^{-1}\psi'(\mathbf{x},t') \\
&= \left(\frac{\not{p}'+m}{2m}\right)\left(\frac{1+\gamma_5\not{s}'}{2}\right)\psi'(\mathbf{x},t')
\end{aligned} \tag{5.16}$$

where $p' = (p_0, -\mathbf{p})$ and $s' = (s_0, -\mathbf{s})$ project a free-particle solution with reversed direction of space momentum $\mathbf{p}$ and spin $\mathbf{s}$.

This operation, known as the Wigner time reversal, was first introduced in 1932.

Since the space and time coordinate inversions P and $\mathcal{T}$ are invariance operations of the theory, we may just as well include them, if we wish, in constructing the positron wave function.

Combining (5.9), (5.14), and (5.15) with (5.5), we find a simple correspondence between a positron wave function

$$\psi_{PCT}(x') \equiv PC\gamma_0(\mathfrak{I}\psi(x))^* = P\mathcal{C}\mathfrak{I}\psi(x) = ie^{i\varphi}\gamma_5\psi(x) \qquad \text{with } x'_\mu = -x_\mu \tag{5.17}$$

and an electron wave function multiplied by $ie^{i\varphi}\gamma_5$ and moving backward in space-time.

For a free-particle spin-momentum eigenstate $\psi(x)$ characterized by (p^μ, s^μ) and $\epsilon = -1$, we see that

$$\begin{aligned}\psi_{PCT}(x') &= ie^{i\varphi}\gamma_5\left(\frac{-\not{p}+m}{2m}\right)\left(\frac{1+\gamma_5\not{s}}{2}\right)\psi(x) \\ &= \left(\frac{\not{p}+m}{2m}\right)\left(\frac{1-\gamma_5\not{s}}{2}\right)\psi_{PCT}(x')\end{aligned} \tag{5.18}$$

Equation (5.18) differs from (5.8) only in the direction of spin and tells us, therefore, that we may picture a positron wave function of positive energy as a negative-energy electron wave function multiplied by $ie^{i\varphi}\gamma_5$ and moving backward in space-time.

For an arbitrary solution in the presence of electromagnetic forces we may explicitly verify this interpretation by returning to the negative-energy eigenvalue equation

$$[\boldsymbol{\alpha}\cdot(-i\boldsymbol{\nabla}-e\mathbf{A})+\beta m+e\Phi]\psi=-E\psi \tag{5.19(}$$

and carrying out the transformation (5.17).

Evidently, by (5.10) and (5.13), $A'_\mu(x') = + A_\mu(x)$ under space-time coordinate inversion, $x'_\mu = -x_\mu$; then (5.19) takes the desired form

$$[\boldsymbol{\alpha}\cdot(-i\boldsymbol{\nabla}'+e\mathbf{A}'(x'))+\beta m-e\Phi'(x')]\psi_{PCT}(x')=+E\psi_{PCT}(x')$$

The interpretation of positrons as negative-energy electrons running backward in space-time forms the basis of the Stückelberg-Feynman form of positron theory.

We shall use it often in the following chapters in developing scattering theory, and we shall find that it offers great advantages there.

In conclusion, we must notice that the structure of the interaction of electrons with the electromagnetic field was dictated by experience with both the classical and the nonrelativistic limit of the electrodynamics of electrons.

The existences of the symmetries we have discussed are dependent upon the form of interaction.

For instance, an anomalous moment interaction of the type discussed in the problems for Chap. 4 for protons and neutrons adds a term of the form $\sigma_{\mu\upsilon}F^{\mu\upsilon}\psi$ to the Dirac equation.

Its presence affects none of the above symmetries.

In extending the Dirac theory to other particles of spin ½, such as μ meson or nucleons, and to other familiar interactions, it is very natural to assume that these symmetries of $\mathcal{T}$, $\mathcal{C}$, P are still preserved.

It was the great contribution of Lee and Yang to realize that this is really an assumption to be verified by experiment and to suggest that interactions such as β decay violate the symmetries of P and $\mathcal{C}$.

However, the symmetry under the operation $\mathcal{TCP}$ is guaranteed by the much weaker assumptions of proper Lorentz invariance and the usual connection between spin and statistics.

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6 Propagator Theory

6.1 Introduction

We turn to a general discussion of scattering processes.

Our aim is to be able to calculate transition rates and cross sections with the Dirac theory of electrons and positrons—in principle, exactly; in practice, to low orders of expansion in the interaction parameters.

The possibility of altering the numbers of particles in such processes as electron-positron pair production or annihilation carries us beyond the scope of the discussions in nonrelativistic theory.

However, we shall delay as long as possible the enormous task of developing the formalism of quantized field theory in order to accommodate this production and annihilation of particles.

To this end we follow Feynman' in developing the propagator approach.

The scattering process is described in terms of integral equations.

The boundary conditions for their solutions incorporate the Stückelberg-Feynman physical interpretation of positrons as negative-energy electrons running backward in time.

From this formulation a working theory with unambiguous rules of calculation for all physical processes emerges.

To begin, we review the propagator approach to the nonrelativistic Schrödinger equation.

6.2 The Nonrelativistic Propagator

In scattering problems our attention is focused upon wave solutions which develop in time from initial conditions imposed in the remote past rather than on stationary energy eigenfunctions, that is, standing waves.

Characteristically, given a wave packet which in the remote past represents a particle approaching a potential, one asks what the wave will look like in the remote future.

We turn to Huygens' principle for a convenient way of viewing this process.

If the wave function $\psi(\mathbf{x},t)$ is known at one particular time t, it may be found at any later time t' by considering at time t each point of space $\mathbf{x}$ as a source of spherical waves which propagate outward from $\mathbf{x}$.

The strength of the wave amplitude arriving at point $\mathbf{x}'$ at time t' from the point $\mathbf{x}$ will be proportional to the original wave amplitude $\psi(\mathbf{x}, t)$. If we denote the constant of proportionality by $iG(\mathbf{x}',t',\mathbf{x},t)$, the total wave arriving at the point $\mathbf{x}'$ at time t′ will, by Huygens' principle, be

$$\psi(\mathbf{x}',t') = i\int d^3x\, G(\mathbf{x}',t';\mathbf{x},t)\psi(\mathbf{x},t) \qquad t' > t \tag{6.1}$$

$G(\mathbf{x}',t',\mathbf{x},t)$ is known as the Green's function or propagator, and it describes to us according to Huygens' principle the influence upon $\psi(\mathbf{x}',t')$ of the magnitude of ψ at $\mathbf{x}$ at time t.

Knowledge of G enables us to construct the physical state which develops in time from any given initial state, and thus is equivalent to a complete solution of the Schrödinger equation.

We must still give a complete formal definition of G.

So far we have only claimed its existence on the basis of physical arguments.

Let us pursue these arguments further in order to develop a better understanding of the propagator approach.

Consider first a free-wave solution.

The motion of a free particle is completely known, and it should not come as a surprise that the corresponding free-particle Green's function G_0 can be constructed explicitly.

If we now introduce a potential, G_0 will be modified.

Let $V(\mathbf{x}_1,t_1)$ represent an interaction potential which is "turned on" for a very brief interval of time Δt_1 about t_1.

For times earlier than t_1, the wave function will be that of a free wave φ, and the corresponding propagator will be G_0.

However, $V(\mathbf{x}_1,t_1)$ acts as a source of new waves according to the Schrödinger equation

$$\left(i\frac{\partial}{\partial t_1} - H_0\right)\psi(\mathbf{x}_1,t_1) = V(\mathbf{x}_1,t_1)\psi(\mathbf{x}_1,t_1) \tag{6.2}$$

The right-hand side differs from zero in the interval Δt_1.

It produces an additional change in ψ during Δt_1 above that taking place in the absence of V.

This additional wave $\Delta\psi(\mathbf{x}_1,t_1)$ is found by integrating (6.2) to first order in Δt_1.

$$\Delta\psi(\mathbf{x}_1,t_1) = -iV(\mathbf{x}_1,t_1)\varphi(\mathbf{x}_1,t_1)\,\Delta t_1 \tag{6.3}$$

This added wave, by Huygens' principle and (6.1), leads at a future time t' to a new contribution to $\psi(\mathbf{x}',t')$, which is

$$\Delta\psi(\mathbf{x}',t') = \int d^3x_1\, G_0(\mathbf{x}',t';\mathbf{x}_1,t_1)V(\mathbf{x}_1,t_1)\varphi(\mathbf{x}_1,t_1)\,\Delta t_1 \tag{6.4}$$

Thus the wave ψ developing from an arbitrary packet φ in the remote past is

$$\begin{aligned}\psi(\mathbf{x}',t') &= \varphi(\mathbf{x}',t') + \int d^3x_1\, G_0(\mathbf{x}',t';\mathbf{x}_1,t_1)V(\mathbf{x}_1,t_1)\varphi(\mathbf{x}_1,t_1)\,\Delta t_1 \\ &= i\int d^3x\,[G_0(\mathbf{x}',t';\mathbf{x},t) \\ &\qquad + \int d^3x_1\,\Delta t_1\, G_0(\mathbf{x}',t';\mathbf{x}_1,t_1)V(\mathbf{x}_1,t_1)G_0(\mathbf{x}_1,t_1;\mathbf{x},t)]\varphi(\mathbf{x},t)\end{aligned} \tag{6.5}$$

Comparing with (6.1), we see that the Green's function here is given by

$$\begin{aligned}G(\mathbf{x}',t';\mathbf{x},t) &= G_0(\mathbf{x}',t';\mathbf{x},t) \\ &\qquad + \int d^3x_1\,\Delta t_1\, G_0(\mathbf{x}',t';\mathbf{x}_1,t_1)V(\mathbf{x}_1,t_1)G_0(\mathbf{x}_1,t_1;\mathbf{x},t)\end{aligned} \tag{6.6}$$

It may be illustrated by the space-time diagram shown in Fig. 6.1.

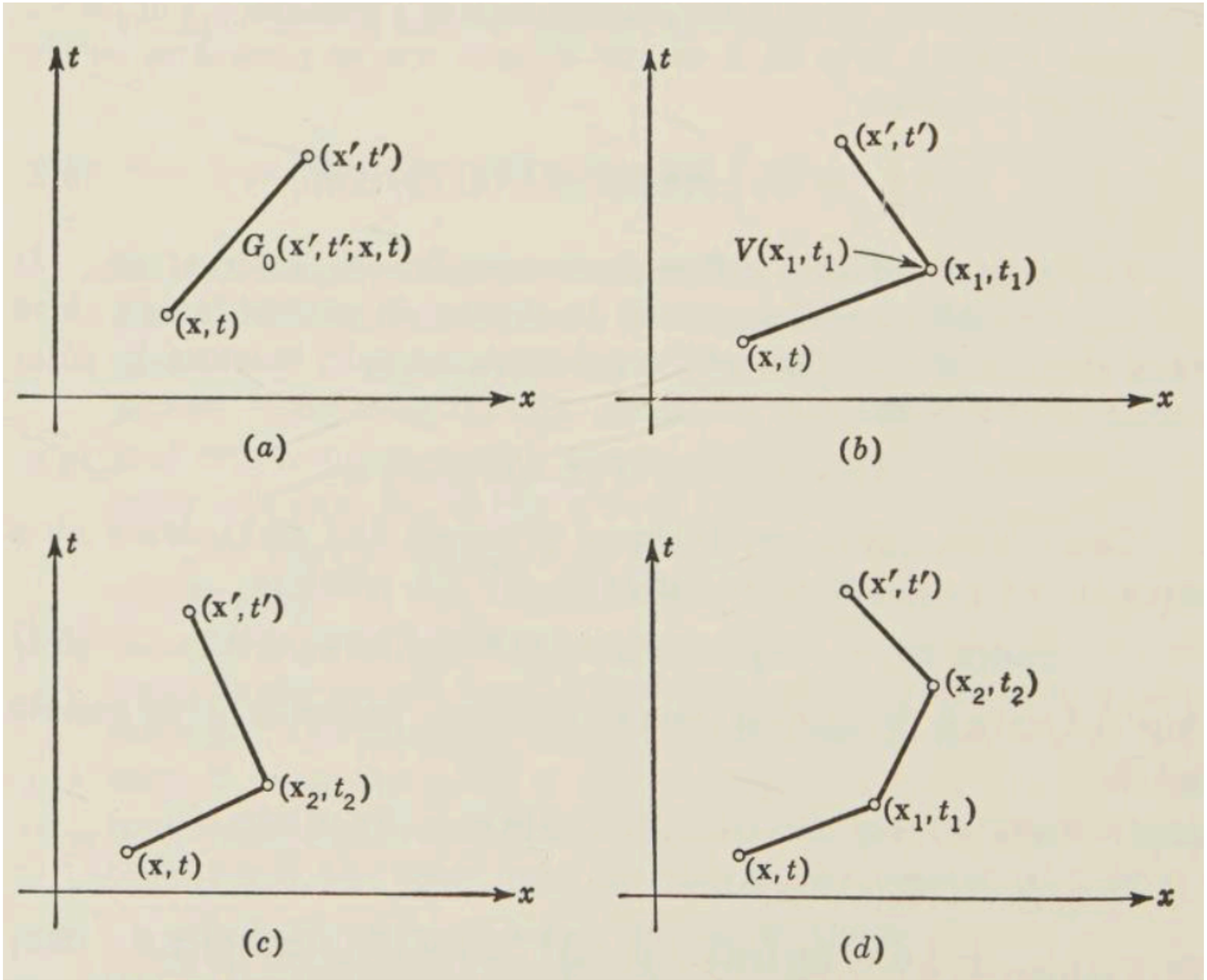

Fig. 6-1 Space-time diagrams for propagation from (**x**,t) to (**x**',t') as (a) a free particle, (b) with one scattering by potential V($\mathbf{x}_1$,t_1) at ($\mathbf{x}_1$,t_1), (c) with single scattering at ($\mathbf{x}_2$,t_2), and (d) with double scattering.

The first term (Fig. 6.la) represents the propagation from (**x**, t) to (**x**', t) as a free particle; Fig. 6.1b represents free propagation from (**x**, t) to ($\mathbf{x}_1$,t_1), a scattering at ($\mathbf{x}_1$,t_1), and free propagation from ($\mathbf{x}_1$,t_1) to (**x**′, t').

If we turn on another potential $V(\mathbf{x}_2,t_2)$ for an interval Δt_2 at time $t_2 > t_1$, the additional contribution to $\psi(\mathbf{x}',t')$ for $t' > t_2$ is, in analogy to (6.4),

$$\begin{aligned}\Delta\psi(x') &= \int d^3x_2\, G_0(x';2)V(2)\psi(2)\,\Delta t_2 \\ &= i\int d^3x\, d^3x_2\,\Delta t_2\, G_0(x';2)V(2) \\ &\qquad \times [G_0(2;x) + \int d^3x_1\,\Delta t_1\, G_0(2;1)V(1)G_0(1;x)]\varphi(x)\end{aligned} \tag{6.7}$$

in an abbreviated notation whose meaning should be clear.

The first term is illustrated in the diagram (Fig. 6.1c) and represents single scattering at t_2; the second term is a double scattering correction illustrated in Fig. 6.1d.

The total wave arriving at (**x**',t′) is then built up by inserting (6.5) for $\psi(2)$ in the right-hand side of (6.7) and adding the resulting $\Delta\psi$ to (6.5):

$$\begin{aligned}\psi(x') = \varphi(x') &+ \int d^3x_1\,\Delta t_1\, G_0(x';1)V(1)\varphi(1) \\ &+ \int d^3x_2\,\Delta t_2\, G_0(x';2)V(2)\varphi(2) \\ &+ \int d^3x_1\,\Delta t_1\, d^3x_2\,\Delta t_2\, G_0(x';2)V(2)G_0(2;1)V(1)\varphi(1)\end{aligned} \tag{6.8}$$

Without further ado, if there are *n* such time intervals when the potential *V* is turned on, the wave arriving at (**x**',t′) will be

$$\psi(x') = \varphi(x') + \sum_i \int d^3x_i\, \Delta t_i\, G_0(x';x_i)\, V(x_i)\varphi(x_i)$$
$$+ \sum_{\substack{ij \\ (t_i>t_j)}} \int d^3x_i\, \Delta t_i\, d^3x_j\, \Delta t_j\, G_0(x';x_i)\, V(x_i) G_0(x_i;x_j)\, V(x_j)\varphi(x_j)$$
$$+ \sum_{\substack{ijk \\ (t_i>t_j>t_k)}} \int d^3x_i\, \Delta t_i\, d^3x_j\, \Delta t_j\, d^3x_k\, \Delta t_k$$
$$\times G_0(x',x_i)\, V(x_i) G_0(x_i;x_j)\, V(x_j) G_0(x_j;x_k)\, V(x_k)\varphi(x_k) + \cdots \tag{6.9}$$

By comparison with (6.5) and (6.6) the corresponding expression for the Green's function G will be

$$G(x';x) = G_0(x';x) + \sum_i \int d^3x_i\, \Delta t_i\, G_0(x';\mathbf{x}_i,t_i)\, V(\mathbf{x}_i,t_i) G_0(\mathbf{x}_i,t_i;x)$$
$$+ \sum_{\substack{ij \\ (t_i>t_j)}} \int d^3x_i\, \Delta t_i\, d^3x_j\, \Delta t_j\, G_0(x';\mathbf{x}_i,t_i)\, V(\mathbf{x}_i,t_i)$$
$$\times G_0(\mathbf{x}_i,t_i;\mathbf{x}_j,t_j)\, V(\mathbf{x}_j,t_j) G_0(\mathbf{x}_j,t_j;x) + \cdots \tag{6.10}$$

We may lift the time-ordering restrictions $t_i > t_j$, etc., if we define $G_0(\mathbf{x}',t';\mathbf{x},t) = 0$ for $t' < t$.

With this boundary condition of propagating waves forward in time only, G_0 is known as the retarded propagator.

Physically this just means that no Huygens wavelets $\Delta\psi$ from the ith iteration (at time t_i) appear until after t_i.

In the limit of a continuous interaction the sums over time intervals may be replaced by integrals over *dt* with the result

$$G(x';x) = G_0(x';x) + \int d^4x_1\, G_0(x';x_1)V(x_1)G_0(x_1;x) + \int d^4x_1\, d^4x_2\, G_0(x';x_1)V(x_1)G_0(x_1;x_2)V(x_2)G_0(x_2;x) + \cdots \tag{6.11}$$

where

$$d^4x = d^3\boldsymbol{x}\, dx_0 = d^3x\, dt$$

This multiple scattering series (6.11) is assumed to converge! and may be summed formally to yield

$$G(x';x) = G_0(x';x) + \int d^4x_1\, G_0(x';x_1)V(x_1)G(x_1;x) \tag{6.12}$$

We notice that not only $G_0(x'; x)$ but also $G(x'; x)$ vanishes for $t' < t$, as demanded by our elementary concept of causality.

Equation (6.11) gives us an iteration procedure for finding *G* in terms of *V* and G_0 and hence for constructing the wave function $\psi(x',t')$ if it is known at an earlier time.

In particular, to solve the scattering problem, we must know the wave in the remote future, given a wave packet $\varphi(\mathbf{x}, t)$ representing a particle in the remote past approaching the interaction region.

In order to define properly the scattering problem, there should be no interaction at this initial time, so that φ is a solution of the free-particle equation which incorporates the required initial conditions.

A mathematically convenient way of accomplishing this is to localize the interaction in time by adiabatically turning off $V(\mathbf{x},t)$ as $t \to -\infty$; the exact solution ψ then approaches φ in the remote past and there is no scattered wave.

In the future the wave $\psi(\mathbf{x}', t')$ is given by (6.1)

$$\psi(\mathbf{x}',t') = \lim_{t \to -\infty} i\int d^3x\, G(\mathbf{x}',t';\mathbf{x},t)\varphi(\mathbf{x},t) \tag{6.13}$$

Expressing G in terms of G_0 by (6.12), we see

$$\begin{aligned}\psi(\mathbf{x}',t') = \lim_{t \to -\infty} i\int d^3x\, [&G_0(\mathbf{x}',t';\mathbf{x},t) \\ &+ \int d^4x_1\, G_0(\mathbf{x}',t';1)V(1)G(1;\mathbf{x},t)]\varphi(\mathbf{x},t) \\ = \varphi(\mathbf{x}',t') &+ \int d^4x_1\, G_0(\mathbf{x}',t';\mathbf{x}_1,t_1)V(\mathbf{x}_1,t_1)\psi(\mathbf{x}_1,t_1)\end{aligned} \tag{6.14}$$

We have really not solved anything, since the unknown ψ appears under the integral on the right.

However, we do have a formulation which includes the desired boundary conditions and which affords an immediate approximation procedure if the perturbing potential V is weak.

We are primarily interested in the form of the scattered wave as $t' \to \infty$.

In this limit the particle emerges from the interaction region and again ψ becomes a solution of the free-particle equation.

As before, we adiabatically turn off the interaction as t'$\rightarrow +\infty$ in order to ensure this condition.

All information about the scattered wave may be obtained from the probability amplitudes for the particle to arrive in various final free states φ_f

$$\varphi_f(\mathbf{x}',t') = \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{k}_f\cdot\mathbf{x}'-i\omega_f t'} \tag{6.15}$$

as t' $\rightarrow +\infty$ from a given incident wave φ_i; in particular, we may work with plane waves.

The probability amplitude for a given pair (f, i) is an element of the S, or scattering, matrix and is given by

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$$\begin{aligned} S_{fi} &= \lim_{t'\to\infty} \int \varphi_f^*(\mathbf{x}',t')\psi_i^{(+)}(\mathbf{x}',t')\, d^3x' \\ &= \lim_{t'\to\infty} \int d^3x'\, \varphi_f^*(\mathbf{x}',t')[\varphi_i(\mathbf{x}',t') \\ &\qquad + \int d^4x\, G_0(\mathbf{x}',t';\mathbf{x},t)\, V(\mathbf{x},t)\psi_i^{(+)}(\mathbf{x},t)] \\ &= \delta^3(\mathbf{k}_f - \mathbf{k}_i) + \lim_{t'\to\infty} \int d^3x'\, d^4x\, \varphi_f^*(\mathbf{x}',t')G_0(\mathbf{x}',t';\mathbf{x},t) \\ &\qquad \times V(\mathbf{x},t)\psi_i^{(+)}(\mathbf{x},t) \end{aligned} \tag{6.16}$$

where $\psi_i^{(+)}(\mathbf{x}, t)$ is that solution of the wave equation (6.14) which reduces to a plane wave of momentum k; as t$\rightarrow -\infty$.

By the shorthand t$\longrightarrow \pm\infty$ we mean t $\longrightarrow$ any large finite time for which the particles are not in the interaction region (or alternatively when V is turned off); in particular, t$\longrightarrow \pm\infty$ may mean the times when the particle is produced and detected.

We may expand $\psi^{(+)}$ in a multiple scattering series by iteration of (6.14) and thus express the S matrix in a multiple scattering series, the terms of which correspond to the diagrams of Fig. 6.1.

6.3 Formal Definitions and Properties of Green's Functions

We have unearthed the physical ingredients for solving a scattering problem.

We now build the formal mathematical machinery to manufacture these solutions.

Our goal is to investigate the differential equation which defines G, and in particular to solve for G_0 explicitly, so that the expansion of G we have outlined can be explicitly carried out.

We start with Eq. (6.1), valid for t' > t, and rewrite it in a form valid for all times:

$$\theta(t' - t)\psi(x') = i\int d^3x\, G(x';x)\psi(x) \tag{6.17}$$

θ(t' - t) is the unit step function defined by

$$\theta(t' - t) = \begin{cases} 1 & t' > t \\ 0 & t' < t \end{cases} \tag{6.18}$$

and has the following very useful integral representation:

$$\theta(\tau) = \lim_{\epsilon \to 0} \frac{-1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\omega\, e^{-i\omega\tau}}{\omega + i\epsilon} \tag{6.19}$$

It is evaluated as a contour integral in the complex ω plane as shown in Fig. 6.2.

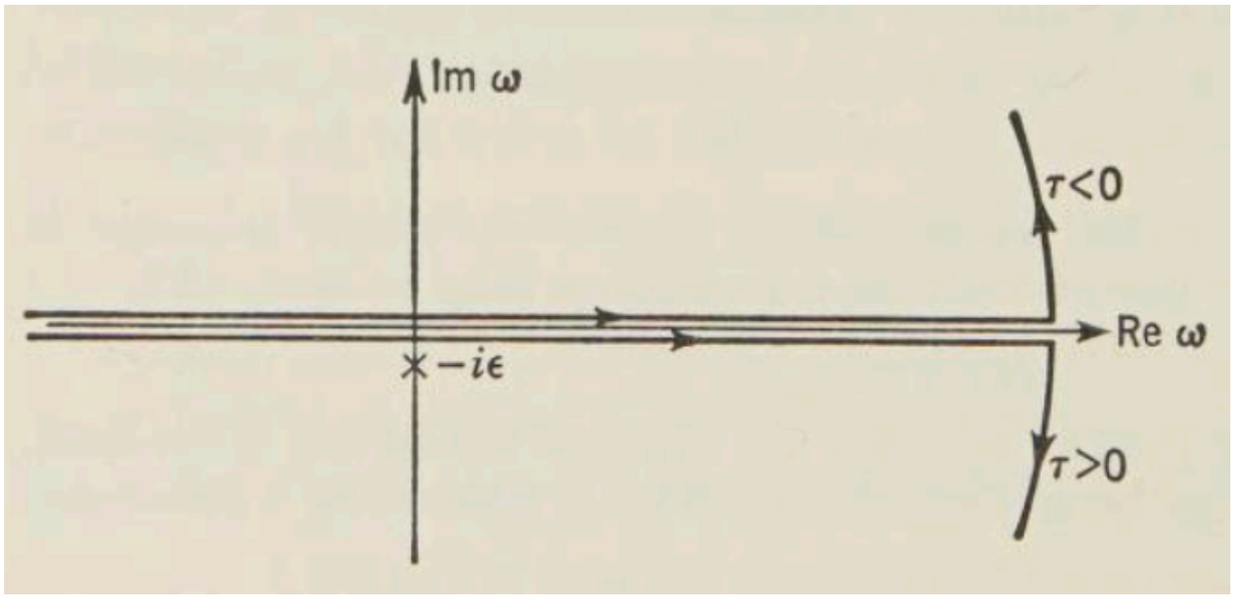


Fig. 6-2 Contour in the complex ω plane for integrating the unit step function $\theta(\tau)$.

For $\tau > 0$ the contour may be closed along an infinite semicircle below the real axis in order to ensure exponential damping of the integrand, and the value of the integral is 1 by Cauchy's theorem.

For $\tau < 0$, the contour is closed above and the integral vanishes because the pole at $-i\epsilon$ now lies outside the contour.

Since $\theta(\tau)$ takes a unit jump at $\tau = 0$, its derivative is a $\hat{o}$ function:

$$\frac{d\theta(\tau)}{d\tau} = \delta(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{i\omega\tau} \tag{6.20}$$

Digression on Complex Integration

Complex Variable — Existence of a Derivative

For a complex function of a complex variable given by $f(z) = u(z) + iv(z)$ where $z = x + iy$ and u, v are real functions, the derivative of $f(z)$ is defined by

$$f'(z) = \frac{df}{dz} = \lim_{\delta z \to 0} \frac{\delta f(z)}{\delta z} = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}$$

provided that the limit is **independent** of the particular approach to the point z in the complex plane.

This is always true provided the following Cauchy-Riemann(C-R) conditions are satisfied:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

If these conditions are true at some point z and in the neighborhood of z, then they are necessary and sufficient conditions for the existence of the derivative of $f(z)$ or we say the function is differentiable in this region of the complex plane.

Analytic Functions

If $f(z)$ is differentiable at $z = z_0$ and in some small region around z_0, then we say that $f(z)$ is analytic at $z = z_0$.

Example:

$$w(z) = z^2 = (x + iy)^2 = x^2 - y^2 + 2xyi$$

$$u(x, y) = x^2 - y^2 \quad , \quad v(x, y) = 2xy$$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

This is true everywhere in the complex plane and thus $w(z)$ is analytic and differentiable everywhere and

$$\frac{dw}{dz} = 2z$$

Another example: Define $z^* = x - iy =$ complex conjugate.

$$w(z) = \frac{1}{z} = \frac{1}{z}\frac{z^*}{z^*} = \frac{x - iy}{x^2 + y^2}$$

$$u(x, y) = \frac{x}{x^2 + y^2} \quad , \quad v(x, y) = -\frac{y}{x^2 + y^2}$$

The Cauchy-Riemann conditions are satisfied everywhere but at $z = 0$.

Thus, the function $1/z$ is analytic everywhere but $z = 0$ and its derivative is

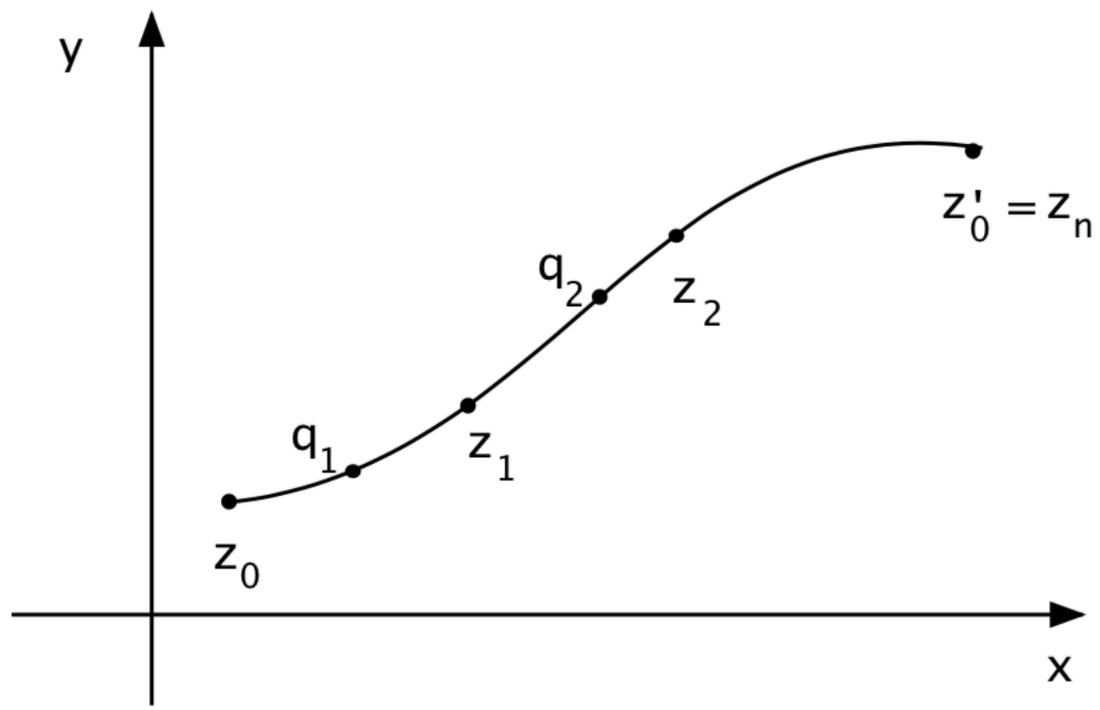
$$\frac{d(1/z)}{dz} = -\frac{1}{z^2}$$

in the analytic region.

Define a Contour Integral

The integral of a complex function over a contour(path) in the complex plane is defined in close analogy with the standard integral of a real function integrated along the real x-axis.

Divide the contour $z_0 \to z_0' = z_n$ into n intervals by picking $n-1$ intermediate points $z_1, z_2, \ldots$. on the contour (see figure)



Now consider the sum

$$S_n = \sum_{j=1}^{n} f(q_j)(z_j - z_{j-1})$$

where q_j is a point on the curve between z_j and z_{j-1}. Now let $n \to \infty$ with

$$|z_j - z_{j-1}| \to 0$$

for all j.

If the $\lim_{n\to\infty} S_n$ exists and is independent of the details of choosing the points z_j and q_j, then

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(q_j)(z_j - z_{j-1}) = \int_{z_0}^{z_0'} f(z)dz$$

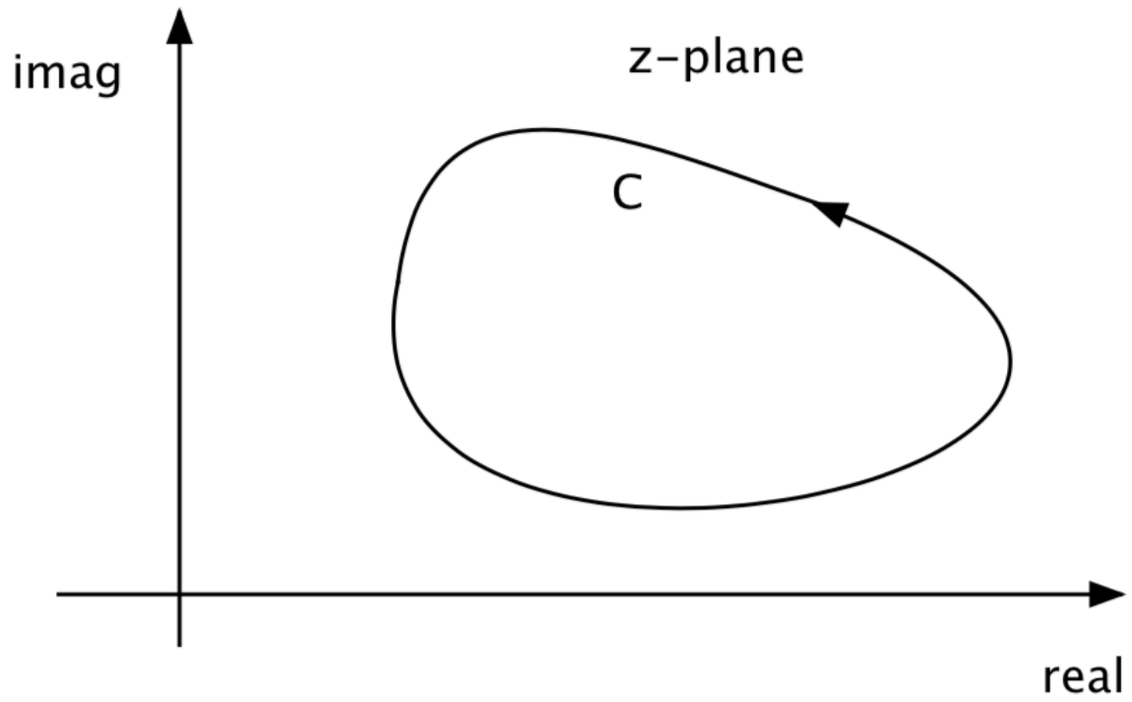
The RHS is called the **contour integral** of $f(z)$ (along the specified contour C from $z = z_0$ to $z = z_0'$.

Cauchy Integral Theorem

Now let $w(z)$ be a complex function which is analytic in some region of the complex plane and consider this integral

$$\oint_C w(z)dz$$

along a closed contour C which lies **entirely** within the analytic region as shown below:



Since $w(z)$ is analytic and obeys the C-R conditions, we have

$$\oint_C w(z)dz = \oint_C (u(z) + iv(z))d(x + iy)$$

$$= \oint_C (udx - vdy) + i \oint_C (vdx + udy)$$

$$= -\iint_R \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dxdy - i\iint_R \left(-\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dxdy = 0$$

where R is are area inside the curve C and we have made use of Green's theorem. $\int_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

So we have the **Cauchy Integral Theorem:**

If $f(z)$ is an analytic function whose derivative $f'(z)$ exists and is continuous at each point within and on the closed contour C, then

$$\oint_C f(z)dz = 0$$

We note that all integrations traverse the contour counterclockwise unless otherwise stated.

Contour Deformation

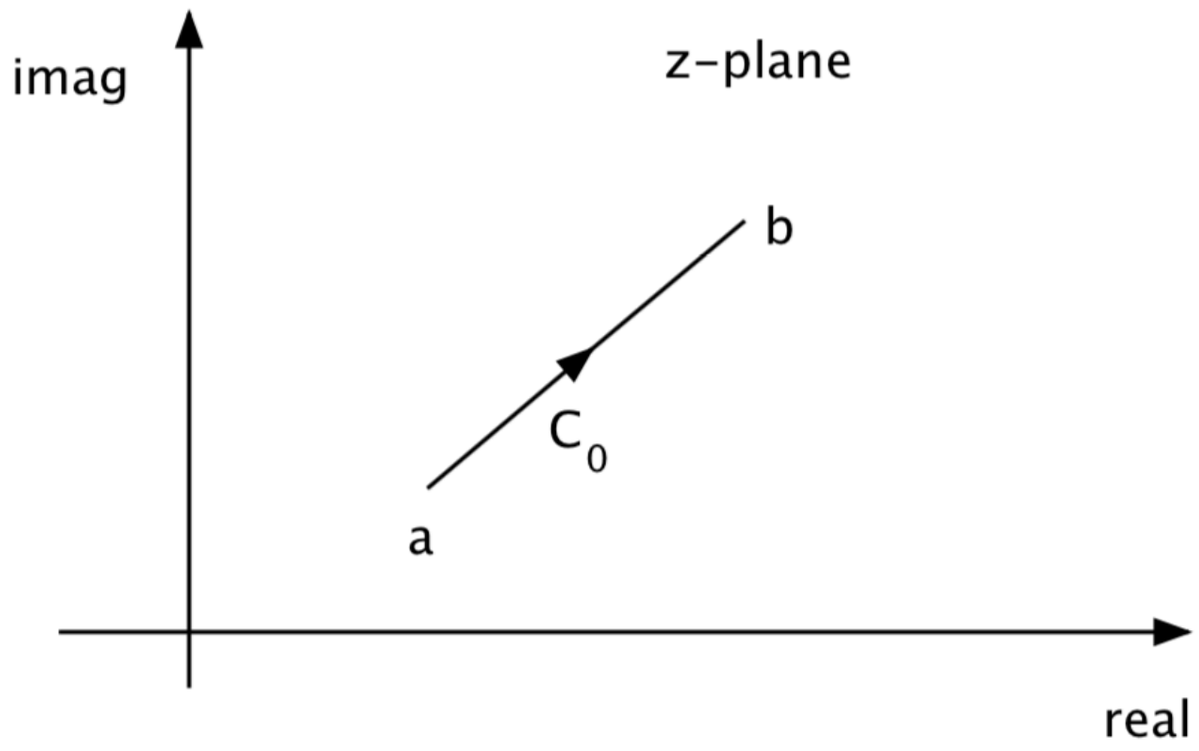
An important consequence of the Cauchy integral theorem is that

any contour of such complex integrals can be arbitrarily deformed within the analytic region without changing the value of the integral.

We can see this as follows: the integral

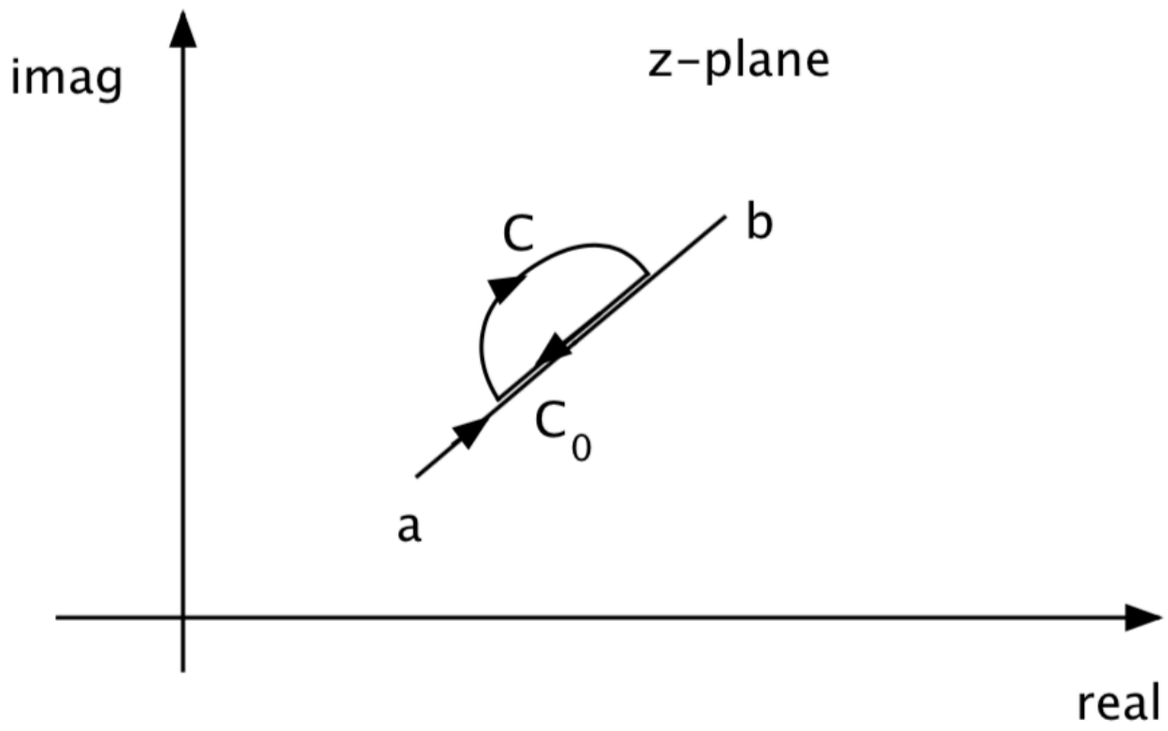
$$I = \int_{C_0} w(z)dz$$

is along the contour C_0 from a to b



We now add an integral with the same integrand but a different contour C

$$I = \int_{C_0} w(z)dz + \oint_C w(z)dz$$



The value is still I since $w(z)$ is analytic and thus

$$\oint_C w(z)dz = 0$$

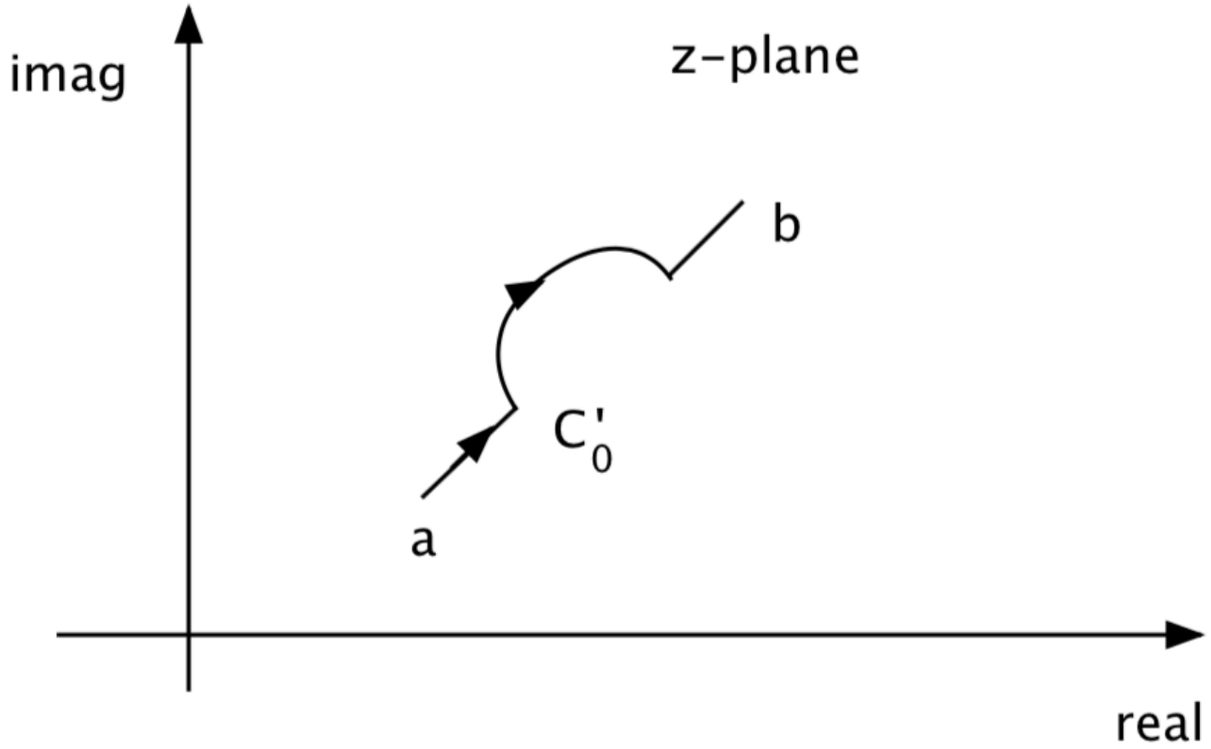
Notice, however, that two parts of the paths overlap.

The integral is the same over both of these paths but we are traversing them in opposite directions so they cancel out.

Therefore

$$I = \int_{C_0} w(z)dz = \int_{C_0'} w(z)dz$$

where C_0' is shown below:

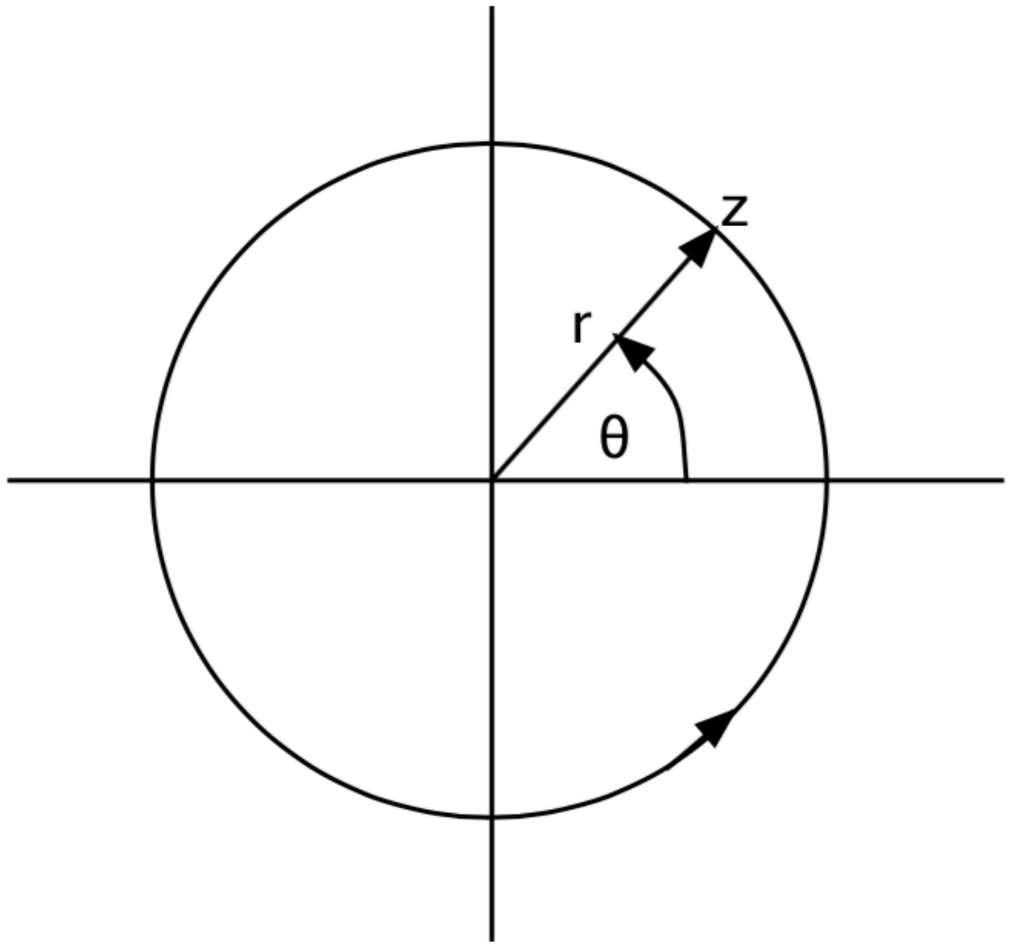


This means we can **freely deform** the contour in analytic regions.

An important example is the contour integral:

$$\int_C z^n dz$$

where C is a circle of radius $r > 0$ around the origin $z = 0$ in the positive mathematical sense (counterclockwise).



In polar coordinates we can parameterize z by

$$z = re^{i\theta} \rightarrow dz = ire^{i\theta}d\theta \text{ (on the circle where r = constant)}$$

For $n \neq 1$, we get

$$\frac{1}{2\pi i}\int_C z^n dz = \frac{r^{n+1}}{2\pi}\int_0^{2\pi} e^{i(n+1)\theta}d\theta = \frac{r^{n+1}}{2\pi i(n+1)}\left[e^{i(n+1)\theta}\right]_0^{2\pi} = 0$$

while for $n = 1$ we have

$$\frac{1}{2\pi i}\int_C = \frac{1}{2\pi}\frac{dz}{z}\int_0^{2\pi} d\theta = 1$$

Both results are **independent of** r.

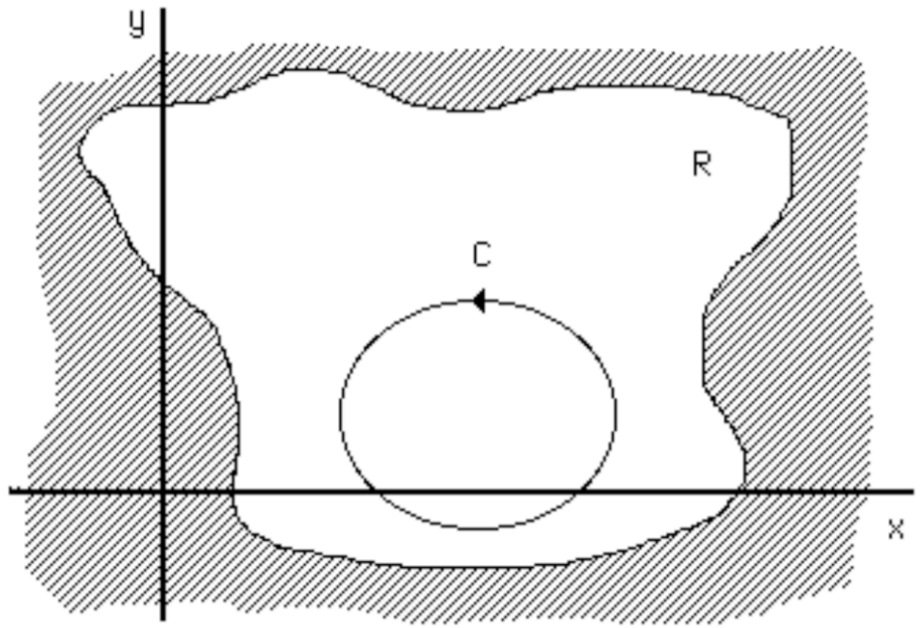
This is an example of the **Cauchy integral theorem**.

The Cauchy integral theorem is the first of **two basic theorems** in the study of complex variables.

Let us elaborate on our earlier definition:

Statement of the Theorem

If a function $f(z)$ is analytic (therefore single-valued) and its partial derivatives are continuous throughout some region R as shown in the figure,



then for every closed path C in R the contour integral of $f(z)$ around C is zero

$$\oint_C f(z)dz = 0$$

The Cauchy Integral Formula

Consider a function $f(z)$ that is analytic on and within a closed contour C.

Let us evaluate

$$\oint_C \frac{f(z)}{z - z_0} dz$$

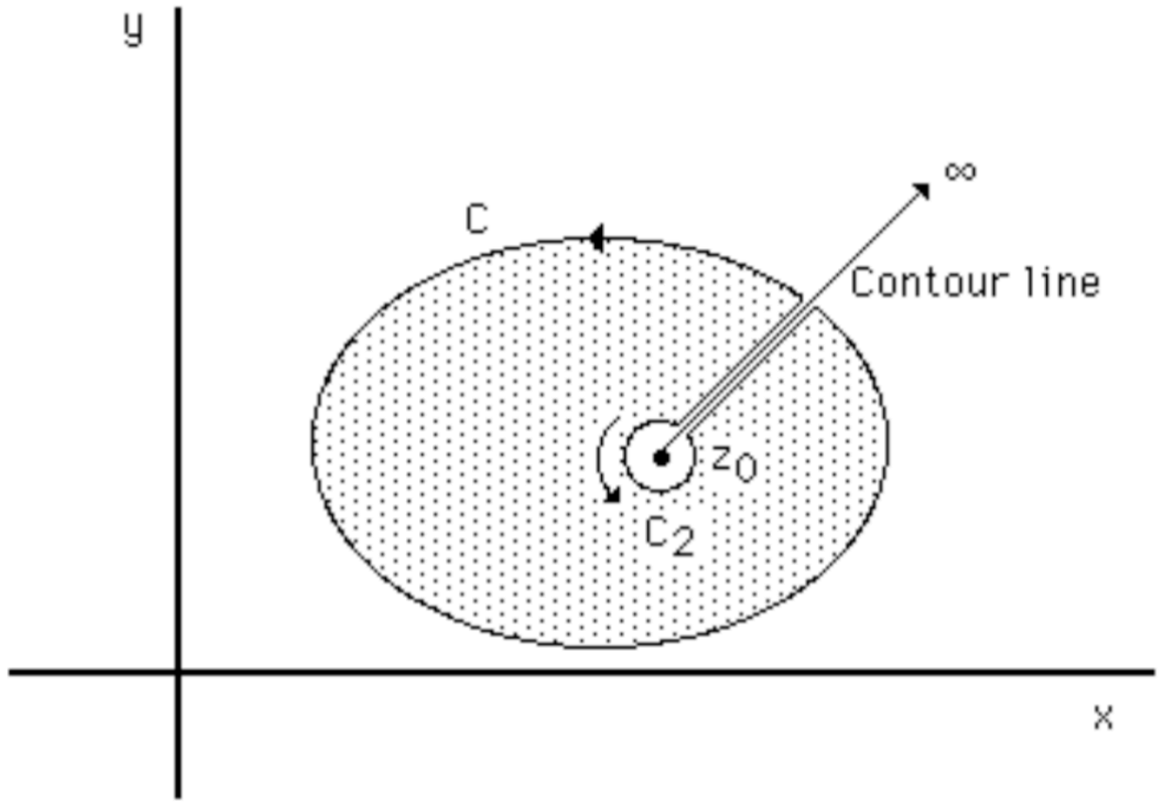
where z_0 is some point in the interior region bounded by C.

Although $f(z)$ is analytic, the integrand

$$\frac{f(z)}{z - z_0}$$

is not analytic at z_0 unless $f(z_0) = 0$.

If the contour is deformed as shown in the figure,



then Cauchy's integral theorem applies and we have

$$\oint_C \frac{f(z)}{z - z_0} dz - \oint_{C_2} \frac{f(z)}{z - z_0} dz = 0$$

where C is the original outer contour and C_2 is the circle surrounding the point z_0 (traversed in a counterclockwise direction).

If we let

$$z = z_0 + re^{i\theta}$$

on C_2, where eventually r will go to zero, then we have

$$\oint_C \frac{f(z)}{z - z_0} dz = \oint_{C_2} \frac{f(z)}{z - z_0} dz = \oint_{C_2} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta} d\theta$$

Taking the limit $r \to 0$, we get

$$\oint_C \frac{f(z)}{z - z_0} dz = if(z_0) \oint_{C_2} d\theta = 2\pi i f(z_0)$$

which is the **Cauchy integral formula**.

Here is a **remarkable result**.

The value of an analytic function $f(z)$ is given(by the formula) at an interior point $z = z_0$ once the values on the boundary C are specified. If z_0 is a point exterior to the region bounded by C , then the integral equals zero.

Summarizing the theorem:

Let $f(z)$ be be an analytic function within a closed contour C and continuous within and on C.

If a is any point within C, then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$$

If a is any point outside C, then

$$0 = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$$

Calculus Of Residues

Singularities: Poles

In the Laurent expansion of $f(z)$ about z_0

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

If $a_n = 0$ for $n < -m < 0$ and $a_m \neq 0$, we say that is a pole of order m.

For instance, if $m = 1$, that is, if

$$\frac{a_{-1}}{z - z_0}$$

is the first non-vanishing term in the Laurent series, then we have a **pole of order one, which is called a simple pole**.

Residue Theorem

If the Laurent expansion of a function

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - a)^n$$

is integrated term by term by using a closed contour that encircles one isolated singular point z_0 once in a counterclockwise sense, we obtain

$$\oint_C f(z)dz = \sum_{n=-\infty}^{\infty} a_n \oint_C (z - z_0)^n dz = \sum_{n=-\infty}^{\infty} a_n \left. \frac{(z - z_0)^{n+1}}{n+1} \right|_{z-1}^{z_1} = 0$$

for all $n \neq 1$.

However, for $n = -1$,

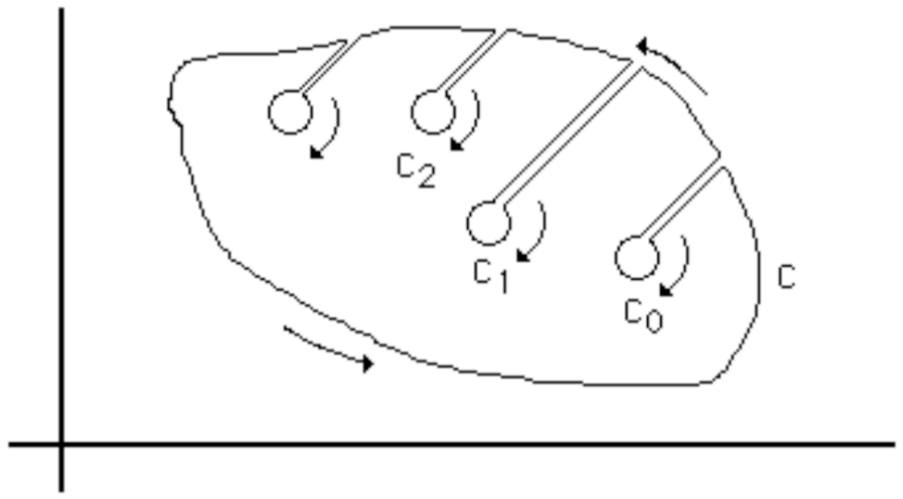
$$a_{-1} \oint_C (z - z_0)^{-1} dz = a_{-1} \oint_C \frac{ire^{i\theta}}{re^{i\theta}} d\theta = 2\pi i a_{-1}$$

Summarizing

$$\frac{1}{2\pi i} \oint_C f(z) dz = a_{-1}$$

The constant a_{-1}, which is the coefficient of $(z - z_0)^{-1}$ in the Laurent expansion , is called the **residue** of $f(z)$ at $z = z_0$.

If we have a set of isolated singularities, then we can handle them by deforming the our contour as shown in the figure



Cauchy's integral theorem then gives

$$\oint_C f(z)dz + \oint_{C_0} f(z)dz + \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + ... = 0$$

The circular integral around any singular point is given by

$$\oint_{C_i} f(z)dz = -2\pi i a_{-1z_i}$$

(negative sign from clockwise integration as shown).

Putting this all together we get,

$$\oint_C f(z)dz = 2\pi i(a_{-1z_0} + a_{-1z_1} + a_{-1z_2} + a_{-1z_3} + ...)$$

$$= 2\pi i(\text{sum of enclosed residues})$$

This is the **residue theorem**.

The problem of evaluating one or more contour integrals is replaced by the algebraic problem of computing residues at the enclosed singular points.

The residues are obviously very important.

We need a general way to calculate them.

If we know the Laurent series for a function, then the residue is simply the coefficient of the $(z - z_0)^{-1}$ term.

But, often, we do not know the Laurent series.

We can always, however, assume that a Laurent series exists and then write

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z)^{n+1}} dz'$$

The coefficient a_{-1} or the residue for a pole of order n is then given by

$$a_{-1} = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (f(z)(z - z_0)^n)$$

Examples:

First-order pole ($n = 1$):

$$f(z) = \frac{a_1}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + ...$$

We have

$$\lim z \to z_0 [f(z)(z - z_0)] = a_{-1}$$

Second-oder pole ($n = 2$):

$$f(z) = \frac{a_2}{(z - z_0)^2} + \frac{a_1}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + ...$$

We have

$$\lim z \to z_0 \frac{d}{dz} [f(z)(z - z_0)^2] = a_{-1}$$

Evaluation of Definite Integrals

now for the useful stuff after all those mathematical proof....

Integrals of the form

$$I = \int_0^{2\pi} f(\sin\theta, \cos\theta) d\theta$$

where f is a finite and single-valued function for all values of θ.

We let

$$z = re^{i\theta} \rightarrow dz = ire^{i\theta} \rightarrow d\theta = -i\frac{dz}{z}$$

$$\sin\theta = \frac{z - z^{-1}}{2i} \quad , \quad \cos\theta = \frac{z + z^{-1}}{2}$$

The integral becomes (choosing $r = 1$)

$$I = -i \oint_{\substack{unit \\ circle}} f\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right) \frac{dz}{z}$$

with the new path of integration(as indicated) the **unit circle**.

By the residue theorem we get

$$I = (-i)2\pi i \sum (\text{residues within the unit circle})$$

where we must find the residues of $f(z)/z$.

Example:

$$I = \int_0^{2\pi} \frac{d\theta}{1 + \epsilon \cos\theta} \quad , \quad |\epsilon| < 1$$

This becomes

$$I = -i \oint_{\substack{unit \\ circle}} \frac{dz}{z\left(1 + \frac{\epsilon}{2}(z + z^{-1})\right)} = i\frac{2}{\epsilon} \oint_{\substack{unit \\ circle}} \frac{dz}{z^2 + \frac{2z}{\epsilon} + 1}$$

The denominator has roots

$$z_{\pm} = -\frac{1}{\epsilon} \pm \sqrt{1 - \epsilon^2}$$

Now z_+ is within the unit circle and z_- is outside.

The integrand looks like

$$\frac{1}{(z - z_+)(z_+ - z_-)}$$

near $z = z_+$, and thus,

$$a_{-1} = \frac{1}{(z_+ - z_-)} = \frac{\epsilon}{2\sqrt{1 - \epsilon^2}}$$

The integral then becomes

$$I = -i\frac{2}{\epsilon}2\pi i\frac{\epsilon}{2\sqrt{1 - \epsilon^2}} = \frac{2\pi}{\sqrt{1 - \epsilon^2}} \quad , \quad |\epsilon| < 1$$

A truly amazing procedure!

Integrals of the form

$$I = \int_{-\infty}^{\infty} f(x)dx$$

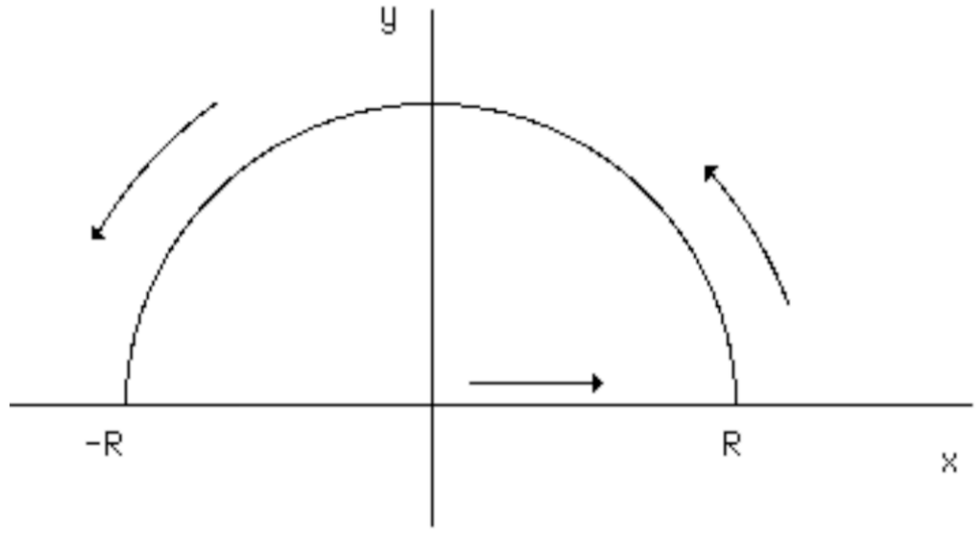
where

(a) $f(z)$ is analytic in the upper half plane except for a finite number of poles(no poles on the real axis)

(b) $f(z)$ vanishes as strongly as

$$\frac{1}{z^2} \quad \text{for} \quad |z| \to \infty \quad , \quad 0 \leq \arg(z) \leq \pi$$

With these conditions, we can take the contour of integration as the real axis $[-R, R]$ + a semi-circle in the upper half-plane as shown in the figure:



where we eventually let the radius R of the semicircle become infinitely large.

Then we have

$$\oint f(z)dz = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx + \lim_{R\to\infty}\int_{0}^{\pi} f(Re^{i\theta})d(Re^{i\theta})$$
$$= 2\pi i \sum(\text{sum of residues in the upper half-plane})$$

where the integral over the semicircle generally vanishes(**see example below**) and we get

$$I = \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum(\text{sum of residues in the upper half-plane})$$

Example:

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\pi i \sum (\text{sum of residues in the upper half-plane})$$

The first question is always...where are the poles?

$$\frac{1}{1+z^2} = \frac{1}{z+i} - \frac{1}{z-i} \rightarrow 2 \text{ simple poles at } \pm i$$

with

$$a_{-1} = \pm \frac{1}{2i}$$

We then get(since only the pole at $z = i$ lies within the contour)

$$I = \int_{-\infty}^{\infty} f(x)dx = 2\pi i \frac{1}{2i} = \pi$$

Show once: Vanishing of integral on upper semi-circle in limit $R \rightarrow \infty$.

On the upper semi-circle, $z = Re^{i\theta}$ so that

$$\lim_{R\to\infty}\int_0^{\pi}\frac{dz}{1+z^2} = \lim_{R\to\infty}\int_0^{\pi}\frac{iRe^{i\theta}}{1+R^2e^{2i\theta}}d\theta = i\lim_{R\to\infty}\frac{1}{R}\int_0^{\pi}d^{-i\theta}d\theta = 0$$

Integrals of the form

$$I = \int_{-\infty}^{\infty} f(x)e^{iax}dx \quad , \quad a \text{ real and positive}$$

This is just the Fourier transform we defined earlier.

We assume two conditions:

(a) $f(z)$ is analytic in the upper half plane except for a finite number of poles(no poles on the real axis)

(b)

$$\lim_{|z|\to\infty} f(z) = 0 \quad , \quad 0 \le \arg(z) \le \pi$$

We use the same contour as in the previous example(the real axis plus a large semicircle).

The application of the residue theorem is the same as before, except that it is more difficult to show that the integral goes to zero over the large semicircle.

On the semicircle the integral becomes

$$I_R = \int_0^\pi f(Re^{i\theta}) e^{iaR\cos\theta - aR\sin\theta} iRe^{i\theta} d\theta$$

We let R be so large that

$$|f(z)| = |f(Re^{i\theta})| < \epsilon$$

Then we get

$$|I_R| \leq \epsilon R \int_0^\pi e^{-aR\sin\theta} d\theta = 2\epsilon R \int_0^{\pi/2} e^{aR\sin\theta} d\theta$$

Now in the range $[0, \pi/2]$, $2\theta/\pi \leq \sin\theta$.

Thus, we have

$$|I_R| \leq \epsilon R \int_0^\pi e^{-2aR\theta/\pi} d\theta = 2\epsilon R \frac{1 - e^{-aR}}{2aR/\pi}$$

Finally,

$$\lim_{R\to\infty} |I_R| \leq \frac{\pi}{a}\epsilon \to 0 \quad \text{as} \quad \epsilon \to 0 (R \to \infty)$$

Using the contour we then have

$$I = \int_{-\infty}^{\infty} f(x) e^{iax} dx + \lim_{R\to\infty} |I_R|$$

$$= 2\pi i \sum (\text{sum of residues in the upper half-plane})$$

or for $a > 0$

$$\int_{-\infty}^{\infty} f(x) e^{iax} dx = 2\pi i \sum (\text{sum of residues in the upper half-plane})$$

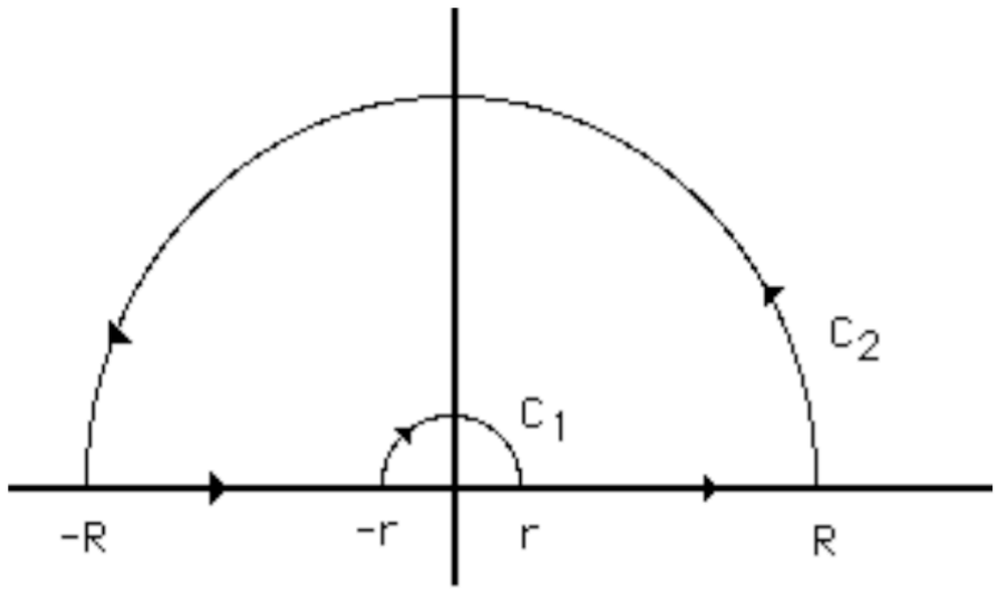
This allows us to evaluate many Fourier transform integrals.

Note that if $a < 0$, then we just close contour in **lower half-plane**.

Example:

$$I = \int_0^\infty \frac{\sin x}{x} dx$$

We use the contour shown in the figure below:



and evaluate

$$I_z = \oint \frac{e^{iz}}{z} dz$$

The original integral I is the **imaginary part** of I_z.

The only pole is a simple pole at $z = 0$ which is outside the chosen contour.

We therefore have

$$\oint \frac{e^{iz}}{z} dz = 0 = \int_{-R}^{-r} \frac{e^{ix}}{x} dx + \int_{C_1} \frac{e^{iz}}{z} dz + \int_{r}^{R} \frac{e^{ix}}{x} dx + \int_{C_2} \frac{e^{iz}}{z} dz$$

As before,

$$\int_{C_2} \frac{e^{iz}}{z} dz = 0$$

In the limits $r \to 0$ and $R \to \infty$, we have

$$I = -\frac{1}{2} \lim_{r \to 0} \int_{C_1} \frac{e^{iz}}{z} dz$$

There are two ways to do the integral around the small semicircle.

First, it is

$$-\pi i \text{ (residue at } z = 0) \text{ (1/2 full circle in wrong direction)} = -\pi i$$

or by explicit integration

$$
\begin{aligned}
-\lim_{r\to 0}\int_{C_1}\frac{e^{iz}}{z}dz &= \lim_{r\to 0}\int_{\pi}^{0}\frac{e^{ir\cos\theta - r\sin\theta}}{re^{i\theta}}ire^{i\theta}d\theta \\
&= -i\lim_{r\to 0}\int_{\pi}^{0} e^{ir\cos\theta - r\sin\theta}d\theta \\
&= -i\int_{\pi}^{0}\lim_{r\to 0} e^{ir\cos\theta - r\sin\theta}d\theta = -i\int_{\pi}^{0} d\theta = \pi i
\end{aligned}
$$

Thus,

$$
I = \int_0^{\infty}\frac{\sin x}{x}dx = \frac{1}{2}\text{Imag}(i\pi) = \frac{\pi}{2}
$$

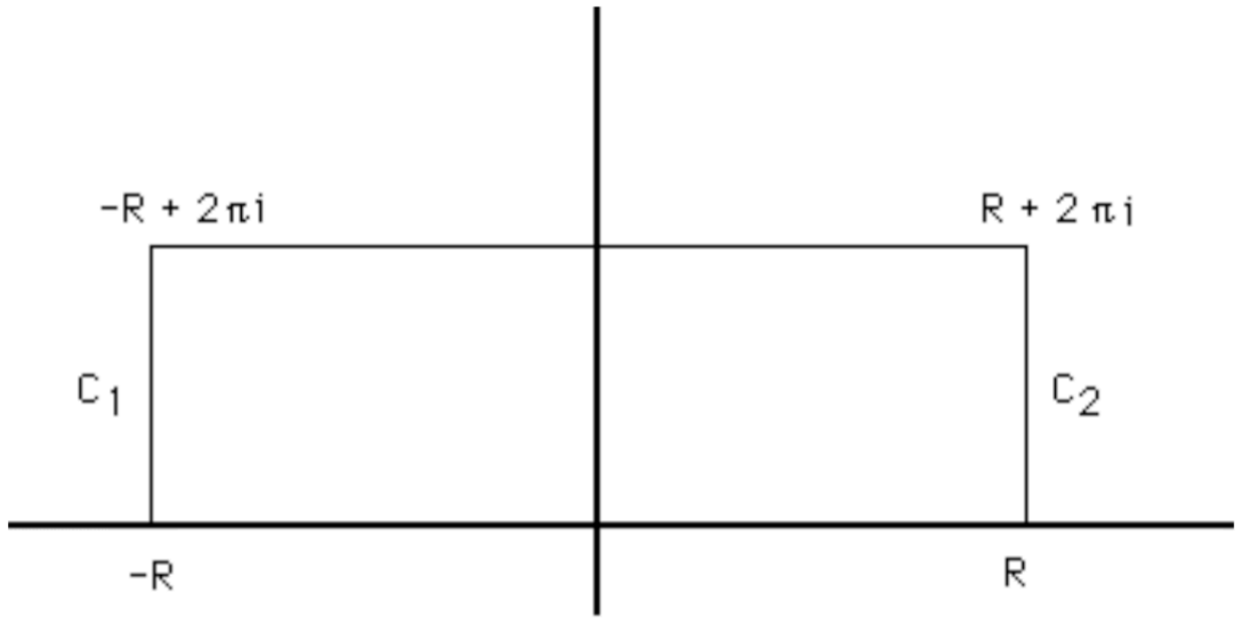
Integrals of exponential form

If we have real exponentials in the integrand, then we must choose a contour specifically for each separate problem.

We illustrate this with an example.

$$I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}} dx \quad , \quad 0 < a < 1$$

The limits on a are necessary to avoid a divergent integral. We choose the contour



and evaluate

$$\oint \frac{e^{az}}{1+e^{z}} dz = \lim_{R\to\infty} \left(\int_{-R}^{R} \frac{e^{ax}}{1+e^{x}} dx + \int_{-R}^{R} \frac{e^{ax}}{1+e^{x}} dx + \int_{C_1} + \int_{C_2} \right)$$

The two vertical contours C_1 and C_2 vanish exponentially as $R \to \infty$.

We then have

$$\oint \frac{e^{az}}{1+e^z} dz = \lim_{R\to\infty} \left(\int_{-R}^{R} \frac{e^{ax}}{1+e^x} dx - e^{2\pi i a} \int_{-R}^{R} \frac{e^{ax}}{1+e^x} dx \right)$$

$$= \lim_{R\to\infty} \left((1 - e^{2\pi i a}) \int_{-R}^{R} \frac{e^{ax}}{1+e^x} dx \right)$$

$$= 2\pi i \sum (\text{residues inside the contour})$$

Where are the poles?

We have a pole when

$$e^z = -1 \to \text{ pole at } z = i\pi$$

A Laurent expansion in powers of $(z - i\pi)$ is

$$1+e^z = 1-e^{z-i\pi} = (z-i\pi)\left(1-\frac{z-i\pi}{2!}+\frac{(z-i\pi)^2}{3!}+....\right)$$

and thus it has a simple pole with residue $-e^{i\pi a}$.

We thus get

$$\lim_{R\to\infty}\left((1-e^{2\pi ia})\int_{-R}^{R}\frac{e^{ax}}{1+e^x}dx\right) = 2\pi i\sum(\text{residues inside the contour})$$
$$= 2\pi i(-e^{i\pi a})$$

and finally

$$\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}dx = \frac{2\pi i(-e^{i\pi a})}{(1-e^{2\pi ia})} = \frac{2\pi i(-e^{i\pi a})}{e^{i\pi a}(e^{-i\pi a}-e^{\pi ia})} = \frac{\pi}{\sin\pi a}$$

That is how many integrals are actually done!!!

End of digression

We now attempt to find the equation and formal properties of $G(x';x)$ from (6.17).

We know only that $\psi(x')$ satisfies the Sehrödinger equation; we are therefore led to apply $\left[i\frac{\partial}{\partial t'} - H(x') \right]$ to (6.17):

$$\left[i\frac{\partial}{\partial t'} - H(x') \right] \theta(t' - t)\psi(x') = i\,\delta(t' - t)\psi(x')$$
$$= i\int d^3x \left[i\frac{\partial}{\partial t'} - H(x') \right] G(x';x)\psi(x) \tag{6.21}$$

Since (6.21) is valid for all solutions ψ we can extract from it the Green's function equation of the Schrödinger theory

$$\left[i\frac{\partial}{\partial t'} - H(x') \right] G(x';x) = \delta^3(\mathbf{x}' - \mathbf{x})\,\delta(t' - t) = \delta^4(x' - x) \tag{6.22}$$

Together with the boundary condition of a forward propagation in time, that is,

$$G(x';x) = 0 \qquad \text{for } t' < t \tag{6.23}$$

Eq. (6.22) defines the retarded Green's function or propagator appropriate for (6.17).

We can solve explicitly for the free-particle propagator when $H_0(x') = -\frac{1}{2m}\nabla^2_{x'}$.

In this case $G_0(x';x)$ can depend only upon the difference of the coordinates $(\mathbf{x}',t')$ and $(\mathbf{x},t)$.

This is because the wave at (**x**',t') emerging from a unit source at **x** which is turned on at t depends only on the interval (**x**' - **x**, t' - t), and G_0(x';x) is precisely the amplitude of this wave.

We consider its Fourier transform

$$G_0(x';x) = G_0(x' - x)$$

$$= \int \frac{d^3p\, d\omega}{(2\pi)^4} e^{i\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})} e^{-i\omega(t'-t)} G_0(\mathbf{p},\omega) \qquad (6.24)$$

In terms of $G_0(\mathbf{p},\omega)$, (6.22) is

$$\left(i\frac{\partial}{\partial t'} + \frac{1}{2m}\nabla^{2\prime}\right) G_0(x';x) = \int \frac{d^3p\, d\omega}{(2\pi)^4}\left(\omega - \frac{p^2}{2m}\right) G_0(p,\omega) e^{-i\omega(t'-t)+i\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})}$$

$$= \int \frac{d^3p\, d\omega}{(2\pi)^4} e^{-i\omega(t'-t)+i\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})}$$

and hence for $\omega \neq p^2/2m$

$$G_0(\mathbf{p},\omega) = \frac{1}{\omega - p^2/2m} \qquad (6.25)$$

A rule for handling the singularity in the denominator is necessary to complete the expression in (6.25).

This is determined by the retarded boundary condition (6.23).

Recalling the discussion of the θ function (6.19), we add a positive infinitesimal imaginary part to the denominator and carry out the ω integration in (6.24) first.

The singularity then lies below the real axis as indicated in Fig. 6.3,

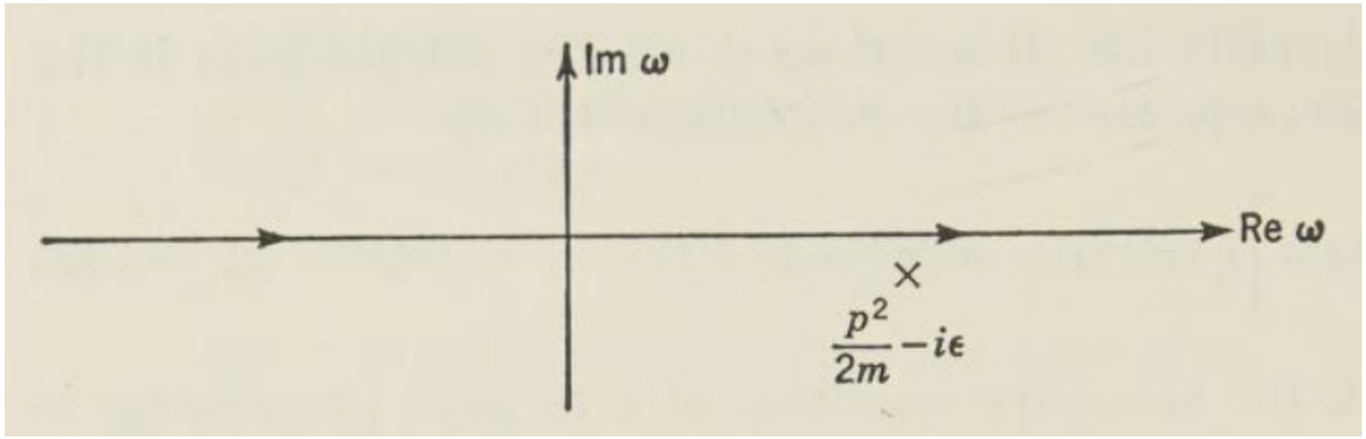


Fig. 6-3 Singularity in $G_0(\mathbf{p},\omega)$.

and we obtain

$$
\begin{aligned}
G_0(x' - x) &= \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t'-t)}}{\omega - p^2/2m + i\epsilon} \\
&= -i \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x}) - i\frac{p^2}{2m}(t'-t)} \theta(t' - t) \qquad (6.26) \\
&= -i\theta(t' - t) \int d^3p\, \varphi_p(\mathbf{x}',t') \varphi_p^*(\mathbf{x},t)
\end{aligned}
$$

where the last form uses the notation of (6.15).

It is - for the special case of plane waves - an example of a useful expression for the Green's function as a sum over a complete set of eigenfunctions of the corresponding differential equation.

In general, if we can construct a complete set of normalized solutions to the Schrödinger equation which satisfy a completeness statement of the form

$$\sum_n \psi_n(\mathbf{x}',t)\psi_n^*(\mathbf{x},t) = \delta^3(\mathbf{x}-\mathbf{x}') \tag{6.27}$$

where $\sum_n$ is a generalized sum and integral over the spectrum of quantum numbers n, then, as is readily verified,

$$G(x';x) = -i\theta(t'-t)\sum_n \psi_n(x')\psi_n^*(x) \tag{6.28}$$

satisfies (6.22) with the desired boundary condition.

The special case (6.26) for G_0 is established by the connection $\sum_n \rightarrow \int d^3p$ when integrating over the continuous momentum spectrum.

From the form of (6.26) and (6.28) it follows that the same Green's function which propagates a solution of the Schrödinger equation forward in time propagates its complex conjugate backward in time.

Multiplying (6.28) by $\psi_m(\mathbf{x})$, integrating over all $\mathbf{x}$, and invoking orthogonality and normalization of the eigenfunctions, we reproduce (6.17)

$$\begin{aligned} i\int d^3x\, G(x';x)\psi_m(x) &= \theta(t'-t)\sum_n \psi_n(x')\int d^3x\, \psi_n^*(x)\psi_m(x) \\ &= \theta(t'-t)\psi_m(x') \end{aligned}$$

Repeating this operation, only multiplying instead by $\psi_m^*(x')$ and integrating over all $\mathbf{x}'$, we obtain the indicated result:

$$i\int d^3x'\ \psi_m^*(x')G(x';x) = \theta(t'-t)\psi_m^*(x) \tag{6.29}$$

We now use these relations to construct various useful forms for the *S* matrix.

From (6.17) and the defining equation (6.16) we can write a compact form for the *S*-matrix elements in terms of the exact propagator:

$$S_{fi} = i\lim_{t'\to\infty}\lim_{t\to-\infty}\int d^3x'\ d^3x\ \varphi_f^*(x')G(x';x)\varphi_i(x) \tag{6.30}$$

This is not yet useful because in general we cannot solve directly for the exact propagator.

As is evident in (6.28) there is an enormous amount of information contained in $G(x' ; x)$.

All the solutions of the Schrödinger equation, including the bound states as required in the completeness relation (6.27), appear with equal weight.

It is no wonder that *G* is difficult to compute.

We proceed, as in our earlier intuitive considerations leading to (6.11), by constructing an iteration procedure starting with the free-particle Green's function.

Regrouping terms in (6.22), we write, with $H = H_0 + V$,

$$\left[i\frac{\partial}{\partial t'} - H_0(x') \right] G(x';x) = \delta^4(x' - x) + V(x')G(x';x)$$

$$= \int d^4x''\, \delta^4(x' - x'')[\delta^4(x'' - x) + V(x'')G(x'';x)] \tag{6.31}$$

where we have expressed the interaction term on the right-hand side as a superposition of δ-function sources.

The integral of (6.31) with the desired boundary conditions is just the corresponding superposition of free propagators:

$$G(x';x) = \int d^4x''\, G_0(x';x'')[\delta^4(x'' - x) + V(x'')G(x'';x)]$$

$$= G_0(x';x) + \int d^4x''\, G_0(x';x'')V(x'')G(x'';x) \tag{6.32}$$

which agrees with (6.12).

Inserting (6.32) into (6.30) and making use of (6.17) and (6.29) for free particles, we arrive at

$$S_{fi} = \int d^3x\, \varphi_f^*(x)\varphi_i(x) + \lim_{t\to-\infty} \int d^4x_1\, d^3x\, \varphi_f^*(x_1)V(x_1)G(x_1;x)\varphi_i(x)$$

$$= \delta_{fi} - i\int d^4x_1\, \varphi_f^*(1)V(1)\varphi_i(1) - i\int d^4x_1\, d^4x_2\, \varphi_f^*(1)V(1)$$

$$\times G_0(1,2)V(2)\varphi_i(2) - i\int d^4x_1\, d^4x_2\, d^4x_3\, \varphi_f^*(1)V(1)$$

$$\times G_0(1,2)V(2)G_0(2,3)V(3)\varphi_i(3) + \cdots \tag{6.33}$$

This multiple-scattering series coincides term by term with that developed from (6.16).

It may also be finally summed up in terms of a solution of the exact Schrödinger equation as in (6.16).

To do this, we note in the first line of (6.33) that we can write

$$\lim_{t\to-\infty}\int d^3x\, G(x'';x)\varphi_i(x) = \lim_{t\to-\infty}\int d^3x\, G(x'';x)\psi_i(x)$$
$$= -i\psi_i(x'')$$

if we refer to (6.17) and turn off V as we did earlier.

Equation (6.33) becomes

$$S_{fi} = \delta_{fi} - i\int d^4x''\ \varphi_f^*(x'')V(x'')\psi_i^{(+)}(x'') \tag{6.34}$$

where the superscript (+) is now appended to y to indicate a solution which reduces to a free wave as t" → -∞ [see (6.14)]

$$\psi_i^{(+)}(x'') = \varphi_i(x'') + \int G_0(x'';x)V(x)\psi_i^{(+)}(x)\ d^4x$$

Equation (6.34) with expansion (6.14) and (6.30) with (6.32) are equivalent forms for the S matrix; both lead to the multiple-scattering series (6.33).

In practice we shall usually calculate only the first or first two nonvanishing contributions to the *S* matrix for a given interaction in (6.33).

The validity of this procedure depends on the weakness of the interaction V and the rapid convergence of this series in powers of the interaction strength.

A general property of the S matrix which results from the conservation of probability is the property of unitarity.

We recall from the introductory remarks of Chap. 1 that hermiticity of the hamiltonian implies conservation of probability and thereby the result that the inner product of two solutions is independent of time.

We can write, therefore,

$$\int d^3x\, \psi_i^{(+)*}(x)\psi_j^{(+)}(x) = \lim_{t\to-\infty} \int d^3x\, \psi_i^{(+)*}(x)\psi_j^{(+)}(x) = \lim_{t\to-\infty} \int d^3x\, \varphi_i^*(x)\varphi_j(x) = \delta_{ji} \tag{6.35}$$

In the particular example of a plane-wave representation

$$\delta_{ji} = \delta^3(\mathbf{k}_j - \mathbf{k}_i)$$

We may also project this inner product into the remote future in which case, by (6.16) and the completeness relation (6.27) for the φ's, we can expand the solutions $\psi_i^{(+)}$ into plane-wave states with the S-matrix elements as the expansion coefficients:

$$\lim_{t'\to+\infty} \psi_i^{(+)}(x') = \sum_n \varphi_n(x')S_{ni} \tag{6.36}$$

Note: ($\sum_n = \int d^3p$ for a plane-wave representation.)

Inserting (6.36) into the left-hand side of (6.35), we find

$$\sum_n S_{ni} S^*_{nj} = \delta_{ji} \tag{6.37}$$

or in matrix notation $S^\dagger S = 1$.

If the $\psi_i^{(+)}$, like the φ_n in (6.36), form a complete set, $S^\dagger = S^{-1}$ and we conclude that S is a unitary matrix.

6.4 The Propagator in Positron Theory

We generalize our propagator development of the nonrelativistic theory and apply it to the relativistic electron theory.

Our starting point is provided by the picture of the nonrelativistic $G(x';x)$ as the probability amplitude for a particle wave originating at x to propagate to x′.

This amplitude, given in (6.11), is a sum of amplitudes, the *n*th such term being a product of factors corresponding to the diagram of Fig. 6.4.

Each line in Fig. 6.4 represents the amplitude $G_0(x_i;x_{i-1})$that a particle wave originating at x_{i-1} propagates freely to x_i.

At the point x_i (represented by a ◦) it is scattered with probability amplitude per unit space-time volume $V(x_i)$ to a new wave propagating forward in time with amplitude $G_0(x_{i+1};x_i)$ to the next interaction.

This amplitude is then summed over all space-time points in which the interactions can occur.

One may say the interaction at the *i*th point, or vertex, destroys the particle propagating up to x_i and creates a particle which propagates on to x_{i+1} with $t_{i+1} > t_i$.

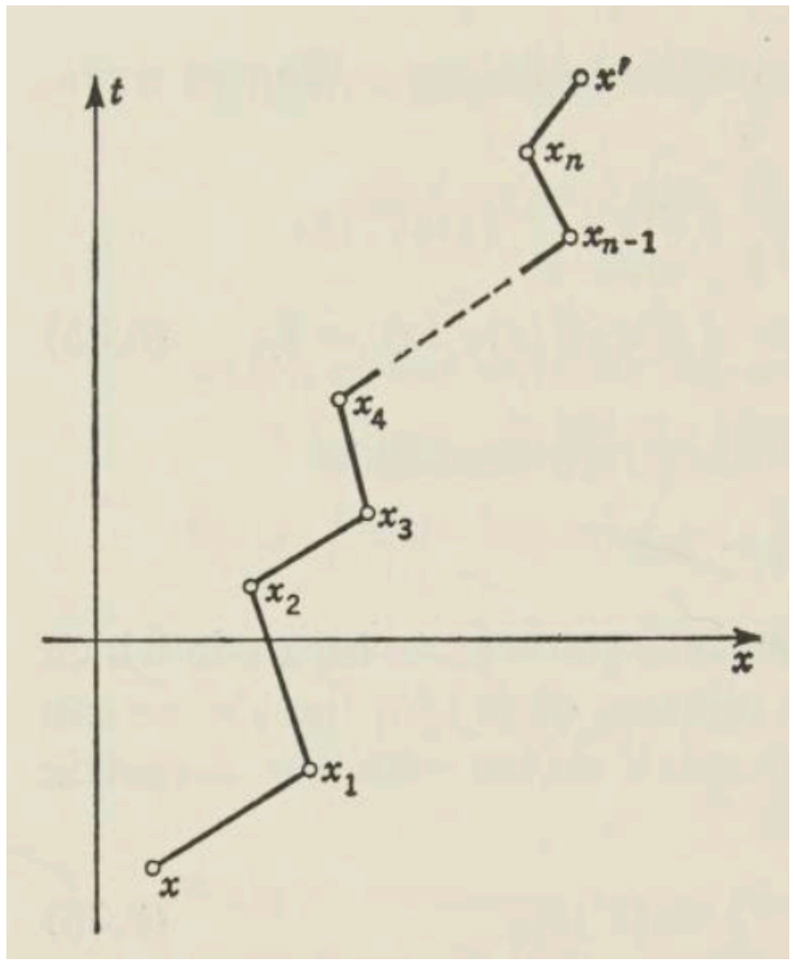


Fig. 6-4 *n*th order contribution to *G*(x,x').

It is this picture which we will keep in the Dirac hole theory.

It is well suited to a relativistic theory because of its emphasis on the overall space-time view of the scattering process, in contrast to a hamiltonian formalism with its emphasis upon the time.

The aim is to construct by analogy with the nonrelativistic propagator theory rules for calculating scattering processes in Dirac hole theory.

However, the existence of pair production and annihilation processes, which we must also describe, complicates matters.

The ground rules which we shall adopt in dealing with this situation are simply that the instructions for calculating with the propagator must be consistent with the dynamics of the Dirac equation and with the general postulates pronounced in Chap. 1 and amended by our discussion of positrons in Chap. 5.

We shall lean heavily on intuitive arguments at the expense of rigor in our developments in this and the following chapters.

Let us look at pictures of typical processes which must be described in positron theory.

There not only are scattering processes of the type illustrated in Fig. 6.4 but also the pair production and annihilation processes illustrated in Fig. 6.5.

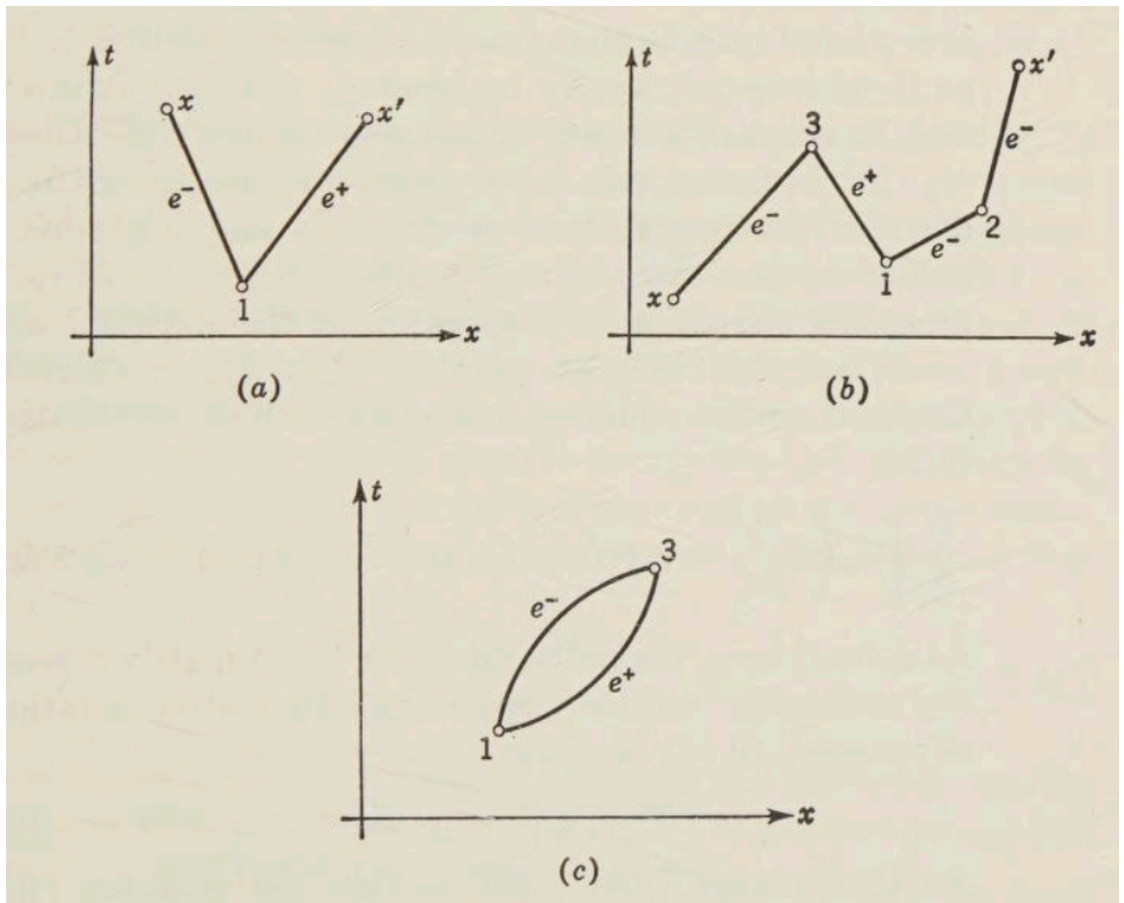


Fig. 6-5 Examples of space-time diagrams in positron theory for (a) pair production, (b) scattering, and (c) a closed loop.

Diagram 6.5a shows the production of an electron-positron pair by a potential acting at point 1; the two particles of the pair then propagate to points x and x', respectively.

Diagram 6.5b shows an electron originating at x and ending up at x'.

Along the way, a pair is produced by a potential acting at 1; the positron of the pair annihilates the initial electron in the field at 3; the electron of the pair propagates up to point 2, where it is destroyed by the potential.

This potential at 2 creates the electron which propagates to x'.

Diagram 6.5c shows a pair produced at 1, propagating up to 3, and being destroyed in the field there.

We see from these diagrams that we need not only the amplitude for an electron to be created, say, at 1, to propagate from 1 to 2, and to be destroyed at 2 as in the nonrelativistic case, but also the amplitude for a positron to be created, to propagate, and to be destroyed.

If this positron amplitude is found, we may then attempt to associate a probability amplitude with each process of the type illustrated in Fig. 6.5 and to construct the total amplitude for any particular process, by summing, or integrating, over all intermediate paths which can contribute to the process.

Thus for a scattering event paths of both types shown in Fig. 6.4 and Fig. 6.56 occur.

We must determine the positron amplitude in accordance with the hole theory formulated in the preceding chapter.

Since the existence of a positron is associated with the absence of a negative-energy electron from the filled sea, we may view the destruction of a positron at 3 in Fig. 6.5 as equivalent to the creation of a negative-energy electron there.

This suggests the possibility that the amplitude for creating the positron at 1 and destroying it at 3 is related to the amplitude for creating a negative-energy electron at 3 and destroying it at 1.

The diagrams of Fig. 6.5 would be interpreted in terms of electrons propagating forward in time with positive energy and backward in time with negative energy.

Diagram 6.5a, which describes pair production, may be considered in terms of a negative-energy electron originating at x', propagating backward in time to 1, where it is destroyed, and a positive-energy electron propagating forward to space-time point x.

In a scattering process the electron propagating up to point 3 has the option of being scattered by the potential forward in space-time as in Fig. 6.4 and propagating on with positive energy or of scattering backward to 1 as in Fig. 6.56 with negative energy.

In addition to electron paths which zigzag forward and backward in time, there is also the possibility of closed loops as illustrated in Fig. 6.5c.

In hole theory one says the potential at 1 scatters an electron in the sea into a positive-energy state; it then scatters back into the sea at 3.

In propagator language, the electron created at 1 is scattered back in time from 3 to destroy itself at 1.

Processes such as these may not simply be ignored.

The formalism requires them, and, as we shall also see, experiment verifies their existence.

As the first step in our program we construct a Green's function to describe the propagation of electrons and positrons.

We shall be guided by the discussion of positron theory in Chap. 5 and by the preceding discussion of propagators in the nonrelativistic theory.

The relativistic propagator, $S'_F(x';x)$, is defined to satisfy a Green's function equation in analogy with the nonrelativistic definition (6.22):

$$\sum_{\lambda=1}^{4}\left[\gamma_\mu\left(i\frac{\partial}{\partial x'_\mu}-eA^\mu(x')\right)-m\right]_{\alpha\lambda}S'_{F\lambda\beta}(x',x)=\delta_{\alpha\beta}\,\delta^4(x'-x) \tag{6.38}$$

As defined here, the propagator is a 4 x 4 matrix corresponding to the dimensionality of the γ matrices.

In matrix notation with indices suppressed, (6.38) becomes

$$(i\not\partial'-e\not A'-m)S'_F(x',x)=\delta^4(x'-x) \tag{6.39}$$

Another change from (6.22) is that the operator $i\,\partial/\partial t' - H(x')$ is multiplied by γ^5 in (6.38) in order to form the covariant operator $(i\not\partial'-e\not A'-m)$.

We can compute the free-particle propagator

$$(i\not\partial'-m)S_F(x',x)=\delta^4(x'-x) \tag{6.40}$$

by Fourier transforming to momentum space.

As in the nonrelativistic case (6.24), $S_F(x',x)$ depends only on the interval $(x' - x)$, so that

$$S_F(x',x) = S_F(x'-x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x'-x)} S_F(p) \tag{6.41}$$

Inserting into (6.40) gives

$$\sum_{\lambda=1}^{4} (\not{p} - m)_{\alpha\lambda} S_{F\lambda\beta}(p) = \delta_{\alpha\beta}$$

Solving for the Fourier amplitude S_F(p) and reverting to matrix shorthand, we find

$$S_F(p) = \frac{\not{p} + m}{p^2 - m^2} \equiv \frac{1}{\not{p} - m} \qquad \text{for } p^2 \neq m^2 \tag{6.42}$$

Instructions on how to handle the singularities at $p^2 = m^2$, that is, at $p_0 = \pm \sqrt{p^2 + m^2} = \pm E$, are needed to complete the definition.

As we recall from the nonrelativistic theory, the answer to this question comes from the boundary conditions put on S_F(x' - x) in integrating (6.41).

The interpretation given to the Green's function S_F(x' - x) is that it represents the wave produced at the point x' by a unit source located at the point x.

The Fourier components of such a localized point source contain many momenta larger than *m*, the reciprocal of the electron Compton wavelength, and we expect that positrons as well as electrons may be created at x by the source.

However, a necessary physical requirement of hole theory is that the wave propagating from x into the future consist only of positive-energy electron and positron components.

Since positive-energy positrons and electrons are represented by wave functions with positive frequency time behavior [for example,

$$\psi_c^{(+)}(x) = C\bar{v}(p)^T e^{-ip\cdot x}$$
$$\psi^{(+)}(x) = u(p)e^{-ip\cdot x} \qquad (6.43)$$

are (unnormalized) wave functions of positrons and electrons, respectively, of momentum p^μ with $p^0 > 0$ $S_F(x' - x)$ can contain in the future, $x'_0 > x_0$, *only positive-frequency components*.

In order to accomplish this, we return to the Fourier expansion of $S_F(x' - x)$ in (6.41) and (6.42) and perform the dp_0 integration along the contour in the complex p_0 plane shown in Fig. 6.6.

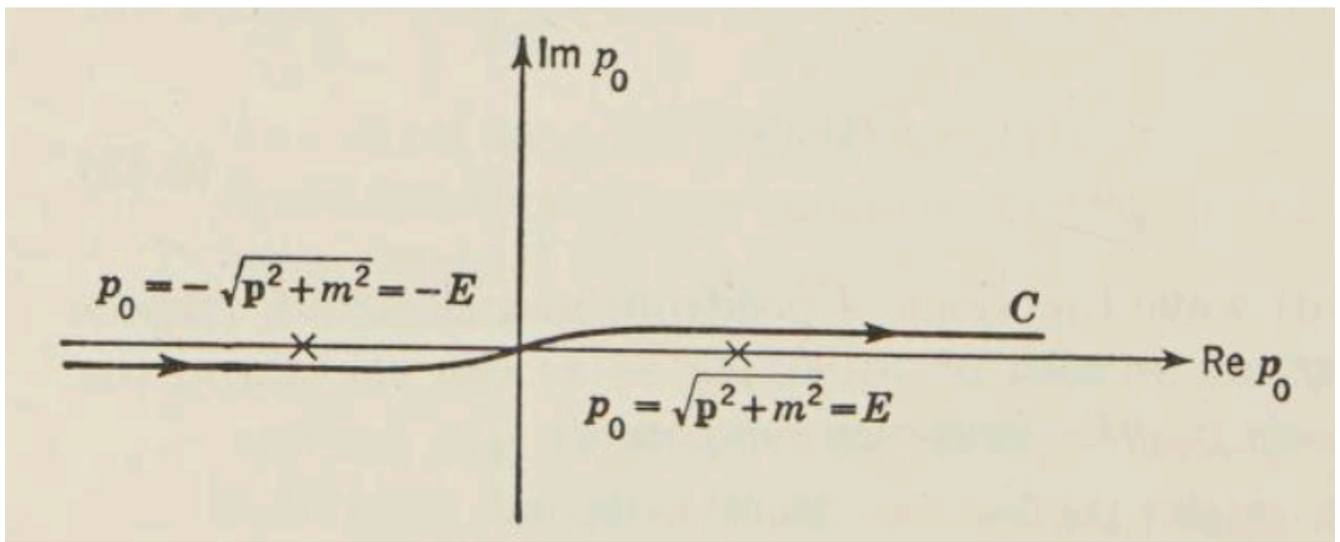


Fig. 6-6 Singularities of an integration contour for $S_F(p)$.

This gives

$$S_F(x'-x) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})} \int_C \frac{dp_0}{2\pi} \frac{e^{-ip_0(t'-t)}}{p^2-m^2} (\not{p}+m)$$

$$= -i \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})} e^{-iE(t'-t)} \frac{E\gamma_0 - \mathbf{p}\cdot\boldsymbol{\gamma} + m}{2E} \qquad t' > t \tag{6.44}$$

so that the wave at (x',t) contains positive-frequency components only.

For $t' < t$, the contour can be closed above, including the pole at $p_0 = \sqrt{\mathbf{p}^2 + m^2}$.

This gives

$$S_F(x'-x) = -i \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})} e^{+iE(t'-t)} \frac{(-E\gamma_0 - \mathbf{p}\cdot\boldsymbol{\gamma} + m)}{2E} \tag{6.45}$$

$$t' < t$$

showing the propagator to consist of negative-frequency waves for $t' < t$.

These negative-energy waves, absent in the nonrelativistic theory, are unavoidable here.

Any other choice of contour C in (6.44) leads to negative-energy waves propagating into the future or positive-energy waves into the past.

Moreover, these negative-energy waves propagating into the past are welcome; they are the positive-energy positrons, as we discussed in the preceding section.

This will become more apparent when we apply the propagator formalism to scattering problems.

The origin of the negative-energy waves is the pole at $p_0 = \sqrt{\mathbf{p}^2 + m^2}$ which was not present in the non-relativistic theory.

The choice of the contour C is summarized by adding a small positive imaginary part to the denominator in (6.42), or simply taking $m^2 \rightarrow m^2 - i\epsilon$, where the limit $\epsilon \rightarrow 0+$ is understood:

$$S_F(x' - x) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x'-x)}}{p^2 - m^2 + i\epsilon} (\not{p} + m) \tag{6.46}$$

The forms (6.44) and (6.45) are combined by introducing projection operators (3.18) and changing **p** to -**p** in the negative-frequency part:

$$S_F(x' - x) = -i \int \frac{d^3p}{(2\pi)^3} \left(\frac{m}{E}\right) [\Lambda_+(p) e^{-ip\cdot(x'-x)} \theta(t' - t) + \Lambda_-(p) e^{ip\cdot(x'-x)} \theta(t - t')] \tag{6.47}$$

with $p_0 = E > 0$.

Equivalently, writing

$$\psi^r_{\mathbf{p}}(x) = \sqrt{\frac{m}{E}} (2\pi)^{-3/2} w^r(\mathbf{p}) e^{-i\epsilon_r p\cdot x}$$

for normalized plane-wave solutions, we find

$$S_F(x' - x) = -i\theta(t' - t) \int d^3p \sum_{r=1}^{2} \psi_p^r(x') \bar{\psi}_p^r(x) + i\theta(t - t') \int d^3p \sum_{r=3}^{4} \psi_p^r(x') \bar{\psi}_p^r(x) \tag{6.48}$$

and verify with the aid of (3.11) that S_F(x' - x) carries the positive-energy solutions $\psi^{(+)}$ forward in time and the negative-energy ones $\psi^{(-)}$— backward:

$$\theta(t' - t)\psi^{(+)}(x') = i\int S_F(x' - x)\gamma_0\psi^{(+)}(x)\, d^3x \tag{6.49}$$

$$\theta(t - t')\psi^{(-)}(x') = -i\int S_F(x' - x)\gamma_0\psi^{(-)}(x)\, d^3x \tag{6.50}$$

S_F(x' - x) as defined here is known as the Feynman propagator.

It was first introduced into positron theory in 1942 by Stückelberg and independently in 1948 by Feynman, who applied it extensively to physical calculations.

From the free propagator S_F(x' - x) we may formally construct the complete Green's function and the *S*-matrix elements, that is, the amplitudes for various scattering processes of electrons and positrons in the presence of force fields.

To accomplish this, we paraphrase the nonrelativistic treatment.

The exact Feynman propagator Sp(x';x) satisfies (6.39) and, in parallel with (6.31) and (6.32), can be expressed in terms of a superposition of free Feynman propagators; that is

$$(i\not\nabla_{x'} - m)S'_F(x';x) = \int d^4y\, \delta^4(x' - y)[\delta^4(y - x) + e\not A(y)S'_F(y;x)]$$

which integrates to

$$S'_F(x',x) = S_F(x' - x) + e\int d^4y\, S_F(x' - y)\not A(y)S'_F(y,x) \tag{6.51}$$

In analogy with (6.14), the exact solution of the Dirac equation

$$(i\not\nabla_x - m)\Psi(x) = e\not A(x)\Psi(x) \tag{6.52}$$

with the Feynman boundary conditions, is

$$\Psi(x) = \psi(x) + e\int d^4y\, S_F(x - y)\not A(y)\Psi(y) \tag{6.53}$$

The scattered wave in (6.53) contains only positive frequencies in the future and negative frequencies in the past according to (6.48):

$$\Psi(x) - \psi(x) \to \int d^3p \sum_{r=1}^{2} \psi^r_p(x)[-ie \int d^4y\, \bar\psi^r_p(y)\not A(y)\Psi(y)] \tag{6.54}$$

$$\text{as } t \to +\infty$$

$$\Psi(x) - \psi(x) \to \int d^3p \sum_{r=3}^{4} \psi^r_p(x)[+ie \int d^4y\, \bar\psi^r_p(y)\not A(y)\Psi(y)] \tag{6.55}$$

$$\text{as } t \to -\infty$$

We thus have a formulation of the scattering problem in (6.54) which is in accord with the requirement from hole theory that electrons cannot fall into the negative-energy sea after scattering by an applied field $A^{\mu}(y)$; only the unfilled positive-energy states are available.

Equation (6.55) shows that the waves scattered backward to earlier times have negative energies.

From (6.54) and (6.55) we identify the *S*-matrix elements as the coefficients of the free-wave solutions $\psi^{r}(x)$, that is,

$$S_{fi} = \delta_{fi} - ie\epsilon_f \int d^4y\, \bar{\Psi}_f(y) \not{A}(y) \Psi_i(y) \tag{6.56}$$

where $\psi_f(y)$ is the final free wave emerging with quantum numbers *f* and with $\epsilon_f = +1$ for positive-frequency solutions in the future and - 1 for negative-frequency ones in the past, respectively; $\Psi_i(y)$ is the incident wave which reduces at $y_0 \to -\infty$ to an incident positive-frequency wave $\psi_i(y)$ with quantum numbers i or at $y_0 \to +\infty$ to an incident negative-frequency wave propagating into the past, according to the Feynman boundary condition on the solution (6.53).

Equations (6.56) and (6.53) contain the rules for calculating the pair production and annihilation amplitudes, shown in Fig. 6.5, as well as for the "ordinary" scattering process of Fig. 6.4.

Consider first the ordinary process of electron scattering.

For this process $\Psi_i(y)$ reduces to an incident plane wave of positive energy $\psi_i^{(+)}(y)$ at $y_0 \to -\infty$ and the *n*th order contribution to (6.56) is

$$\begin{aligned} -ie^n \int\int d^4y_1 \cdots d^4y_n\, \bar{\Psi}_f^{(+)}(y_n) \not{A}(y_n) S_F(y_n - y_{n-1}) \not{A}(y_{n-1}) \cdots \\ \times S_F(y_2 - y_1) \not{A}(y_1) \psi_i^{(+)}(y_1) \end{aligned} \tag{6.57}$$

Graphs such as Fig. 6.4 as well as Fig. 6.5b are contained in the series (6.57).

To calculate pair production, we insert for $\Psi_i(y)$ in (6.56) a solution which reduces as $t \to +\infty$ to a free plane wave of negative energy.

Specifically, for production of an electron-positron pair with quantum numbers ($\mathbf{p}_-$,s_-) and ($\mathbf{p}_+$,s_+), respectively (where $p_{0\pm} > 0$), we insert for $\Psi_i(y)$ in (6.56) the solution of (6.53), which reduces as $t \to +\infty$ to a negative-energy plane wave with quantum numbers ($+\mathbf{p}_+$, $+s_+$, $\epsilon = -1$); i.e.,

$$\psi_i^{(-)}(y) = \sqrt{\frac{m}{E_+}}\,(2\pi)^{-3/2} v(p_+, s_+) e^{+ip_+ \cdot x}$$

For ψ_f we take a positive-energy solution labeled by ($\mathbf{p}_-$, s_-, $\epsilon = 1$).

By the ground rules of our discussion of hole theory, it is the absence of a negative-energy electron with four-momentum $-p_+$ and spin $-s_+$ that we record as the presence of a positron with four-momentum p_+ and polarization s_+.

In the propagator formulation we have identified the amplitude for producing the positron at x and propagating it forward in space-time out of the interaction volume and into a given plane-wave state ($\mathbf{p}_+$,s_+) at x' with the amplitude for a negative-energy electron with four-momentum $-p_+$ and spin $-s_+$ to propagate from x' back into the interaction volume and be destroyed at x.

We thus associate a transition amplitude with the pair-production process by tracing the path of a negative-energy electron backward in time into the interaction region where it scatters in the field and emerges in a positive-energy state propagating forward in time.

The two lowest-order Feynman diagrams are shown in Fig. 6.7, and the second-order amplitude is further decomposed by time ordering of the two scatterings as indicated.

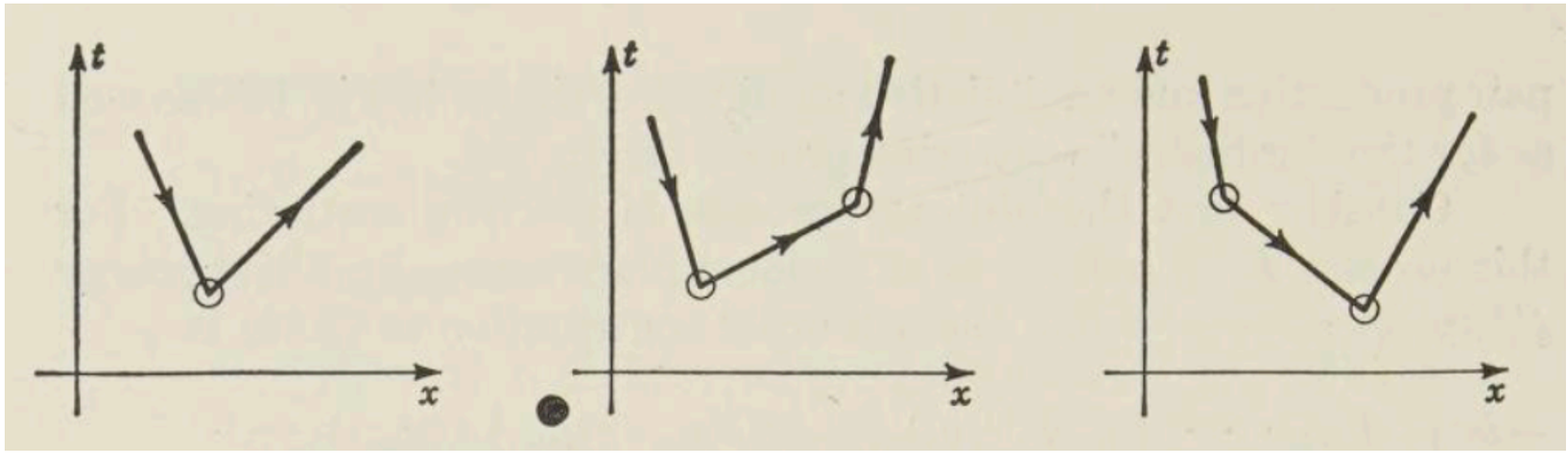


Fig. 6-7 Space-time Feynman diagrams of pair production to first and second orders. The second-order contribution is further decomposed by time ordering of the two scatterings.

In a similar way, to calculate the pair annihilation amplitude, we insert for $\Psi_i(y)$ a solution of (6.53) which reduces to it $\psi_i^{(+)}(y)$ at $y \to -\infty$.

This positive-energy electron propagates forward in time into the interaction to be scattered backward in time and emerge in a negative-energy state.

The *n*th-order amplitude that the electron scatters into a given "final" state $\psi_f^{(-)}$ labeled by quantum numbers ($\mathbf{p}_+$, s_+, $\epsilon = -1$) is

$$ie^n \int d^4y_1 \cdots d^4y_n \, \bar{\Psi}_f^{(-)}(y_n) \not{A}(y_n) S_F(y_n - y_{n-1}) \cdots \times \not{A}(y_1) \psi_i^{(+)}(y_1) \tag{6.58}$$

In hole-theory language this is the *n*th-order amplitude that an electron scatters into a negative-energy state of momentum $-p_+$ and spin $-s_+$.

This state at $t = -\infty$ must have been empty; that is, there must have been a hole, or positron, present with four-momentum p_+ and spin, or polarization, s_+.

Finally, in order to describe positron scattering, the "incident" positive-frequency wave in (6.56) and (6.58) is replaced by a negative-frequency solution with quantum numbers ($\mathbf{p}'_+$, s'_+, $\epsilon = -1$).

This represents the outgoing positron with momentum and spin ($\mathbf{p}'_+$, s'_+).

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7 Applications

7.1 Coulomb Scattering of Electrons

In this chapter we apply the propagator formalism just developed to various practical calculations.

As we gain experience with these propagator amplitudes we shall extend them in a natural and plausible fashion to include interactions between several particles.

Our program is the same as that of the original Feynman papers: to establish rules for calculating transition rates and cross sections for general processes of physical interest before resorting to the formal manipulations of quantum field theory.

We shall begin with Rutherford scattering of an electron from a fixed Coulomb potential.

The transition matrix element for this process, as given by (6.56),

$$S_{fi} = -ie \int d^4x \, \bar{\Psi}_f(x) \not{A}(x) \Psi_i(x) \qquad (f \neq i) \tag{7.1}$$

must now be translated into plain English; here e < 0 is the electron charge.

In lowest order $\Psi_i(x)$ reduces to the incident plane wave $\psi_i(x)$ describing an electron of momentum p_i and spin s_i:

$$\psi_i(x) = \sqrt{\frac{m}{E_i V}} \, u(p_i, s_i) e^{-ip_i \cdot x} \tag{7.2}$$

where we normalize $\psi(x)$ to unit probability in a box of volume V.

In the same way

$$\psi_f(x) = \sqrt{\frac{m}{E_f V}}\, \bar{u}(p_f, s_f) e^{ip_f \cdot x} \tag{7.3}$$

The Coulomb potential is given by

$$A_0(x) = \frac{-Ze}{4\pi|\mathbf{x}|} \qquad \mathbf{A}(x) = 0$$

for a point charge -Ze > 0; thus

$$S_{fi} = \frac{iZe^2}{4\pi} \frac{1}{V} \sqrt{\frac{m^2}{E_f E_i}}\, \bar{u}(p_f, s_f) \gamma^0 u(p_i, s_i) \int \frac{d^4x}{|\mathbf{x}|} e^{i(p_f - p_i)\cdot x} \tag{7.4}$$

The integration over the time coordinate yields $2\pi\delta(E_f - E_i)$ and expresses energy conservation between initial and final states in a static potential.

The space integral is the Fourier transform of the Coulomb potential, well known to be

$$\int \frac{d^3x}{|\mathbf{x}|} e^{-i\mathbf{q}\cdot\mathbf{x}} = \frac{4\pi}{|\mathbf{q}|^2}$$

where $\mathbf{q} = \mathbf{p}_f - \mathbf{p}_i$.

Our S-matrix element becomes

$$S_{fi} = iZe^2 \frac{1}{V} \sqrt{\frac{m^2}{E_f E_i}} \frac{\bar{u}(p_f,s_f)\gamma^0 u(p_i,s_i)}{|\mathbf{q}|^2} 2\pi\delta(E_f - E_i) \tag{7.5}$$

The number of final states in momentum interval d^3p_f is $V\, d^3p_f/(2\pi)^3$, and thus the transition probability *per particle* into these states is

$$|S_{fi}|^2 \frac{V\, d^3p_f}{(2\pi)^3} = \frac{Z^2(4\pi\alpha)^2 m^2}{E_i V} \frac{|\bar{u}(p_f,s_f)\gamma^0 u(p_i,s_i)|^2}{|\mathbf{q}|^4} \frac{d^3p_f}{(2\pi)^3 E_f} \times [2\pi\delta(E_f - E_i)]^2 \tag{7.6}$$

The square of the δ function requires some explanation.

Were we to consider transitions in a given time interval $(-T/2, T/2)$, the energy δ function would be smeared out; that is

$$2\pi\delta(E_f - E_i) \Rightarrow \int_{-T/2}^{T/2} dt\, e^{i(E_f - E_i)t} = 2 \frac{\sin (T/2)(E_f - E_i)}{E_f - E_i} \tag{7.7}$$

From (7.7) we see that for large but finite T,

$$[2\pi\delta(E_f - E_i)]^2 \Rightarrow 4 \frac{\sin^2 (T/2)(E_f - E_i)}{(E_f - E_i)^2}$$

Considered as a function of E_f, the area under such a curve is $2\pi T$, so that we may identify

$$[2\pi\delta(E_f - E_i)]^2 = [2\pi\delta(0)]2\pi\delta(E_f - E_i) = 2\pi T\delta(E_f - E_i) \tag{7.8}$$

or simply

$$2\pi\delta(0) = T \tag{7.9}$$

A heuristic way of seeing this is from the definition

$$2\pi\delta(E_f - E_i) \underset{(T\to\infty)}{=} \int_{-T/2}^{T/2} dt\, e^{i(E_f - E_i)t}$$

hence

$$2\pi\delta(0) \underset{(T\to\infty)}{=} \int_{-T/2}^{T/2} dt = T$$

Returning to (7.6) and dividing out the time, we find the number R of transitions per unit time into momentum interval d^3p_f, to be

$$R = \frac{4Z^2\alpha^2m^2}{E_iV} \frac{|\bar{u}(p_f,s_f)\gamma^0u(p_i,s_i)|^2}{|\mathbf{q}|^4} \frac{d^3p_f}{E_f} \delta(E_f - E_i)$$

the

A cross section is defined as the transition rate R divided by the flux of incident particles, $J^a_{\text{inc}} = \bar{\psi}_i(x)\gamma^a\psi_i(x)$, where a denotes the vector component along the incident velocity $\mathbf{v}_i = \mathbf{p}_i/E_i$.

With the normalization adopted in (7.2) the flux is $|J_{inc}| = |\mathbf{v}_i|/V$.

Thus the differential cross section do per unit solid angle $d\Omega$ is

$$\frac{d\sigma}{d\Omega} = \int \frac{4Z^2\alpha^2m^2}{|\mathbf{v}_i|\,E_i} \frac{|\bar{u}(p_f,s_f)\gamma^0u(p_i,s_i)|^2}{|\mathbf{q}|^4} \frac{p_f^2\,dp_f}{E_f}\,\delta(E_f - E_i) \tag{7.10}$$

Using the identity

$$p_f\,dp_f = E_f\,dE_f$$

we finally obtain

$$\frac{d\sigma}{d\Omega} = \frac{4Z^2\alpha^2m^2}{|\mathbf{q}|^4}\,|\bar{u}(p_f,s_f)\gamma^0u(p_i,s_i)|^2 \tag{7.11}$$

which agrees with Rutherford in the nonrelativistic limit.

In general one does not observe the polarization of the final particle and one does not know the initial polarizations.

If the incident beam has net polarization, there is usually a good reason why; and the experimentalist eventually will discover it, as in the case of polarized electrons from β decay.

In the absence of such information one assigns equal a priori probabilities to the different initial polarization states.

This means that the actual cross section observed will be a sum of (7.11) over final spin states and an average over initial states, that is,

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{4Z^2\alpha^2m^2}{2|\mathbf{q}|^4} \sum_{\pm s_f, s_i} |\bar{u}(p_f,s_f)\gamma^0 u(p_i,s_i)|^2 \tag{7.12}$$

This spin sum can be rewritten as follows:

$$\sum_{\pm s_f, s_i} \bar{u}_\alpha(p_f,s_f)\gamma^0_{\alpha\beta}u_\beta(p_i,s_i)u^\dagger_\lambda(p_i,s_i)\gamma^{0\dagger}_{\lambda\delta}\gamma^{0\dagger}_{\delta\sigma}u_\sigma(p_f,s_f)$$

$$= \sum_{\pm s_f, s_i} \bar{u}_\alpha(p_f,s_f)\gamma^0_{\alpha\beta}u_\beta(p_i,s_i)\bar{u}_\delta(p_i,s_i)\gamma^0_{\delta\sigma}u_\sigma(p_f,s_f)$$

with, as usual, sums over repeated indices implied.

This is a special case of the general form which we shall often encounter:

$$|\bar{u}(f)\Gamma u(i)|^2 = [\bar{u}(f)\Gamma u(i)][\bar{u}(i)\bar{\Gamma}u(f)] \tag{7.13}$$

where $\bar{\Gamma} = \gamma^0\Gamma^\dagger\gamma^0$, and in particular,

and

$$\overline{\gamma^\mu} = \gamma^\mu$$

$$\overline{i\gamma^5} = i\gamma^5$$

$$\overline{\gamma^\mu\gamma^5} = \gamma^\mu\gamma^5$$

$$\overline{\not{a}\not{b}\not{c}\cdots\not{p}} = \not{p}\cdots\not{c}\not{b}\not{a}$$

The spin sums can now be reduced to traces if we use the energy projection operators from (3.18):

$$\sum_{\pm s_i} u_\beta(p_i,s_i)\bar{u}_\lambda(p_i,s_i) = \sum_{r=1}^{4} \epsilon_r w_\beta^r(\mathbf{p}_i)\bar{w}_\gamma^r(\mathbf{p}_i)\left(\frac{\not{p}_i + m}{2m}\right)_{\gamma\lambda}$$

$$= \left(\frac{\not{p}_i + m}{2m}\right)_{\beta\lambda} = [\Lambda_+(p_i)]_{\beta\lambda}$$

The spin sum in (7.12) becomes

$$\sum_{\alpha,\beta}\sum_{\pm s_f} \bar{u}_\alpha(p_f,s_f)\left(\gamma^0\frac{\not{p}_i + m}{2m}\gamma^0\right)_{\alpha\beta} u_\beta(p_f,s_f)$$

$$= \sum_{\alpha,\beta}\left(\gamma^0\frac{\not{p}_i + m}{2m}\gamma^0\right)_{\alpha\beta}\left(\frac{\not{p}_f + m}{2m}\right)_{\beta\alpha}$$

where again we use the same technique.

This last expression is the trace, that is, the sum of the diagonal elements, of the matrix

$$\gamma_0 \frac{(\not{p}_i + m)}{2m} \gamma_0 \frac{(\not{p}_f + m)}{2m}$$

Therefore, (7.12) may be written as

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{4Z^2\alpha^2 m^2}{2|\mathbf{q}|^4} \operatorname{Tr} \gamma_0 \frac{(\not{p}_i + m)}{2m} \gamma_0 \frac{(\not{p}_f + m)}{2m} \tag{7.14}$$

7.2 Some Trace Theorems; the Spin-averaged Coulomb Cross Section

We must now digress and establish useful properties of traces of Dirac matrices.

These properties will allow us to calculate cross sections without ever looking directly at a Dirac matrix.

They are derived from the commutation algebra of the γ's and are valid independently of the choice of representations for the γ's.

We summarize these properties in a series of theorems:

Theorem 1

The trace of an odd number of γ matrices is zero.

Proof. For n odd

$$\mathrm{Tr}\ \not{a}_1 \cdots \not{a}_n = \mathrm{Tr}\ \not{a}_1 \cdots \not{a}_n \gamma_5 \gamma_5 = \mathrm{Tr}\ \gamma_5 \not{a}_1 \cdots \not{a}_n \gamma_5$$

where we have used the cyclic property of the trace, namely Tr AB = Tr BA.

Moving the first γ to the right, we pick up n minus signs from $\gamma_\mu \gamma_5 + \gamma_5 \gamma_\mu = 0$ and obtain

$$\mathrm{Tr}\ \not{a}_1 \cdots \not{a}_n = (-)^n\, \mathrm{Tr}\ \not{a}_1 \cdots \not{a}_n \gamma_5 \gamma_5 = 0 \tag{7.15}$$

for n odd.

Theorem 2

$$\begin{aligned} \mathrm{Tr}\ 1 &= 4 \\ \mathrm{Tr}\ \not{a}\not{b} &= \mathrm{Tr}\ \not{b}\not{a} = \tfrac{1}{2}\,\mathrm{Tr}\,(\not{a}\not{b} + \not{b}\not{a}) = a \cdot b\ \mathrm{Tr}\ 1 \\ &= 4a \cdot b \end{aligned} \tag{7.16}$$

Theorem 3

$$\begin{aligned} \mathrm{Tr}\ \not{a}_1 \cdots \not{a}_n = a_1 \cdot a_2\ \mathrm{Tr}\ \not{a}_3 \cdots \not{a}_n - a_1 \cdot a_3\ \mathrm{Tr}\ \not{a}_2 \not{a}_4 \cdots \not{a}_n \\ + \cdots + a_1 \cdot a_n\ \mathrm{Tr}\ \not{a}_2 \cdots \not{a}_{n-1} \end{aligned} \tag{7.17}$$

In particular

$$\mathrm{Tr}\ \not{a}_1 \not{a}_2 \not{a}_3 \not{a}_4 = 4[a_1 \cdot a_2\ a_3 \cdot a_4 + a_1 \cdot a_4\ a_2 \cdot a_3 - a_1 \cdot a_3\ a_2 \cdot a_4]$$

Proof. Using did = $\phi_1 \phi_2 = -\phi_2 \phi_1 + 2a_1 \cdot a_2$, we move ϕ_1 to the right of ϕ_2, that is,

$$\mathrm{Tr}\, \not{a}_1 \not{a}_2 \cdots \not{a}_n = 2a_1 \cdot a_2\, \mathrm{Tr}\, \not{a}_3 \cdots \not{a}_n - \mathrm{Tr}\, \not{a}_2 \not{a}_1 \not{a}_3 \cdots \not{a}_n$$

Continuing the process, we obtain

$$\mathrm{Tr}\, \not{a}_1 \cdots \not{a}_n = 2a_1 \cdot a_2\, \mathrm{Tr}\, \not{a}_3 \cdots \not{a}_n - \cdots + 2a_i \cdot a_n\, \mathrm{Tr}\, \not{a}_2 \cdots \not{a}_{n-1} - \mathrm{Tr}\, \not{a}_2 \cdots \not{a}_n \not{a}_1$$

Finally, we use the cyclic property of the trace to get ϕ_1 back on the left of the other γ matrices; the theorem then follows.

This last theorem is exceedingly useful in order to reduce a complicated trace, although for any $n > 6$ it pays if possible to use some subtlety to avoid an avalanche of terms.

Theorem 4

$$\begin{aligned} \mathrm{Tr}\, \gamma_5 &= 0 \\ \mathrm{Tr}\, \gamma_5 \not{a} \not{b} &= 0 \\ \mathrm{Tr}\, \gamma_5 \not{a} \not{b} \not{c} \not{d} &= 4i \epsilon_{\alpha\beta\gamma\delta} a^\alpha b^\beta c^\gamma d^\delta \end{aligned} \tag{7.18}$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is +1 for $(\alpha, \beta, \gamma, \delta)$, an even permutation of (0,1,2,3); is - 1 for an odd permutation; and is 0 if two indices are the same.

Proof. Since $\gamma_5 = i\,\gamma^0\gamma^1\gamma^2\gamma^3$, the first two identities are immediate.

The third follows by looking at the components.

For a non-vanishing contribution all components of a, b, c, d must be different and the total contribution is the sum of the various combinations of components multiplied by the sign of the permutation.

To fix the overall sign take

$$\begin{aligned}\mathrm{Tr}\,\gamma_5\gamma_0\gamma_1\gamma_2\gamma_3 a^0 b^1 c^2 d^3 &= i\epsilon_{0123} a^0 b^1 c^2 d^3\,\mathrm{Tr}\,\gamma_5^2 \\ &= 4i\epsilon_{0123} a^0 b^1 c^2 d^3\end{aligned}$$

Theorem 5

$$\begin{aligned}\gamma_\mu\gamma^\mu &= 4 \\ \gamma_\mu \not{a}\gamma^\mu &= -2\not{a} \\ \gamma_\mu \not{a}\not{b}\gamma^\mu &= 4a\cdot b \\ \gamma_\mu \not{a}\not{b}\not{c}\gamma^\mu &= -2\not{c}\not{b}\not{a} \\ \gamma_\mu \not{a}\not{b}\not{c}\not{d}\gamma^\mu &= 2[\not{d}\not{a}\not{b}\not{c} + \not{c}\not{b}\not{a}\not{d}]\end{aligned} \tag{7.19}$$

Although this is not strictly a trace theorem, it is often used in conjunction with trace calculations, and it is included here for convenience.

The proof is a straightforward exercise.

Theorem 6

$$\mathrm{Tr}\ \not{a}_1\not{a}_2 \cdots \not{a}_{2n} = \mathrm{Tr}\ \not{a}_{2n} \cdots \not{a}_1 \tag{7.20}$$

Proof. From the charge conjugation discussion of Chap. 5, recall that there exists a matrix C such that $C\gamma_\mu C^{-1} = -\gamma^{\mathrm{T}}{}_\mu$.

Then

$$\begin{aligned}\mathrm{Tr}\ \not{a}_1 \cdots \not{a}_{2n} &= \mathrm{Tr}\ C\not{a}_1 C^{-1} C\not{a}_2 C^{-1} \cdots C\not{a}_{2n} C^{-1} \\ &= (-)^{2n}\ \mathrm{Tr}\ \not{a}_1^T \not{a}_2^T \cdots \not{a}_{2n}^T \\ &= \mathrm{Tr}\ [\not{a}_{2n} \cdots \not{a}_1]^T = \mathrm{Tr}\ \not{a}_{2n} \cdots \not{a}_1\end{aligned}$$

Returning to our problem of Coulomb scattering, and using Theorem 1, (7.14) becomes

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{Z^2\alpha^2}{2|\mathbf{q}|^4}\left[\mathrm{Tr}\ \gamma^0 \not{p}_i \gamma^0 \not{p}_f + m^2\ \mathrm{Tr}\ (\gamma^0)^2\right]$$

Using Theorem 3 and Theorem 2, the final answer is obtained:

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{Z^2\alpha^2}{2|\mathbf{q}|^4}\left(8E_iE_f - 4p_i\cdot p_f + 4m^2\right) \tag{7.21}$$

The differential cross section can be put in terms of the scattering energy $E = E_{\mathrm{i}} = E_f$ and scattering angle θ, using the kinematical relations

$$p_i \cdot p_f = E^2 - \mathbf{p}^2 \cos\theta = m^2 + 2\beta^2 E^2 \sin^2\frac{\theta}{2} \qquad \text{and} \qquad |\mathbf{q}|^2 = 4\mathbf{p}^2 \sin^2\frac{\theta}{2}$$

We find

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{Z^2\alpha^2}{4\mathbf{p}^2\beta^2 \sin^4(\theta/2)}\left(1 - \beta^2 \sin^2\frac{\theta}{2}\right) \tag{7.22}$$

This is the Mott cross section'; it reduces to the Rutherford formula as $\beta \to 0$.

7.3 Coulomb Scattering of Positrons

Turning next to the scattering of positrons in a Coulomb field, we note that to lowest order in a the cross section is identical to electron scattering.

This is most simply seen by writing down the matrix element.

From (6.56) and the discussion following it,

$$S_{fi} = ie\int d^4x\, \bar{\psi}_f(x) \not{A}(x) \Psi_i^{(-)}(x) \tag{7.23}$$

Here the incoming state is in the future and is to be interpreted as a negative-energy electron of four-momentum -p_f running backward in time as drawn in Fig. 7.1.

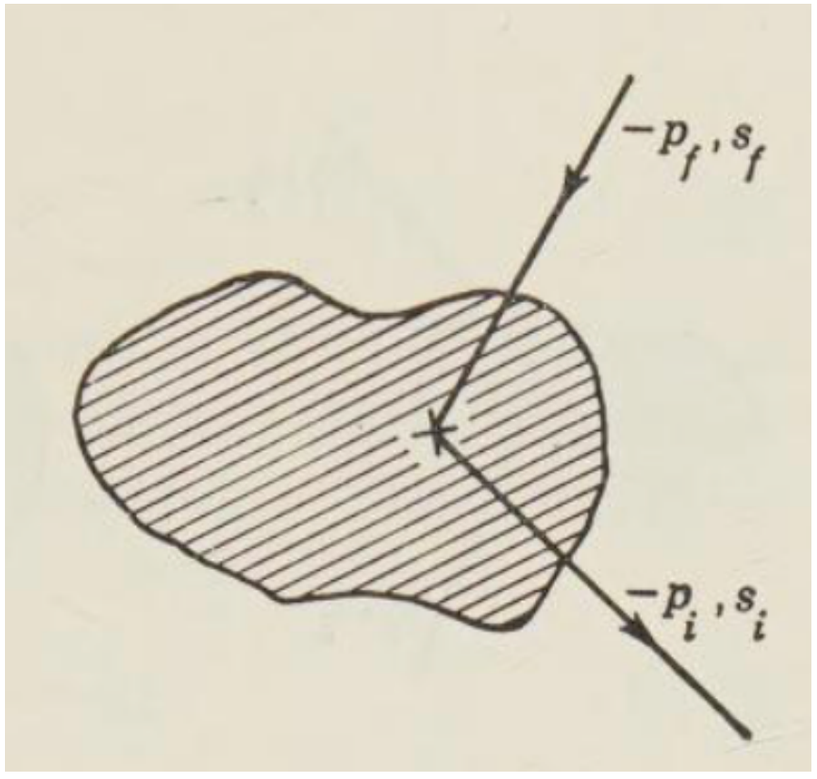

Fig. 7-1 Coulomb scattering of positrons.

Putting in plane waves to lowest order, the wave function is

$$\psi_i(x) = \sqrt{\frac{m}{E_f V}}\, v(p_f, s_f) e^{+ip_f \cdot x} \tag{7.24}$$

Similarly, the outgoing state in (7.23) is the negative-energy electron running backward into the past.

Its wave function is

$$\psi_f(x) = \sqrt{\frac{m}{E_i V}}\, v(p_i, s_i) e^{+ip_i \cdot x} \tag{7.25}$$

representing the incident positron with momentum p_i and polarization s_i before the scattering.

Introducing (7.24) and (7.25) into the S matrix, we have

$$S_{fi} = -\frac{iZe^2}{4\pi}\frac{1}{V}\sqrt{\frac{m^2}{E_f E_i}}\,\bar{v}(p_i,s_i)\gamma^0 v(p_f,s_f)\int\frac{d^4x}{|\mathbf{x}|}\,e^{i(p_f-p_i)\cdot x}$$

in analogy with (7.4).

Because of charge conjugation invariance, we could equally well write for (7.23) to this order in e

$$\begin{aligned} S_{fi} &= +ie\int d^4x\,\bar{\psi}_{ci}(x)\not{A}\psi_{cf}(x) \\ &= -ie\int d^4x\,\psi_i^T(x)C^{-1}\not{A}C\bar{\psi}_f^T(x) \\ &= +ie\int d^4x\,\bar{\psi}_f(x)\not{A}\psi_i(x) \end{aligned}$$

which leads to the same results as before.

In this picture the positron runs forward in time and $\psi_{cf}(\mathrm{x}) = C\gamma^0\,\psi^*_f(\mathrm{x})$ is the wave function of the initial positron.

By the same calculation as leading to (7.12) we find the differential cross section

$$\left(\frac{d\bar{\sigma}}{d\Omega}\right)_{e^+} = \frac{2Z^2\alpha^2m^2}{|\mathbf{q}|^4}\sum_{\pm s_f,s_i}|\bar{v}(p_i,s_i)\gamma^0 v(p_f,s_f)|^2 \qquad (7.26)$$

Again the spin sum may be reduced to a trace, using the relation for positron spinors [see (3.9)]

so that

$$\sum_{\pm i} v_\alpha(p_i,s_i)\bar{v}_\beta(p_i,s_i) = -\left(\frac{-\not{p}_i + m}{2m}\right)_{\alpha\beta}$$

$$\left(\frac{d\bar{\sigma}}{d\Omega}\right)_{e^+} = \frac{Z^2\alpha^2}{2|\mathbf{q}|^4}\,\mathrm{Tr}\,\gamma^0(\not{p}_i - m)\gamma^0(\not{p}_f - m)$$

This is the same as (7.14) with *m* replaced by -*m*.

Since our answer for electron scattering was even in *m*, this confirms that the positron scattering cross section is equal to the electron scattering cross section in lowest order of α.

We could have anticipated this result from charge conjugation invariance.

We saw in Chap. 5 that to each solution of an electron in a potential A_μ there is a corresponding solution of the positron in the potential - A_μ that is, the scattering of an electron from the potential -e/4πr is the same as that of a positron from potential +e/4πr; however, since the calculated cross section depends only on e , the sign of A^μ does not matter.

This is not true for the e corrections which come from the product of the first- and second-order scattering amplitudes in Fig. 7.2 and which have opposite signs for electrons and positrons.

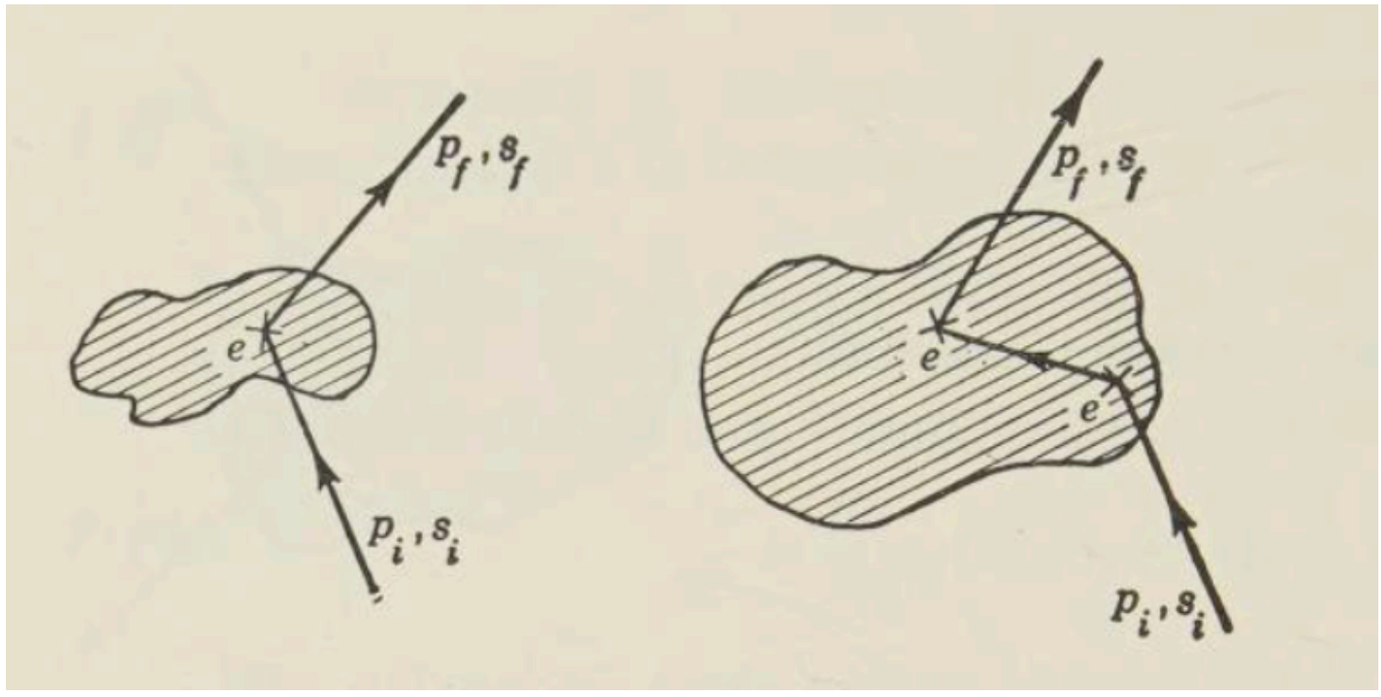

Fig. 7-2 Coulomb scattering of electrons.

We may also observe that the positron cross section is obtained from that of the electron by replacing $p_i \longleftrightarrow -p_f$; this is a general feature of positron theory called the "substitution rule," which is closely related to the propagator picture we have developed.

We shall see more examples of this rule as we go along.

7.4 Electron Scattering from a Dirac Proton

Now suppose that we consider electron scattering from a free, live proton instead of from a fixed Coulomb field. (Temporarily we shall treat the proton as a structureless Dirac particle.)

How would our result be modified?

If we know the current of the proton $J^\mu(x)$, we can calculate by Maxwell's equations the field it generates.

The *S* matrix (7.1) gives the amplitude for scattering of the electron in this field and leads to the transition rate and scattering cross section to lowest order in α if we follow the discussion of the preceding example.

Our first step is to find the electromagnetic field produced by the proton.

The potential is calculated from

$$\Box A^{\mu}(x) = J^{\mu}(x) \tag{7.27}$$

where for convenience we have chosen to work in the Lorentz gauge.

In order to integrate (7.27) for $A^{\mu}(x)$, we introduce a Green's function, or propagator, just as in the case of the electron.

The propagator $D_F(x - y)$ is defined by the equation

$$\Box D_F(x - y) = \delta^4(x - y) \tag{7.28}$$

and has the Fourier representation

$$D_F(x - y) = \int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot(x-y)} D_F(q^2)$$

where $D_F(q) = -1/q^2$ for $q^2 \neq 0$.

As in the Fermion case, we must decide what happens at the pole in D_F at q' = 0.

In analogy with the discussion of the Dirac propagator in Chap. 6 we append an infinitesimally small positive imaginary part to q^2; this is equivalent to adding a small negative imaginary mass as done in (6.46):

$$D_F(q^2) = \frac{-1}{q^2 + i\epsilon} \tag{7.29}$$

This prescription for handling the pole ensures only positive-frequency, or -energy, radiation propagating forward in time.

When we consider the scattering of radiation by matter, perhaps the refraction of light in passing through a bubble chamber, we must make certain that positive-frequency waves-representing positive-energy quanta—emerge unaccompanied by negative frequencies.

The Feynman propagator for electromagnetic radiation is then given by

$$D_F(x - y) = \int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot(x-y)} \left(\frac{-1}{q^2 + i\epsilon}\right) \tag{7.30}$$

and the solution for the potential according to (7.27) is

$$A^\mu(x) = \int d^4y \, D_F(x - y) J^\mu(y) \tag{7.31}$$

Introducing this into the *S*-matrix element (7.1) along with plane-wave electron solutions gives

$$S_{fi} = -i\int d^4x\, d^4y[e\bar{\psi}_f(x)\gamma_\mu\psi_i(x)]D_F(x-y)J^\mu(y) \tag{7.32}$$

Our problem now is to decide what to choose for the proton current J^μ(y).

A physically appealing choice suggested by the correspondence principle is the transition matrix element of the current

$$J^\mu(y) = e_p\bar{\psi}^p_f(y)\gamma^\mu\psi^p_i(y) \tag{7.33}$$

where $e_p = -e > 0$ is the proton charge and ψ^p_i(y) and $\bar{\psi}^p_f(y)$ represent initial and final plane-wave solutions for free protons.

They are of the same form as (7.2) and (7.3), differing only in the substitution of P_i and P_f for the initial and final proton four-momenta and of M for its mass.

With this choice the current is

$$J^\mu(y) = -\sqrt{\frac{M^2}{E_fE_i}}\frac{e}{V}e^{i(P_f-P_i)\cdot y}\,\bar{u}(P_f,S_f)\gamma^\mu u(P_i,S_i) \tag{7.34}$$

Equations (7.31) and (7.33) define what is usually referred to as the Møller potential of a Dirac proton.

In the nonrelativistic approximation, this choice of the transition matrix element of the current as the source of A^μ(x) was adopted by Heisenberg and applied to the electron transitions in his calculations of radiation from atoms with matrix mechanics.

Introducing (7.34) into (7.32) and using (7.30), we readily compute the integrals, and the *S*-matrix element is

$$S_{fi} = \frac{-ie^2}{V^2}(2\pi)^4\delta^4(P_f - P_i + p_f - p_i)\sqrt{\frac{m^2}{E_fE_i}}\sqrt{\frac{M^2}{E_fE_i}}$$
$$\times [\bar{u}(p_f,s_f)\gamma_\mu u(p_i,s_i)]\frac{1}{(p_f - p_i)^2 + i\epsilon}[\bar{u}(P_f,S_f)\gamma^\mu u(P_i,S_i)] \tag{7.35}$$

The symmetric form of this result in electron and proton variables bolsters our faith in the choice (7.33).

Had we started this calculation by applying (7.1) for the scattering amplitude of a proton in the field generated by the electron current and making the guess (7.33) for the electron current, we should have come to the same result.

Comparison with (7.5) shows that the difference between scattering an electron from a Coulomb field and from a proton is contained in replacement of the factors $Z\gamma^0/|\mathbf{q}|^2$ by

$$\gamma_\mu\left(\frac{-1}{q^2 + i\epsilon}\right)\sqrt{\frac{M^2}{E_fE_i}}\,\bar{u}(P_f,S_f)\gamma^\mu(P_i,S_i)$$

and V by

$$(2\pi)^3\delta^3(\mathbf{P}_f - \mathbf{P}_i + \mathbf{p}_f - \mathbf{p}_i)$$

expressing momentum conservation.

Equation (7.35) gives the electron-proton scattering amplitude to lowest order in a; higher-order interaction effects which distort the plane waves that were inserted in the currents have been ignored.

We may conveniently represent this expression by a Feynman graph, Fig. 7.3.

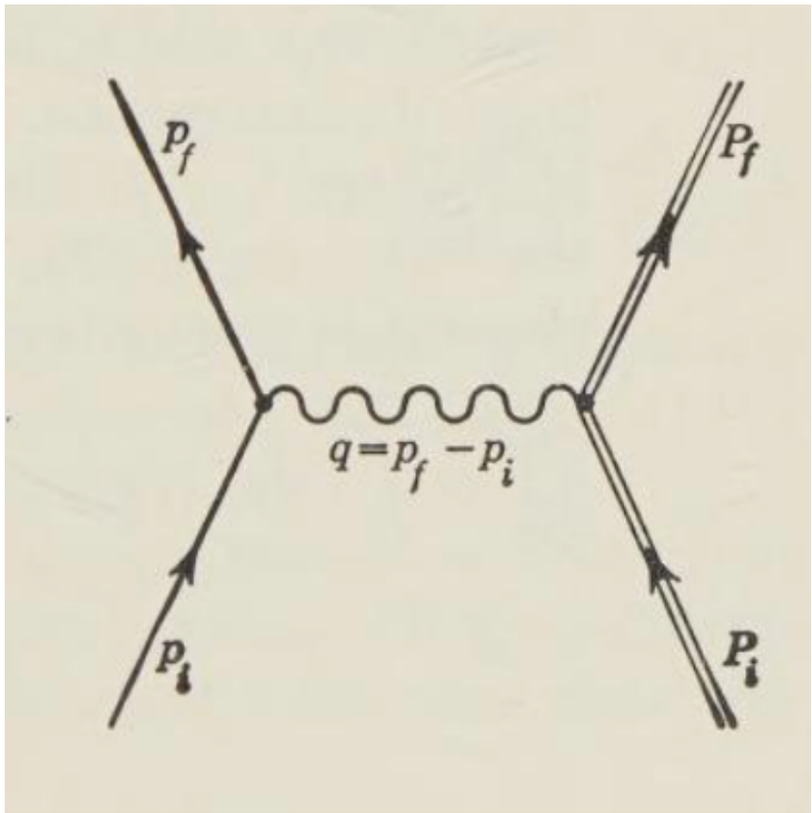

Fig. 7-3 Electron-proton scattering

A solid line with an arrow pointing toward positive time represents the electron and a double line the proton.

The wavy line represents the influence of the electromagnetic interaction, which is expressed in the matrix element by the reciprocal of the square of the momentum transfer, or the inverse of the d'Alembertian of (7.27) in momentum space.

We refer to this line as representing a “virtual photon" exchanging four-momentum $q = p_f - p_i = P_f - P_i$ between the electron and proton.

The amplitude for the virtual photon to propagate between the two currents is $-(q2 + i\epsilon)^{-1}$.

At the points— or vertices—on which the photon lands there operate factors $e\gamma^{\mu}$ sandwiched between spinors $\sqrt{m/E}\, u(p,s)$ representing the free, real incident and outgoing particles.

For each line and intersection of the graph there corresponds a unique factor in the S matrix.

In addition, S_{fi} always contains a four-dimensional 6 function expressing overall energy-momentum conservation.

Returning to Fig. 7.3 and Eq. (7.35) to calculate a cross section, we first form a transition rate per unit volume by dividing $|S_{fi}|^2$ by the time interval of observation T and by the spatial volume of the interaction region.

This gives

$$\left.\begin{aligned} w_{fi} = \frac{|S_{fi}|^2}{VT} &= (2\pi)^4\delta^4(P_f + p_f - P_i - p_i) \\ &\quad \times \frac{1}{V^4}\frac{m^2}{E_f E_i}\frac{M^2}{\mathsf{E}_f \mathsf{E}_i}\,|\mathfrak{M}_{fi}|^2 \\ \text{where} \\ \mathfrak{M}_{fi} &= [\bar{u}(p_f,s_f)\gamma_\mu u(p_i,s_i)]\,\frac{e^2}{q^2 + i\epsilon}\,[\bar{u}(P_f,S_f)\gamma^\mu u(P_i,S_i)] \end{aligned}\right\} \tag{7.36}$$

is a Lorentz invariant matrix element and will be called the invariant amplitude.

In forming ω_{fi} from (7.35) we have extended the treatment of the square of a δ function given below [see (7.6)] to include the space as well as time interval:

$$[(2\pi)^4\delta^4(P_f + p_f - P_i - p_i)]^2$$
$$= (2\pi)^4\delta^4(0)(2\pi)^4\delta^4(P_f + p_f - P_i - p_i) \rightarrow$$
$$\times VT(2\pi)^4\delta^4(P_f + p_f - P_i - p_i) \tag{7.37}$$

Next we divide the transition rate per unit volume by the flux of incident particles $|J_{\text{inc}}|$ and by the number of target particles per unit volume, which is just $1/V$ according to the normalization used in (7.2).

Finally, to get a physical cross section, we must sum over a given group of final states of the electron and proton corresponding to laboratory conditions for observing the process.

The number of final states of a specified spin in the momentum interval $d^3p_f\, d^3P_f$ is

$$V\frac{d^3p_f}{(2\pi)^3}\,V\frac{d^3P_f}{(2\pi)^3} \tag{7.38}$$

and so the cross section for transitions to final states in interval τ is

$$d\sigma = \int_\tau V^2\frac{d^3p_f}{(2\pi)^3}\frac{d^3P_f}{(2\pi)^3}\frac{V}{|J_{\text{inc}}|}\,w_{fi}$$
$$= \int_\tau \frac{d^3p_f}{(2\pi)^3}\frac{d^3P_f}{(2\pi)^3}\frac{mM}{E_fE_f}\frac{mM}{E_iE_i}\frac{(2\pi)^4\delta^4(P_f + p_f - P_i - p_i)}{|J_{\text{inc}}|V}\,|\mathfrak{M}_{fi}|^2 \tag{7.39}$$

We may still sum this over spin states of the final particles and average over the initial spins for unpolarized cross sections.

At this stage we can identify some features which will be common to all scattering processes.

The physics lies in $|\mathfrak{M}_{fi}|^2$, the square of the invariant amplitude.

There is a factor m/E for each external fermion line, that is, for each Dirac particle incident upon or emerging from the interaction.

The phase-space factor for each final particle is $d^3p_f/(2\pi)^3$.

We observe, then, that each final particle gives rise to the factor $(m/E)[d^3p/(2\pi)^3]$.

This forms a Lorentz invariant volume element in momentum space, as we see with the following identity:

$$\frac{d^3p}{2E} = \int_0^\infty dp_0\, \delta(p_\mu p^\mu - m^2)\, d^3p = \int_{-\infty}^{\infty} d^4p\, \delta(p_\mu p^\mu - m^2)\theta(p_0) \tag{7.40}$$

$$\theta(p_0) = \begin{cases} 1 & \text{for } p_0 > 0 \\ 0 & \text{for } p_0 < 0 \end{cases}$$

is the step function introduced in (6.18) and is Lorentz invariant provided p^μ is time-like, as is the case here.

Overall energy-momentum conservation in the interaction comes from the factor $(2\pi)^4\delta^4(P_f+p_f-P_i-p_i)$.

Finally, there appears the factor $1/V|J_{inc}|$; $|J_{inc}|$, the flux, is for collinear beams the number of particles per unit area which run by each other per unit time, that is

$$|J_{\text{inc}}| = \frac{|\mathbf{v}_i - \mathbf{V}_i|}{V}$$

When $V|J_{inc}|$ is combined with the normalization factor for the two incident particles it forms a Lorentz invariant expression:

$$\frac{mM}{E_i E_i |\mathbf{v}_i - \mathbf{V}_i|} = \frac{mM}{|\mathbf{p}_i| E_i + |\mathbf{P}_i| E_i} = \frac{mM}{\sqrt{(p_i \cdot P_i)^2 - m^2 M^2}} \tag{7.41}$$

This shows that the total cross section is invariant under Lorentz transformation along the direction of motion of the incident beams.

Equation (7.39) may now be written in the invariant form

$$d\sigma = \int_\tau \frac{mM}{\sqrt{(p_i \cdot P_i)^2 - m^2 M^2}} |\mathfrak{M}_{fi}|^2 (2\pi)^4 \delta^4(P_f - P_i + p_f - p_i) \times \frac{m\, d^3 p_f}{(2\pi)^3 E_f} \frac{M\, d^3 P_f}{(2\pi)^3 E_f} \tag{7.42}$$

These factors are of very general origin; hereafter we shall omit details of how they appear.

The box normalization volume V has now disappeared from our final result, which indeed could also be obtained from other normalization conventions not involving V.

Under circumstances not involving collinear beams it is more convenient to consider directly the number of events per unit time dN/ dt, which is found from (7.36) to be

$$\frac{dN}{dt} = \int d^3x\, \rho_e(\mathbf{x},t)\rho_p(\mathbf{x},t) \int \frac{mM}{E_i E_i} |\mathfrak{M}_{fi}|^2 (2\pi)^4 \delta^4(P_f - P_i + p_f - p_i) \times \frac{m\, d^3p_f}{(2\pi)^3 E_f} \frac{M\, d^3P_\zeta}{(2\pi)^3 E_f}$$

The $\rho_e(\mathbf{x},t)$ and $\rho_p(\mathbf{x},t)$ are the number of electrons and protons per unit volume, respectively, and they replace the two factors (1/V) which normalized (7.36) to one per volume V.

As written, the cross section (7.42) is for a transition from given initial spins to final spins of the electron and proton.

If the polarizations are not observed, we must average over initial and sum over final spins; then

$$|\overline{\mathfrak{M}}_{fi}|^2 = \frac{1}{4} \sum_{S_f,S_i,s_f,s_i} \left| \bar{u}(p_f,s_f)\gamma^\mu u(p_i,s_i) \frac{e^2}{q^2 + i\epsilon} \bar{u}(P_f,S_f)\gamma_\mu u(P_i,S_i) \right|^2$$

$$= \frac{1}{4} \mathrm{Tr} \frac{(\not{p}_f + m)}{2m} \gamma^\mu \frac{(\not{p}_i + m)}{2m} \gamma^\nu \mathrm{Tr} \frac{(\not{P}_f + M)}{2M} \gamma_\mu \frac{(\not{P}_i + M)}{2M} \gamma_\nu \frac{e^4}{(q^2)^2}$$

The calculation of the first trace using the trace theorems of Sec. 7.2 yields

$$\mathrm{Tr} \frac{(\not{p}_f + m)}{2m} \gamma^\mu \frac{(\not{p}_i + m)}{2m} \gamma^\nu = \frac{1}{4m^2} \mathrm{Tr}\, (\not{p}_f \gamma^\mu \not{p}_i \gamma^\nu + m^2 \gamma^\mu \gamma^\nu)$$

$$= \frac{1}{m^2} [p_f^\mu p_i^\nu + p_i^\mu p_f^\nu - g^{\mu\nu}(p_f \cdot p_i - m^2)]$$

For the unpolarized cross section we insert this into (7.42).

For a useful result we evaluate do in the laboratory frame of reference in which the initial proton is at rest and we let $p_f = (E', \mathbf{p}')$, $p_i = (E, \mathbf{p})$, and $P_i = (M,0)$.

It is convenient to apply (7.40) in carrying out the phase-space integrals for the differential cross section for the electron to emerge into a given solid angle $d\Omega'$ about an angle θ.

Writing $d^3p' = p'^2\, dp' d\Omega' = p'E'\, dE'\, d\Omega'$, we have

$$\frac{d\bar{\sigma}}{d\Omega'} = \frac{2}{|\mathbf{p}|}\int \frac{m^2 M p'\, dE'}{(2\pi)^2} |\overline{\mathfrak{M}}_{fi}|^2\, d^4P_f\, \delta(P_f^2 - M^2)\theta(P_f^0) \times \delta^4(P_f + p' - P_i - p)$$

$$= \frac{2}{|\mathbf{p}|}\frac{m^2 M}{4\pi^2}\int p'\, dE'\, |\overline{\mathfrak{M}}_{fi}|^2 \delta((p' - P_i - p)^2 - M^2) \times \theta(P_i^0 + E - E')$$

$$= \frac{m^2 M}{2\pi^2 p}\int_M^{M+E} p'\, dE'\, |\overline{\mathfrak{M}}_{fi}|^2 \delta(2m^2 - 2(E' - E)M - 2E'E + 2pp' \cos\theta) \tag{7.44}$$

$$= \frac{m^2 M}{4\pi^2}\frac{p'}{p}\frac{|\overline{\mathfrak{M}}_{fi}|^2}{M + E - (pE'/p')\cos\theta}$$

where the requirement of energy conservation coming from the ô function is

$$E'(M + E) - p'p\cos\theta = EM + m^2 \tag{7.45}$$

and in obtaining the final form we used $\int dx\, \delta(f(x)) = |df(x)/dx|^{-1}$.

For electrons of energy E << M, the proton rest energy, we resurrect our earlier result of scattering in a static Coulomb field by neglecting E/M << 1.

In this limit (7.44) reduces to the Mott cross section (7.21), since

$$\frac{d\bar{\sigma}}{d\Omega'} = \frac{m^2}{4\pi^2} |\overline{\mathfrak{M}}_{fi}|^2 \qquad \frac{E}{M} \ll 1$$

with E' = E from (7.45).

From (7.43)

$$|\overline{\mathfrak{M}}_{fi}|^2 = \frac{8\pi^2\alpha^2}{m^2 q^4} (2E^2 + m^2 - p_f \cdot p_i) \qquad \frac{E}{M} \ll 1$$

When the proton recoil becomes important, the electron may be treated as extreme relativistic and correction terms proportional to the electron mass neglected.

As is apparent from (7.43) and (7.44), there are no linear (or odd) terms in m/E, so that the correction terms are of order $(m/E)^2$:

$$\frac{d\bar{\sigma}}{d\Omega'} = \frac{m^2}{4\pi^2} \frac{E'/E}{1 + (2E/M)\sin^2(\theta/2)} |\overline{\mathfrak{M}}_{fi}|^2 \qquad \frac{m}{E} \ll 1$$

In computing $|\overline{\mathfrak{M}}_{fi}|^2$ from (7.43) it is convenient to reexpress P_f in terms of the electron recoil using energy-momentum conservation $P_f = P_i + p_i - p_f$.

Doing this gives

$$|\overline{\mathfrak{M}}_{fi}|^2 = \frac{8\pi^2\alpha^2}{m^2M^2q^4}\,[2P_i{\cdot}p_f\,P_i{\cdot}p_i + p_i{\cdot}p_f(P_i{\cdot}p_i - P_i{\cdot}p_f - M^2)]$$

$$= \frac{8\pi^2\alpha^2EE'}{m^2 16E^2E'^2\sin^4(\theta/2)}\left[2 + 2\sin^2\frac{\theta}{2}\left(-\frac{1}{2}\frac{q^2}{M^2} - 1\right)\right]$$

$$= \frac{\pi^2\alpha^2}{m^2EE'\sin^4(\theta/2)}\left(\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right) \qquad \frac{m}{E} \ll 1$$

where we recall $q^2 = (p_f - p_i)^2 = -4EE' \sin^2(\theta/2)$.

The differential cross section is thus

$$\frac{d\bar{\sigma}}{d\Omega'} = \frac{\alpha^2}{4E^2}\frac{\cos^2(\theta/2) - (q^2/2M^2)\sin^2(\theta/2)}{\sin^4(\theta/2)[1 + (2E/M)\sin^2(\theta/2)]} \qquad \frac{m}{E} \ll 1 \tag{7.46}$$

where use is made of (7.45) in the limit $m^2 \to 0$:

$$E'E(1 - \cos\theta) = M(E - E')$$

Equation (7.46) was derived under the assumption that the proton behaves just like a heavy electron of mass M.

This description is incomplete, however, since it fails to take into account the proton structure and anomalous magnetic moment, which are of mesonic origin.

A complete description of the proton leads to modifications in (7.46) which are important at large energies exceeding several hundred MeV.

We shall return to this point in a later discussion (see Chap. 10).

7.5 Higher-order Corrections to Electron-Proton Scattering

The calculation of electron-proton scattering we have carried out thus far is valid only to the lowest nonvanishing order in e^2.

To obtain the next higher order corrections in e we must return to (6.57) and consider the amplitude for second-order interactions between the electron and proton.

This is given by

$$S_{fi}^{(2)} = -ie^2 \int d^4x\, d^4y\, \bar{\psi}_f(x) \not{A}(x) S_F(x-y) \not{A}(y) \psi_i(y) \tag{7.47}$$

where the electromagnetic potential is again to be generated by the current of the proton.

To determine the form of this current, we look at the form of the second-order electron current interacting with $A_\mu(x)$ and $A_\mu(y)$ in (7.47).

As in the first-order calculation (7.35), S_{fi}^2 should be symmetric in form between this second-order electron current,

$$i\bar{\psi}_f(x)\gamma_\mu S_F(x-y)\gamma_\nu\psi_i(y) = \bar{\psi}_f(x)\gamma_\mu \Big\{ \sum_{n;p_0>0} \theta(x_0-y_0)\psi_n(x)\bar{\psi}_n(y) - \sum_{n;p_0<0} \theta(y_0-x_0)\psi_n(x)\bar{\psi}_n(y) \Big\} \gamma_\nu\psi_i(y)$$

and the proton current.

The factor i is necessary to make the current a superposition of products of two transition currents.

This suggests that we write, following (7.31),

$$A_\mu(x)A_\nu(y) = e_p^2 \int d^4w\, d^4z\, D_F(x-w)D_F(y-z)\bar{\psi}_f^p(w)\gamma_\mu \times \Big\{ \sum_{n;p_0>0} \theta(w_0-z_0)\psi_n^p(w)\bar{\psi}_n^p(z) - \sum_{n;p_0<0} \theta(z_0-w_0)\psi_n^p(w)\bar{\psi}_n^p(z) \Big\} \gamma_\nu \psi_i^p(z)$$

$$= ie_p^2 \int d^4w\, d^4z\, D_F(x-w)D_F(y-z) \times \bar{\psi}_f^p(w)\gamma_\mu S_F^p(w-z)\gamma_\nu \psi_i^p(z) \qquad (7.48)$$

The factors $D_F(x - w)$ $D_F(y - z)$ are the Feynman propagators for the two internal photon lines in Fig. 7.4.

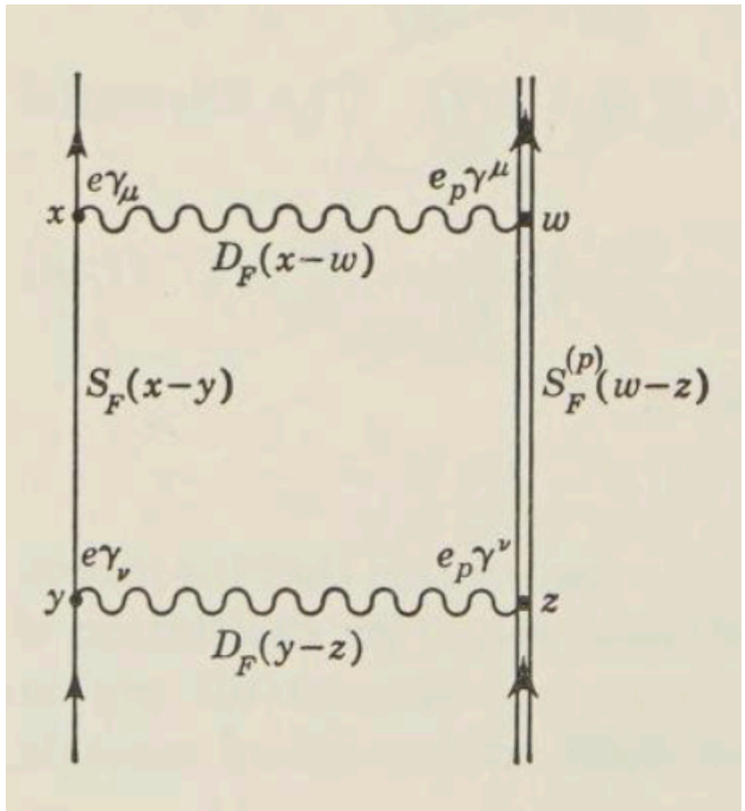

Fig. 7-4 Contribution to fourth-order electron-proton scattering.

They propagate between the electron and proton vertices, represented as dots, which contribute $e\gamma_\lambda$ and $e_p\gamma_\sigma$, respectively.

The internal electron and proton lines have the fermion propagators $S_F(x-y)$ and $S_F(w-z)$.

These factors are an example of the correspondence between Feynman graphs and S matrix elements written in coordinate space.

To complete the expression for $S_{fi}^{(2)}$, another term must be added to the proton current expressing the indistinguishability of the two photons.

The electron does not know whether the photon interacting at x originated at *w* or at z, and the possibility illustrated in Fig. 7.5 must be included along with that corresponding to Fig. 7.4.

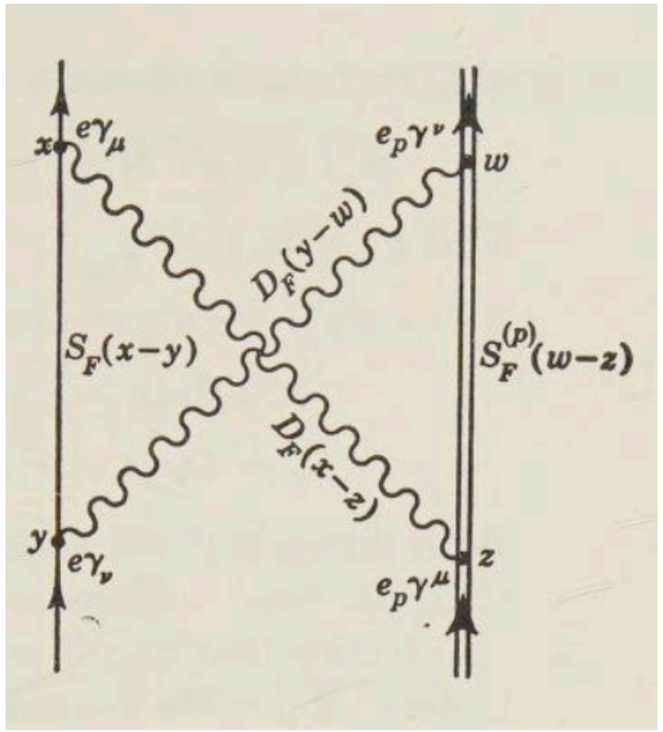

Fig. 7-5 Contribution to fourth-order electron-proton scattering.

The Feynman propagators for the photons assure that positive frequencies only are propagated forward in time.

However, all relative time orderings of the four points x, y, *w*, and z occur in the interaction and the photon at *w* could be equally well the first or second one emitted or absorbed by the electron.

In order to symmetrize the variables of the two indistinguishable exchanged photons, we add to (7.48) the term

$$ie_p^2 \int d^4w\, d^4z\, D_F(x-z) D_F(y-w) \bar{\psi}_f^p(w) \gamma_\nu S_F^p(w-z) \gamma_\mu \psi_i^p(z)$$

This gives in (7.47)

$$\begin{aligned} S_{fi}^{(2)} = e^2 e_p^2 \int d^4x\, d^4y\, d^4z\, d^4w\, \bar{\psi}_f(x) \gamma_\mu S_F(x-y) \gamma_\nu \psi_i(y) \\ \times \{ D_F(x-w) D_F(y-z) \bar{\psi}_f^p(w) \gamma^\mu S_F^p(w-z) \gamma^\nu \psi_i^p(z) \\ + D_F(x-z) D_F(y-w) \bar{\psi}_f^p(w) \gamma^\nu S_F^p(w-z) \gamma^\mu \psi_i^p(z) \} \end{aligned} \tag{7.49}$$

Both terms of (7.49) satisfy the same rules for writing down *S*-matrix elements from corresponding Feynman diagrams in coordinate space.

Notice that as yet our rules are somewhat unclear with regard to factors' of i.

We associated an overall factor (-i) with the *S* matrix and then a factor iS^p_F with the *proton* propagator.

In higher orders all the proton propagators S^p_F will be accompanied by an i for the same reason as in our present example.

We may make the rule regarding fermion propagators uniform if we write iS_F also for each electron line and associate with each $\not{A}$ a factor - i, that is,

$$-ie\not{A} S_F e\not{A} S_F \cdots e\not{A} = (-ie\not{A}) iS_F (-ie\not{A}) \cdots (-ie\not{A})$$

The overall (-i) is absorbed into the extra factor of $\not{A}$.

Thus to each γ_μ in the electron line we associate a factor - i.

We can also associate a — i with each proton vertex γ_μ if we compensate by writing an i in front of each photon propagator D_F.

Then we obtain a uniform rule for factors of i: — i for each vertex and i for each line in the graph.

We shall hereafter assume this rule.

For practical calculations it is useful to go over into momentum space, and so we Fourier-transform everything in sight in (7.49).

The wave functions of the external particles (that is, the incident and final electron and proton) are presumed to be plane waves as in (7.2), (7.3), and (7.34).

Then the first term of (7.49), for instance, becomes

$$\frac{e^4}{V^2}\int d^4x\, d^4y\, d^4z\, d^4w \sqrt{\frac{m^2}{E_f E_i}}\sqrt{\frac{M^2}{E_f E_i}}\frac{d^4q_1\, d^4q_2\, d^4p\, d^4P}{(2\pi)^4(2\pi)^4(2\pi)^4(2\pi)^4}$$
$$\times\frac{e^{-iq_1\cdot(x-w)}}{q_1^2+i\epsilon}\frac{e^{-iq_2\cdot(y-z)}}{q_2^2+i\epsilon}\left[e^{ip_f\cdot x}\bar{u}(p_f,s_f)\gamma_\mu\frac{e^{-ip\cdot(x-y)}}{\not{p}-m+i\epsilon}\gamma_\nu u(p_i,s_i)e^{-ip_i\cdot y}\right] \tag{7.50}$$
$$\times\left[e^{iP_f\cdot w}\bar{u}(P_f,S_f)\gamma^\mu\frac{e^{-iP\cdot(w-z)}}{\not{P}-M+i\epsilon}\gamma^\nu u(P_i,S_i)e^{-iP_i\cdot z}\right]$$

Carrying out the integrations over all coordinates yields a factor of $(2\pi)^4$ times a four-dimensional δ function for energy-momentum conservation at the vertex associated with each coordinate.

The momentum integrals may now be done; (7.50) then reduces to

$$\frac{e^4}{V^2}\sqrt{\frac{m^2}{E_f E_i}}\sqrt{\frac{M^2}{E_f E_i}}(2\pi)^4\delta^4(P_f+p_f-P_i-p_i)$$
$$\times\int\frac{d^4q_1}{(2\pi)^4}\frac{1}{q_1^2+i\epsilon}\frac{1}{(q-q_1)^2+i\epsilon}$$
$$\times\left[\bar{u}(p_f,s_f)\gamma_\mu\frac{1}{\not{p}_f-\not{q}_1-m+i\epsilon}\gamma_\nu u(p_i,s_i)\right]$$
$$\times\left[\bar{u}(P_f,S_f)\gamma^\mu\frac{1}{\not{P}_f+\not{q}_1-M+i\epsilon}\gamma^\nu u(P_i,S_i)\right] \qquad (7.51)$$

where $q \equiv p_f - p_i$ as earlier.

Notice the appearance of the overall energy-momentum conserving δ function and of the integral over the four-momentum q_i running around the closed loop in the momentum space Feynman diagram shown in Fig. 7.6.

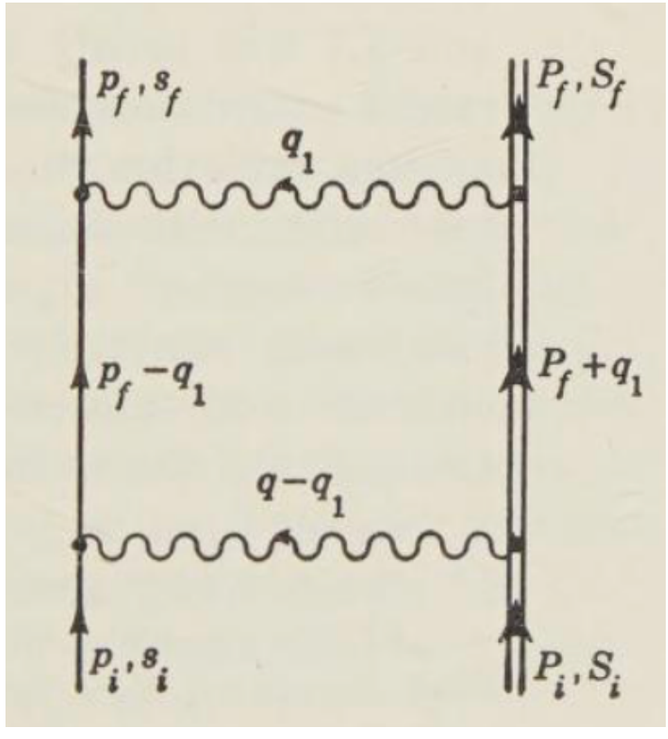

Fig. 7-6 Contribution to fourth-order electron-proton scattering.

There has been a systematic cancellation of factors of $(2\pi)^4$ except for the $(2\pi)^4$ we associate with the δ^4 and the compensating $(2\pi)^{-4}$ that goes with the integral over d^4q_i.

Other factors in (7.51) may be associated with the Feynman diagram in the same way as done for the lowest order calculation in Fig. 7.3.

Each vertex contributes a factor $-ie\gamma_\mu$ and each external particle a factor $\sqrt{m/E}$.

The new feature here is the factor $i[\not{p} - m + i\epsilon]^{-1}$, which is inserted in matrix order between vertices, coming from the propagator for the virtual intermediate fermion line.

With the aid of a little experience it is possible to associate forms such as (7.51) with given Feynman diagrams by inspection.

The Feynman graph of Fig. 7.7 in momentum space corresponds to

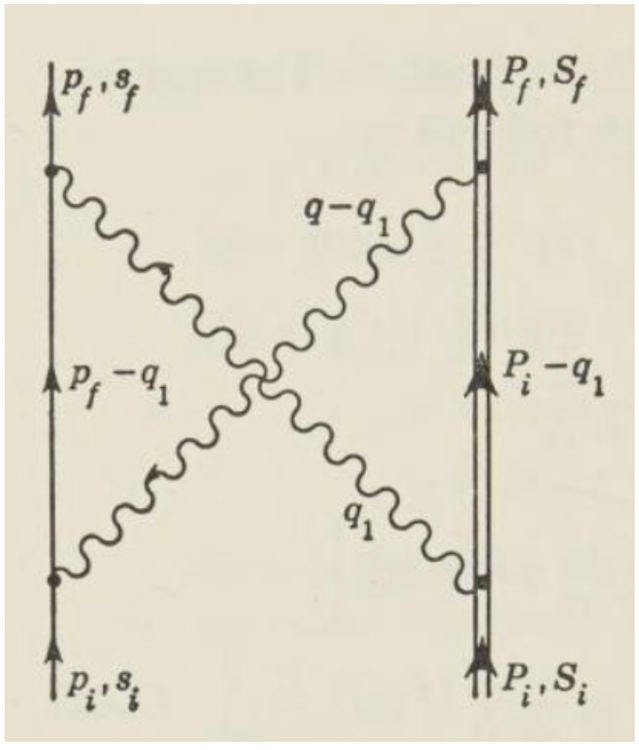

Fig. 7-7 Contribution to fourth-order electron-proton scattering.

Fig. 7.5, and the amplitude differs from (7.51) only in the replacement of the proton spin factor there by

$$\bar{u}(P_f,S_f)\gamma^\nu \frac{1}{\not{P}_i - \not{q}_1 - M + i\epsilon} \gamma^\mu u(P_i,S_i) \tag{7.52}$$

The rest of the evaluation of (7.51) and (7.52) is nontrivial; it involves a difficult four-dimensional integration.

For the static limit of the proton as a point Coulomb source it has been calculated by Dalitz.

Special difficulties arise in this example from the infinite range of the Coulomb interaction.

We shall not carry this calculation further here.

7.6 Bremsstrahlung

It is quite possible that one of the two quanta exchanged in Figs. 7.6 and 7.7 will satisfy the Einstein condition $q^2 = 0$.

In this case it might escape in transit between the electron and proton and emerge as free radiation, or bremsstrahlung.

To study the effect of this interaction with the radiation field on the scattering process, we again turn to heuristic arguments similar to those used by Schiff.

These yield with relatively little labor useful experimental results which coincide with answers obtained on the basis of a strict quantum treatment of the radiation as discussed in the companion volume.

The four-vector potential of a "photon" with momentum k_μ and polarization ϵ^μ is written as a plane wave

$$A^{\mu}(x;k) = \frac{\epsilon^{\mu}}{\sqrt{2kV}}\,(e^{-ik\cdot x} + e^{ik\cdot x}) \tag{7.53}$$

with $k_\mu k^\mu = 0$. ϵ^μ is the unit polarization vector and satisfies the transversality condition

$$\epsilon_\mu k^\mu = 0 \tag{7.54}$$

as required by the nature of the radiation field.

In a special Lorentz frame ϵ^μ is pure space-like, that is, $\epsilon^\mu = (0,e)$, with $\epsilon \cdot \epsilon = 1$; in an arbitrary Lorentz frame ϵ^μ is space-like and normalized to

$$\epsilon_\mu \epsilon^\mu = -1 \tag{7.55}$$

The normalization constants in (7.53) are so chosen that the energy in the wave A^μ is just $\omega = k_0 = |\mathbf{k}|$ as desired.

To verify this, we compute

$$U = \tfrac{1}{2}\int d^3x\,(\mathbf{E}^2 + \mathbf{B}^2) = \int d^3x\,\mathbf{B}^2$$

Since

$$\mathbf{B} = \operatorname{curl}\mathbf{A} = i\sqrt{k/2V}\,\hat{\mathbf{k}} \times \boldsymbol{\varepsilon}(e^{-ik\cdot x} - e^{ik\cdot x}) = \sqrt{2k/V}\,\hat{\mathbf{k}} \times \boldsymbol{\varepsilon}\sin k\cdot x$$

and, according to (7.54) and (7.55),

$$\hat{\mathbf{k}} \times \boldsymbol{\varepsilon} \cdot \hat{\mathbf{k}} \times \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} - (\hat{\mathbf{k}} \cdot \boldsymbol{\varepsilon})^2 = \boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} - (\epsilon^0)^2 = +1$$

we have

$$U = \frac{2k}{V} \int d^3x \sin^2 (\omega t - \mathbf{k} \cdot \mathbf{x}) = k = \omega$$

We consider the scattering amplitude describing the radiation of such a "photon" during a scattering event.

For simplicity we return to the static approximation, replacing the proton by a static Coulomb field as in (7.4), and calculate S_{fi} to lowest nonvanishing order in ϵ.

The Feynman diagrams for this process, shown in Fig. 7.8, correspond to a second-order process with one vertex for the interaction of the electron with the Coulomb field and one for the emission of the bremsstrahlung quantum. There can be no first-order emission of radiation by a free electron in the absence of the external field. This is kinematically forbidden, since it is impossible to conserve energy and momentum: $k^2 = 0 \neq (p_f - p_i)^2 < 0$.

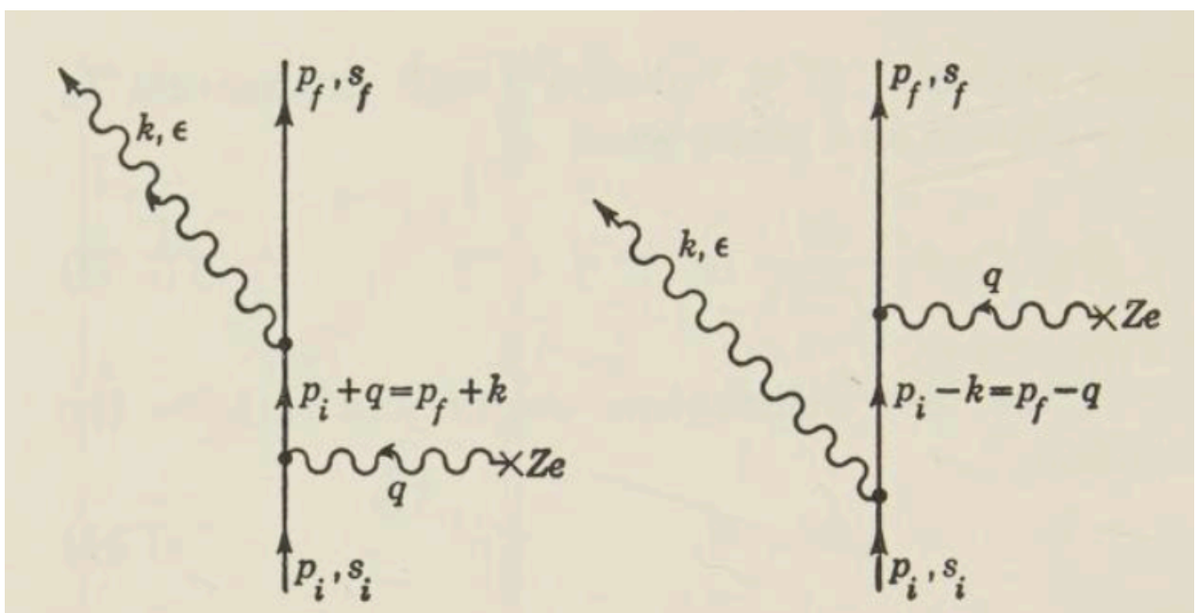

Fig. 7-8 Bremsstrahlung of an electron in a Coulomb field.

The second-order *S*-matrix element is

$$S_{fi} = e^2 \int d^4x\, d^4y\, \bar{\psi}_f(x) \{ -i\not{A}(x;k) i S_F(x-y)(-i\gamma^0) A_0^{\text{coul}}(y) + (-i\gamma^0) A_0^{\text{coul}}(x) i S_F(x-y)[-i\not{A}(y;k)] \} \psi_i(y) \tag{7.56}$$

where

$$A_0^{\text{coul}}(x) = \frac{-Ze}{4\pi|\mathbf{x}|}$$

and the two terms correspond to the two orderings of the vertices shown in Fig. 7.8.

As usual, it is convenient to transform (7.56) to momentum space by Fourier-expanding all factors and carrying out the coordinate integrations.

The result of this by-now-routine operation is

$$S_{fi} = \frac{-Ze^3}{V^{3/2}} 2\pi\delta(E_f + k - E_i) \frac{1}{\sqrt{2k}} \sqrt{\frac{m^2}{E_f E_i}} \frac{1}{|\mathbf{q}|^2} \times \bar{u}(p_f, s_f) \left[(-i\not{\epsilon}) \frac{i}{\not{p}_f + k - m} (-i\gamma_0) + (-i\gamma_0) \frac{i}{\not{p}_i - k - m} (-i\not{\epsilon}) \right] u(p_i, s_i) \tag{7.57}$$

where $\mathbf{q} = \mathbf{p}_f + \mathbf{k} - \mathbf{p}_i$.

There is an additional contribution coming from the first term in (7.53) for which the energy δ function is $\delta(E_i + k - E_f)$.

This term describes absorption of energy in the scattering process and does not contribute to the process of interest here, in which the incident electron gives up energy to the radiation field and emerges with $E_f = E_i - k < E_i$.

We note the new feature in (7.57) which is to be added to our growing lore in Feynman amplitudes: the factor (-iϵ) appears at the vertex where a free photon of polarization ϵ_μ is emitted, and $1/\sqrt{2kV}$ appears as the normalization factor for a photon "wave function."

The bremsstrahlung cross section can now be calculated from the *S* matrix (7.57).

We limit the derivation to $k \to 0$, that is, to the emission of a very soft photon.

The more general result, known as the Bethe-Heitler formula, appears in many texts.

In this limit the factor within the brackets in (7.57) can be approximated as follows:

$$-i\bar{u}(p_f,s_f)\left[\not{\epsilon}\frac{(\not{p}_f + k + m)\gamma_0}{(p_f + k)^2 - m^2} + \gamma_0\frac{(\not{p}_i - k + m)\not{\epsilon}}{(p_i - k)^2 - m^2}\right]u(p_i,s_i)$$

$$\cong -i\bar{u}(p_f,s_f)\left\{\frac{[2\epsilon\cdot p_f - (\not{p}_f - m)\not{\epsilon}]\gamma_0}{2k\cdot p_f} + \frac{\gamma_0[2\epsilon\cdot p_i - \not{\epsilon}(\not{p}_i - m)]}{-2k\cdot p_i}\right\}u(p_i,s_i)$$

$$= -i\bar{u}(p_f,s_f)\gamma_0 u(p_i,s_i)\left(\frac{\epsilon\cdot p_f}{k\cdot p_f} - \frac{\epsilon\cdot p_i}{k\cdot p_i}\right) \qquad (7.57)$$

where we have dropped a factor $k \to 0$ in the numerator and, in the last step, used the properties (3.9) of Dirac spinors.

The bremsstrahlung matrix element in the $k \to 0$ limit is just a multiple of the elastic scattering amplitude.

Proceeding to the cross section, we square S_{fi} of (7.57) and (7.58), divide by the flux $|\mathbf{v}|/V$ and by $2\pi\delta(0)$ to form a rate, and sum over final states $(V^2\, d^3k\, d^3p_f)/(2\pi)^6$ in the observed interval of phase space.

For a cross section for unpolarized electrons we should also sum over final and average over initial electron spin states.

We obtain

$$d\sigma = \frac{Z^2e^6m^2}{2k|\mathbf{v}_i|E_fE_i}\int\left(\frac{\epsilon\cdot p_f}{k\cdot p_f}-\frac{\epsilon\cdot p_i}{k\cdot p_i}\right)^2\frac{|\bar{u}(p_f,s_f)\gamma_0 u(p_i,s_i)|^2}{|\mathbf{q}|^4}$$
$$\times\, 2\pi\delta(E_f+k-E_i)\quad\frac{d^3k\,d^3p_f}{(2\pi)^6}$$

Identifying terms with the elastic scattering cross section in (7.11), we find

$$\frac{d\sigma}{d\Omega_f}=\left(\frac{d\sigma}{d\Omega_f}\right)_{\text{elastic}}\frac{e^2}{2k(2\pi)^3}\,k^2\,d\Omega_k\,dk\left(\frac{\epsilon\cdot p_f}{k\cdot p_f}-\frac{\epsilon\cdot p_i}{k\cdot p_i}\right)^2\theta(E_i-m-k) \tag{7.59}$$

This is the cross section for the electron to be observed in a solid angle $d\Omega_f$ and for a photon of polarization ϵ to emerge with $\mathbf{k}$ in the interval $d\Omega_k dk$.

Thus in the limit of soft-photon emission we can express the inelastic cross section as a multiple of the elastic cross section at the same energy and angle of scattered electron.

We observe in (7.59) that the photon energy spectrum behaves as dk/k and therefore the probability to emit a zero-energy photon is infinite.

This is the "infrared catastrophe."

It requires a careful analysis of actual experimental conditions for observing bremsstrahlung in order to remedy the difficulty presented by this infinity.

The crucial point is that every detecting apparatus has a finite energy resolution; and if it accepts inelastically scattered electrons in a finite energy interval including $k = 0$, it also accepts the elastically scattered ones.

For a consistent comparison with experiment we must therefore include both elastic and inelastic cross sections calculated to the same order in e^2.

Since the bremsstrahlung contribution (7.59) is of order e^2 relative to the elastic scattering, we must also include radiative corrections to $(d\sigma/d\Omega_f)_{\text{elastic}}$ to the same order e^2.

These arise from two types of terms.

There are those illustrated in Figs. 7.4 and 7.5 corresponding to a second-order scattering of the electron in the Coulomb field.

In addition, we must take into account the interaction of the electron with itself via the radiation field.

The Feynman graphs for these contributions in Fig. 7.9 show a virtual photon originating on the electron and boomeranging back instead of landing at the Coulomb source (or proton) as in Fig. 7.4.

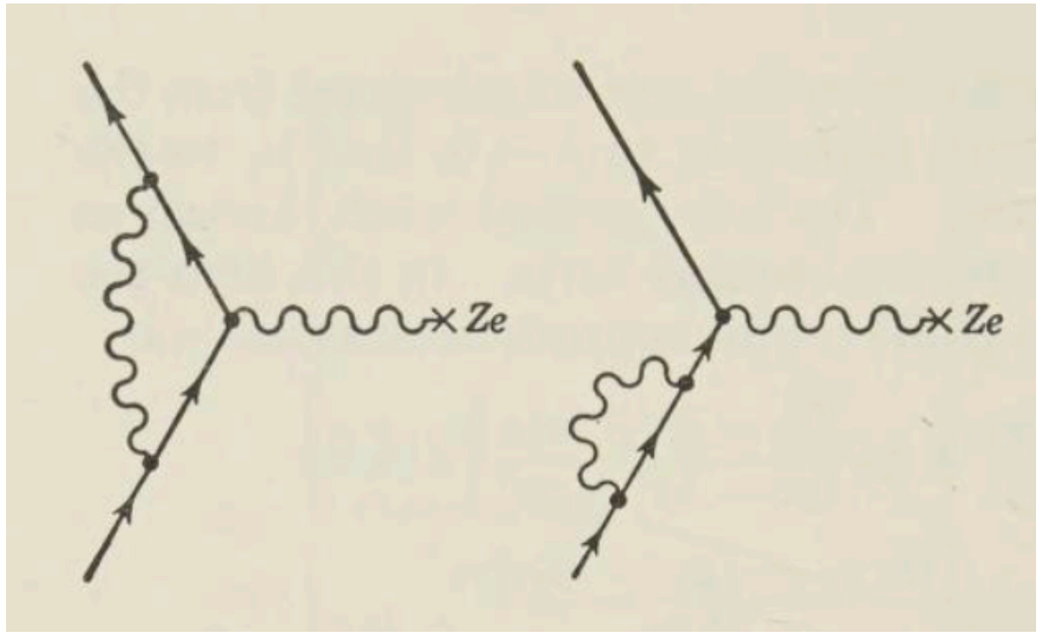

Fig. 7-9 Radiative corrections to Coulomb scattering.

The amplitude coming from these graphs contains a divergent term which precisely cancels the divergence in (7.59) at k = 0.

We shall arm ourselves with more training and experience before undertaking the delicate task of calculating it.

Before leaving (7.59), however, we shall evaluate the cross section for emission of soft bremsstrahlung in an interval ΔE which excludes the elastic limit.

We begin by summing over photon polarizations by using a very convenient technique due to Feynman.

Notice that the exact scattering matrix element in (7.57) vanishes if we replace the photon polarization ϵ by its four-momentum k^{μ}.

This property is also valid for the soft-photon approximation to the cross section (7.59).

It is a consequence of current conservation $\partial j_{\mu}(x)/\partial x_{\mu} = 0$ which has as its momentum space analogue $k_{\mu}j^{\mu}(x)(k) = 0$.

It is required of a gauge invariant calculation in electrodynamics, since the momentum space analogue of a change of gauge is $A^\mu(k) \rightarrow A^\mu(k) + k^\mu \Lambda(k)$ and the added factor proportional to k^μ must not change the final answer.

To take advantage of this result, we write

$$\epsilon_\mu \epsilon_\nu J^{\mu\nu} = \left(\frac{\epsilon \cdot p_f}{k \cdot p_f} - \frac{\epsilon \cdot p_i}{k \cdot p_i} \right)^2 \tag{7.60}$$

and orient the coordinates such that $k^\mu = (k^0, k^1, 0, 0)$, where $k^1 = k^0 = k$.

Since (7.60) is a scalar, we can evaluate it in an arbitrary Lorentz frame; in particular, we choose one in which the scalar potential vanishes, $A^0(x) = 0$ in (7.53).

In this system $\mathbf{A}(x)$ is transverse and the two independent transverse polarizations may be specified in accord with (7.54) and (7.55) by

and

$$\epsilon^{(1)} = (0,0,1,0)$$
$$\epsilon^{(2)} = (0,0,0,1)$$

Summing (7.60) over polarizations gives

$$\sum_{\text{pol}} \epsilon_\mu \epsilon_\nu J^{\mu\nu} = J^{22} + J^{33} = J^{00} - J^{11} - \sum_{\mu=0}^{3} J^\mu{}_\mu$$

Since $k_\mu J^{\mu\upsilon} = k_\upsilon J^{\mu\upsilon} = 0$, as observed above, it follows that $J^{0\upsilon} = J^{1\upsilon}$, $J^{\upsilon 0} = J^{\upsilon 1}$, and therefore $J^{\upsilon 0} = J^{1\upsilon}$; hence

$$\sum_{\text{pol}} \epsilon_\mu \epsilon_\nu J^{\mu\nu} = -J^{\mu}{}_{\mu}$$

where we again revert to the summation convention.

The polarization sum has now been replaced by a manifestly covariant expression which carries a quite general instruction based only upon current conservation: the result of performing a polarization sum is

$$\sum_{\text{pol}} [\epsilon_\mu(k) a^\mu(k)][\epsilon_\nu(k) b^\nu(k)] = -a\cdot b \tag{7.61}$$

provided a^μ and b^μ are conserved currents; that is, $k\cdot a(k) = k\cdot b(k) = 0$.

Applying (7.61) gives the bremsstrahlung cross section (7.59) summed over polarizations.

Integrating this over all photon emission angles and energies in the interval $0 < k_{\text{min}} \le k \le k_{\text{max}} << E_i$, we write

$$\begin{aligned}\frac{d\sigma}{d\Omega_f} &= \left(\frac{d\sigma}{d\Omega_f}\right)_{\text{elastic}} \frac{\alpha}{4\pi^2} \int_{k_{\text{min}}}^{k_{\text{max}}} k\, dk \int d\Omega_k \left[\frac{2p_f\cdot p_i}{k\cdot p_f\, k\cdot p_i} - \frac{m^2}{(k\cdot p_f)^2} - \frac{m^2}{(k\cdot p_i)^2}\right] \\ &= \left(\frac{d\sigma}{d\Omega_f}\right)_{\text{elastic}} \frac{\alpha}{\pi} \ln\frac{k_{\text{max}}}{k_{\text{min}}} \times \int \frac{d\Omega_k}{4\pi} \left[\frac{2(1-\beta_f\cdot\beta_i)}{(1-\hat{\mathbf{k}}\cdot\beta_f)(1-\hat{\mathbf{k}}\cdot\beta_i)}\right. \\ &\qquad \left. - \frac{m^2}{E_f^2(1-\hat{\mathbf{k}}\cdot\beta_f)^2} - \frac{m^2}{E_i^2(1-\hat{\mathbf{k}}\cdot\beta_i)^2}\right]\end{aligned} \tag{7.62}$$

where β_i and β_f are the initial and final electron velocities, respectively, with $\beta_i = \beta_f = \beta$ are the initial in the soft-photon limit and $\beta_i \cdot \beta_f = \beta^2 \cos\theta_f$ for scattering angle θ_f.

Integration of the last two terms in (7.62) is elementary.

$$\int \frac{d\Omega_k}{4\pi} \frac{m^2}{E^2(1 - \beta \cdot \hat{\mathbf{k}})^2} = \frac{m^2}{E^2} \int_{-1}^{1} \frac{1}{2} dz \frac{1}{(1 - \beta z)^2} = 1$$

The first integral is readily evaluated with the aid of yet another trick introduced and exploited by Feynman.

This consists of combining the two denominators with a parameter integral

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax + b(1 - x)]^2} \tag{7.63}$$

Applied here it gives

$$2 \int \frac{d\Omega_k}{4\pi} \frac{1 - \beta_f \cdot \beta_i}{(1 - \hat{\mathbf{k}} \cdot \beta_f)(1 - \hat{\mathbf{k}} \cdot \beta_i)}$$

$$= 2(1 - \beta_f \cdot \beta_i) \int_0^1 dx \int \frac{d\Omega_k}{4\pi} \frac{1}{[1 - \hat{\mathbf{k}} \cdot (\beta_f x + \beta_i(1 - x))]^2}$$

$$= 2(1 - \beta_f \cdot \beta_i) \int_0^1 dx \frac{1}{1 - |\beta_f x + \beta_i(1 - x)|^2}$$

$$= 2(1 - \beta^2 \cos\theta) \int_0^1 \frac{dx}{[1 - \beta^2 + 4\beta^2 \sin^2(\theta/2) x(1 - x)]}$$

$$\approx \begin{cases} 2\left(1+\frac{4}{3}\beta^2\sin^2\frac{\theta}{2}\right)+0(\beta^4) & \beta \ll 1 \\ 2\ln\left(\frac{-q^2}{m^2}\right)+0\left(\frac{m^2}{q^2}\right) & \frac{m^2}{q^2} \ll 1 \end{cases}$$

$$q^2 = (p_f - p_i)^2 \cong -4E^2\sin^2\frac{\theta}{2}$$

The soft bremsstrahlung cross section is then

$$\frac{d\sigma}{d\Omega_f} = \left(\frac{d\sigma}{d\Omega_f}\right)_{\text{elastic}} \frac{2\alpha}{\pi}\ln\frac{k_{\max}}{k_{\min}} \begin{cases} \frac{4}{3}\beta^2\sin^2\frac{\theta}{2}+0(\beta^4) & \text{NR} \\ \ln\left(\frac{-q^2}{m^2}\right)-1+0\left(\frac{m^2}{q^2}\right) & \text{ER} \end{cases} \tag{7.64}$$

where the two forms apply in the nonrelativistic (NR) and extreme relativistic limit (ER) of electron energies.

We must join (7.64) onto the radiative corrections to $(d\sigma/d\Omega)_{\text{elastic}}$ in order to obtain a finite result when $k_{\min} \to 0$.

7.7 Compton Scattering

We may now include second-order interactions with plane-wave fields (7.53).

Let (7.53) represent an incident photon which is absorbed by an electron at one vertex and

$$A'_\mu(x';k') = \frac{1}{\sqrt{2k'V}}\epsilon'_\mu(e^{-ik'\cdot x'}+e^{ik'\cdot x'}) \tag{7.65}$$

a final photon emitted at the second vertex.

This process, known as Compton scattering, conserves energy and momentum in scattering from a free electron according to the relation

$$k + p_i = k' + p_f \tag{7.66}$$

The second-order Compton amplitude differs from (7.56) by the substitution of $\gamma_\mu A^\mu(z;k')$ for $\gamma_0 A_0^{\text{coul}}(z)$.

Inserting (7.53) and (7.65) into the second-order S matrix and carrying out the Fourier transformation to momentum space gives

$$\begin{aligned} S_{fi}^{(\text{Compton})} = {} & \frac{e^2}{V^2}\sqrt{\frac{m^2}{E_f E_i}}\,\frac{1}{\sqrt{2k\cdot 2k'}}\,(2\pi)^4\delta^4(p_f + k' - p_i - k) \\ & \times \bar{u}(p_f,s_f)\left[(-i\not{\epsilon}')\,\frac{i}{\not{p}_i + \not{k} - m}\,(-i\not{\epsilon})\right. \\ & \left. + (-i\not{\epsilon})\,\frac{i}{\not{p}_i - \not{k}' - m}\,(-i\not{\epsilon}')\right] u(p_i,s_i) \end{aligned} \tag{7.67}$$

corresponding to the Feynman graphs of Fig. 7.10.

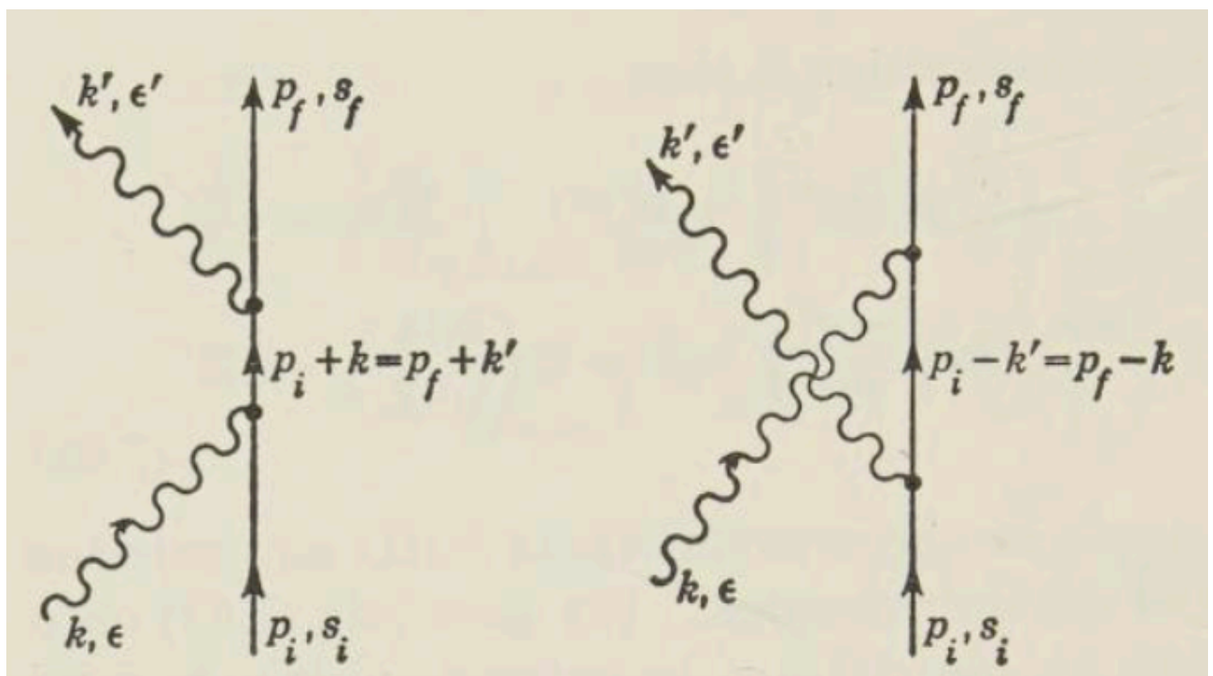

Fig. 7-10 Compton scattering.

Three additional terms with changed signs of k and/or k' have been dropped from (7.67).

Two of these vanish, since they lead to δ^4 functions corresponding to the energy-momentum conditions

$$p_i = p_f + k + k' \qquad \text{and} \qquad p_f = p_i + k + k'$$

which are impossible to satisfy.

These conditions correspond to the process of a free electron decaying to a final state of one free electron plus two photons, which is kinematically forbidden.

A third term with the two photons k and k' interchanged leads to the condition $k' + p_i = k + p_f$ corresponding to scattering of an incident photon with k' to a final one with k.

The kinematical constraint cannot be satisfied simultaneously with (7.66), which describes our present conditions, and so can be dropped.

The term retained in (7.67) comes from the first term of (7.53), $e^{ik\cdot x}$, which corresponds to absorption at x of a photon of four-momentum k^μ from the radiation field, and from the second term of (7.65), $e^{+ik'\cdot x'}$, representing emission at x' of a photon with four-momentum k′.

Notice that $S_{fi}^{\text{Compton)}}$ in (7.67) is symmetric under the substitution

$$k,\epsilon \leftrightarrow -k',\epsilon'$$

This is known as crossing symmetry, and it persists as an exact symmetry in all higher orders of interaction.

It plays an important role in particle physics.

Calculation of the Compton scattering cross section proceeds along the lines developed earlier; it is only the spinor algebra that presents a somewhat more formidable hurdle.

We form the cross section do by squaring the amplitude of (7.67), dividing by $(2\pi)^4\delta^4(0)$ to form a rate, dividing by an incident flux $|\mathbf{v}|/ V$ and by the number of target particles per unit volume 1/V, and summing over the phase space $[V^2/(2\pi)^6]\, d^3p_f\, d^3k'$:

$$d\sigma = \frac{e^4 m}{(2\pi)^2 2kE_i|\mathbf{v}|} \times \int \left| \bar{u}(p_f,s_f) \left(\not\epsilon' \frac{1}{\not p_i + \not k - m} \not\epsilon + \not\epsilon \frac{1}{\not p_i - \not k' - m} \not\epsilon' \right) u(p_i,s_i) \right|^2 \times \delta^4(p_f + k' - p_i - k) \frac{m\, d^3p_f}{E_f} \frac{d^3k'}{2k'} \tag{7.68}$$

The factor $m/kE_i|\mathbf{v}|$ is just (1/k) if we work in the laboratory frame in which the initial electron is at rest; and the integral over all recoil electrons and over photons scattered into a solid angle $d\Omega_{k'}$ about a laboratory angle θ gives, with the aid of (7.40),

$$\begin{aligned} d\Omega_{k'} \int \frac{k'^2\, dk'}{2k'} \int \frac{m\, d^3p_f}{E_f} \delta^4(p_f + k' - p_i - k) &= m\, d\Omega_{k'} \int_0^\infty k'\, dk'\, \delta([k + p_i - k']^2 - m^2)\theta(k + m - k') \\ &= m\, d\Omega_{k'} \int_0^{k+m} k'\, dk'\, \delta[2m(k - k') - 2kk'(1 - \cos\theta)] \\ &= \frac{k'^2}{2k}\, d\Omega_{k'} \end{aligned} \tag{7.69}$$

where k' and k are related by the Compton condition according to the δ function in (7.69):

$$k' = \frac{k}{1 + (k/m)(1 - \cos\theta)} = \frac{k}{1 + (2k/m)\sin^2(\theta/2)} \tag{7.70}$$

Equation (7.68) reduces now to

$$\frac{d\sigma}{d\Omega} = \alpha^2 \left(\frac{k'}{k}\right)^2 \left| \bar{u}(p_f,s_f) \left(\not{\epsilon}' \frac{1}{\not{p}_i + \not{k} - m} \not{\epsilon} + \not{\epsilon} \frac{1}{\not{p}_i - \not{k}' - m} \not{\epsilon}' \right) u(p_i,s_i) \right|^2 \tag{7.71}$$

which describes the differential cross section for electrons and photons polarized initially and finally.

We can further simplify the spinor matrix element considerably by choosing the special gauge in which both the initial and final photon are transversely polarized in the laboratory frame of reference; that is, we choose

$$\epsilon^\mu = (0,\boldsymbol{\varepsilon}) \qquad \text{with } \boldsymbol{\varepsilon}\cdot\mathbf{k} = 0$$
$$\epsilon^{\mu'} = (0,\boldsymbol{\varepsilon}') \qquad \text{with } \boldsymbol{\varepsilon}'\cdot\mathbf{k}' = 0$$

It follows from this that $\epsilon\cdot p_i = \epsilon'\cdot p_i = 0$ and the spinor factors collapse to

$$\bar{u}(p_f,s_f) \left(\not{\epsilon}' \frac{\not{p}_i + \not{k} + m}{2k\cdot p_i} \not{\epsilon} + \not{\epsilon} \frac{\not{p}_i - \not{k}' + m}{-2k'\cdot p_i} \not{\epsilon}' \right) u(p_i,s_i)$$
$$= -\bar{u}(p_f,s_f) \left(\frac{\not{\epsilon}'\not{\epsilon}\not{k}}{2k\cdot p_i} + \frac{\not{\epsilon}\not{\epsilon}'\not{k}'}{2k'\cdot p_i} \right) u(p_i,s_i)$$

where we anticommuted spin matrices and used the property of the Dirac spinors as before:

$$(\not{p}_i + m)\not{\epsilon} u(p_i,s_i) = \not{\epsilon}(-\not{p}_i + m)u(p_i,s_i) = 0.$$

Inserting this result into (7.71) and taking the sum over final spins s_f and the average over initial spins s_i for an unpolarized electron cross section, we find, with the aid of (7.13),

$$\begin{aligned}\frac{d\bar{\sigma}}{d\Omega} &= \frac{1}{2}\sum_{\pm s_i,s_f}\frac{d\sigma}{d\Omega} \\ &= \frac{\alpha^2}{2}\left(\frac{k'}{k}\right)^2 \mathrm{Tr}\,\frac{\not{p}_f + m}{2m}\left(\frac{\not{\epsilon}'\not{\epsilon}\not{k}}{2k\cdot p_i} + \frac{\not{\epsilon}\not{\epsilon}'\not{k}'}{2k'\cdot p_i}\right)\frac{\not{p}_i + m}{2m} \\ &\qquad \times\left(\frac{k\not{\epsilon}\not{\epsilon}'}{2k\cdot p_i} + \frac{k'\not{\epsilon}'\not{\epsilon}}{2k'\cdot p_i}\right)\end{aligned} \tag{7.72}$$

which presents us with traces containing up to eight γ matrices.

There are three distinct traces to be evaluated; the two cross terms with the denominator factor $(k\cdot p_i)\,(k'\cdot p_i)$ are identical according to trace theorem 6 and the cyclic properties of traces.

In reducing such complicated traces which contain the same vector more than once it is usually desirable to anticommute factors until the identical vectors are alongside each other; then the identity $\not{a}\not{a} = a^2$ removes two γ matrices.

Applying this technique here, we reduce the above traces as follows:

$$
\begin{aligned}
T_1 &= \mathrm{Tr}\,(\not{p}_f + m)\not{\epsilon}'\not{\epsilon}\not{k}(\not{p}_i + m)\not{k}\not{\epsilon}\not{\epsilon}' \\
&= \mathrm{Tr}\,\not{p}_f\not{\epsilon}'\not{\epsilon}\not{k}\not{p}_i\not{k}\not{\epsilon}\not{\epsilon}' \qquad \text{terms proportional to } m^2 \text{ vanishing because } k^2 = 0 \\
&= \mathrm{Tr}\,2k\cdot p_i\,\not{p}_f\not{\epsilon}'\not{\epsilon}\not{k}\not{\epsilon}\not{\epsilon}' = 2k\cdot p_i\,\mathrm{Tr}\,\not{p}_f\not{\epsilon}'\not{k}\not{\epsilon}' \\
&= 8k\cdot p_i(k\cdot p_f + 2k\cdot\epsilon'\; p_f\cdot\epsilon') \qquad \text{according to Theorem 3} \\
&= 8k\cdot p_i[k'\cdot p_i + 2(k\cdot\epsilon')^2]
\end{aligned}
$$

where we have used energy-momentum conservation $k + p_i = k' + p_f$, so that

$$
k'\cdot p_i = k\cdot p_f \qquad \text{and} \qquad \epsilon'\cdot p_f = \epsilon'\cdot k \tag{7.73}
$$

In the same way we evaluate

$$
T_2 = \mathrm{Tr}\,(\not{p}_f + m)\not{\epsilon}\not{\epsilon}'\not{k}'(\not{p}_i + m)\not{k}'\not{\epsilon}'\not{\epsilon}
$$

which differs from T_1 only in the substitution $\epsilon, k \longleftrightarrow \epsilon', -k'$, so that

$$
T_2 = 8k'\cdot p_i[k\cdot p_i - 2(k'\cdot\epsilon)^2]
$$

For the last trace we find, by the various tricks indicated,

$$
\begin{aligned}
T_3 &= \mathrm{Tr}\,(\not p_f + m)\not\epsilon'\not\epsilon k(\not p_i + m)k'\not\epsilon'\not\epsilon \\
&= \mathrm{Tr}\,(\not p_i + m)\not\epsilon'\not\epsilon k(\not p_i + m)k'\not\epsilon'\not\epsilon + \mathrm{Tr}\,(k - k')\not\epsilon'\not\epsilon k\not p_i k'\not\epsilon'\not\epsilon \\
&= \mathrm{Tr}\,(\not p_i + m)k(\not p_i + m)k'\not\epsilon'\not\epsilon\not\epsilon'\not\epsilon + 2k\cdot\epsilon'\,\mathrm{Tr}\,(-1)k\not p_i k'\not\epsilon' \\
&\qquad\qquad - 2k'\cdot\epsilon\,\mathrm{Tr}\,(-1)\not\epsilon k\not p_i k' \\
&= 2k\cdot p_i\,\mathrm{Tr}\,\not p_i k'\not\epsilon'\not\epsilon\not\epsilon'\not\epsilon - 8(k\cdot\epsilon')^2\,k'\cdot p_i + 8(k'\cdot\epsilon)^2\,k\cdot p_i \\
&= 8(k\cdot p_i)(k'\cdot p_i)[2(\epsilon'\cdot\epsilon)^2 - 1] - 8(k\cdot\epsilon')^2\,k'\cdot p_i + 8(k'\cdot\epsilon)^2\,k\cdot p_i
\end{aligned}
$$

Putting the traces all together in (7.72) we find the Klein-Nishina formula for Compton scattering

$$
\frac{d\bar\sigma}{d\Omega} = \frac{\alpha^2}{4m^2}\left(\frac{k'}{k}\right)^2\left[\frac{k'}{k} + \frac{k}{k'} + 4(\epsilon'\cdot\epsilon)^2 - 2\right] \tag{7.74}
$$

where k' and k are related through the scattering angle according to (7.70).

In the low-energy limit of k → 0 this reduces to the classical Thomson scattering

$$
\left(\frac{d\bar\sigma}{d\Omega}\right)_{k\to 0} = \frac{\alpha^2}{m^2}(\epsilon\cdot\epsilon')^2
$$

where

$$
\frac{\alpha}{m} = \frac{e^2}{4\pi mc^2} = 2.8\times 10^{-13}\ \mathrm{cm}
$$

is the classical electron radius.

As the scattering angle $\theta \to 0$, $k \to k'$ and we find the Thomson cross section to be valid for all energies in the forward direction.

Finally, we can sum over final photon polarizations ϵ and average over initial e for the unpolarized cross section.

The procedure is just the same one used in classical electrodynamics for scattering of light, and we borrow the result:

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{\alpha^2}{2m^2}\left(\frac{k'}{k}\right)^2\left(\frac{k'}{k} + \frac{k}{k'} - \sin^2\theta\right)$$

This is readily integrated over the photon solid angle to a total cross section.

Introducing $z = \cos\theta$ and using (7.70)

$$\bar{\sigma} = \frac{\pi\alpha^2}{m^2}\int_{-1}^{1} dz \left\{\frac{1}{[1 + (k/m)(1-z)]^3} + \frac{1}{[1 + (k/m)(1-z)]} - \frac{1 - z^2}{[1 + (k/m)(1-z)]^2}\right\} \tag{7.75}$$

The Thomson cross section again emerges at low energies:

$$\bar{\sigma} = \frac{8\pi}{3}\frac{\alpha^2}{m^2} \qquad \text{for } \frac{k}{m} \to 0$$

At high energies the total cross section is

$$\bar{\sigma} \cong \frac{\pi\alpha^2}{km}\left[\ln\frac{2k}{m} + \frac{1}{2} + O\left(\frac{m}{k}\ln\frac{k}{m}\right)\right]$$

with the dominant logarithm coming from the second term in (7.75).

7.8 Pair Annihilation into Gamma Rays

If we turn the Feynman diagrams for Compton scattering on their sides as in Fig. 7.11, we come upon another process of considerable physical interest.

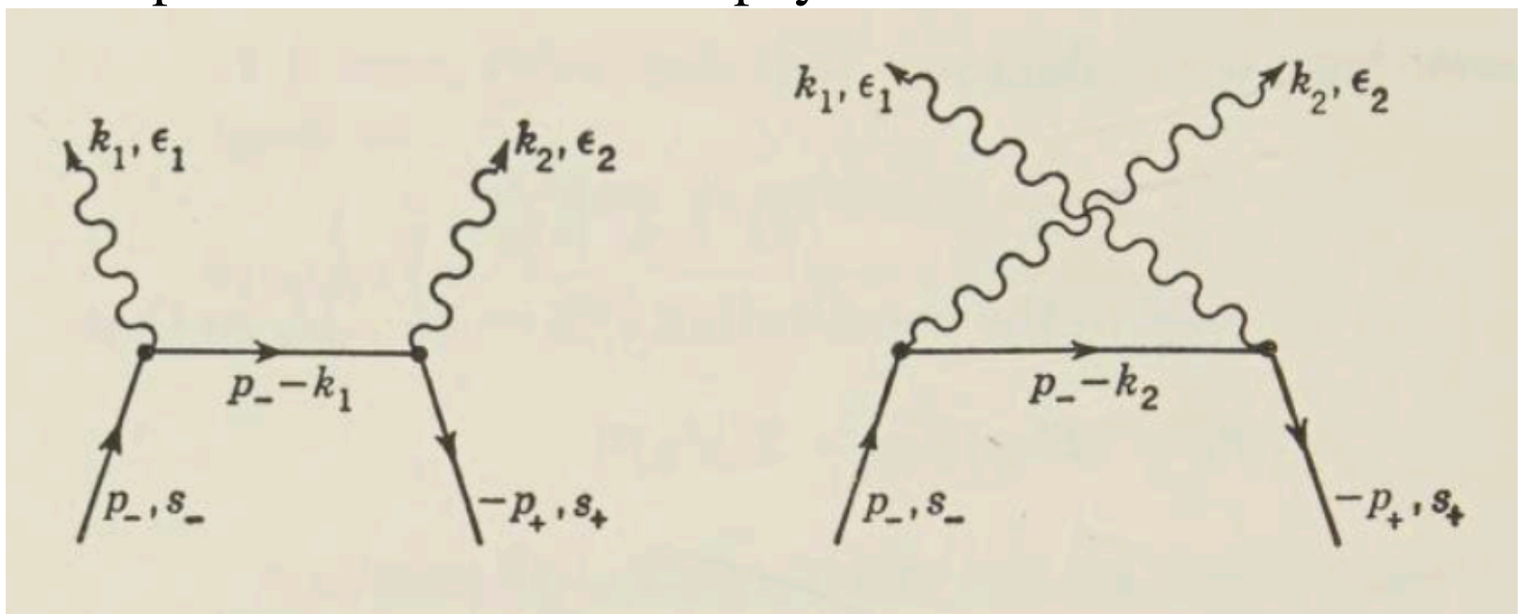

Fig. 7-11 Pair annihilation.

This is the annihilation of an electron-positron pair into two photons.

The relevant S-matrix element in momentum space, with kinematics as shown in the figure, is

$$S_{fi}^{(\text{pair})} = \frac{e^2}{V^2}\sqrt{\frac{m^2}{E_+E_-2k_12k_2}}\,(2\pi)^4\delta^4(k_1+k_2-p_+-p_-)\bar{v}(p_+,s_+)$$
$$\times\left[(-i\not{\epsilon}_2)\frac{i}{\not{p}_- - \not{k}_1 - m}(-i\not{\epsilon}_1) + (-i\not{\epsilon}_1)\frac{i}{\not{p}_- - \not{k}_2 - m}(-i\not{\epsilon}_2)\right] \tag{7.76}$$
$$\times\, u(p_-,s_-)$$

and is symmetric under the interchange of the two photons as required by the Bose statistics.

According to our description with Feynman propagators, this process corresponds to the picture of an electron produced in the past scattering into a state of negative energy $-p_+$ and propagating back into the past.

Along the way it produces two photons, that is, it gives up energy twice to the radiation field.

This is the lowest order in e^2 in which this process can occur, since pair annihilation to a single photon cannot conserve energy and momentum.

Both graphs must be included in order to ensure the required symmetry of $S_{fi}^{(\text{pair})}$ under interchange of the two photons.

Looking back at the Compton scattering amplitude, we notice a very strong similarity between (7.76) and (7.67).

Indeed, the substitutions

$$\begin{aligned} &\epsilon, k \leftrightarrow \epsilon_1, -k_1 \\ &\epsilon', k' \leftrightarrow \epsilon_2, +k_2 \\ &p_i, s_i \leftrightarrow p_-, s_- \\ &p_f, s_f \leftrightarrow -p_+, +s_+ \end{aligned} \tag{7.77}$$

transform the two amplitudes into each other.

This is an example of a general substitution rule which is valid to arbitrary orders and which relates processes of the type

$$A + B \to C + D$$

for instance, to the processes

$$A + \bar{C} \to \bar{B} + D$$

where $\bar{B}$ denotes the antiparticle to B, etc.

Another example of this substitution rule applies to the relation of the bremsstrahlung amplitude (7.56) corresponding to the graph of Fig. 7.8 with the amplitude of pair production in a Coulomb field as shown in Fig. 7.12

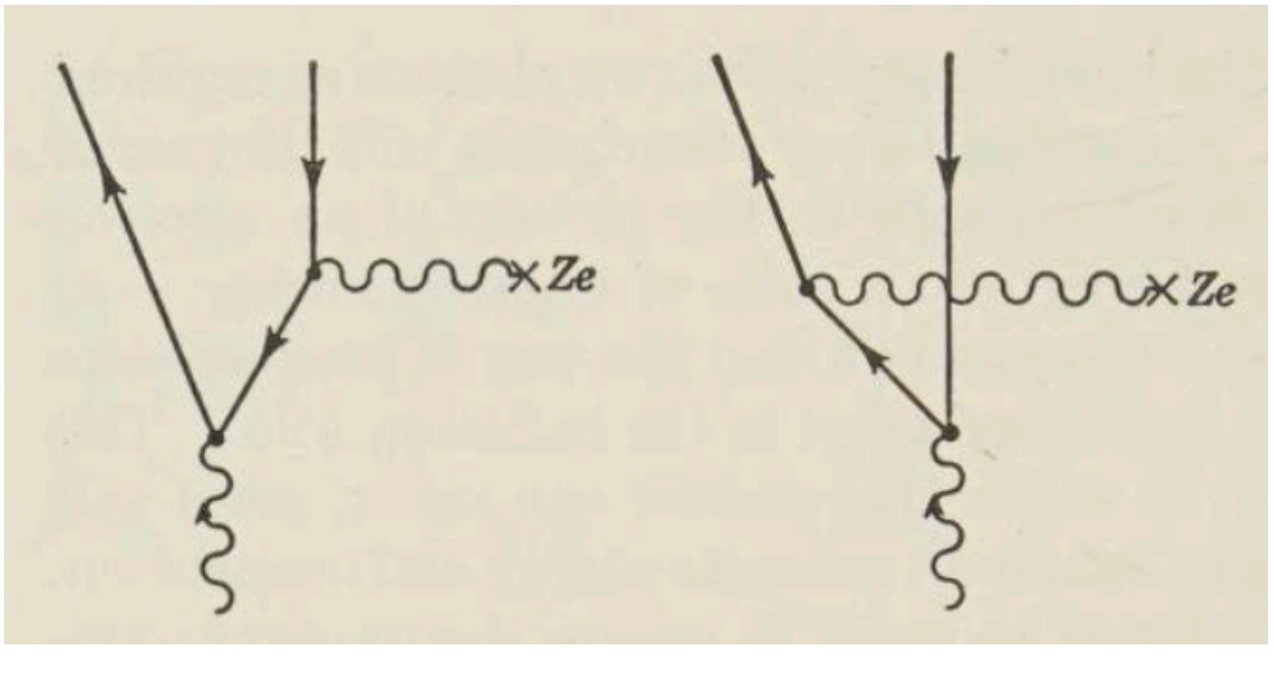

Fig. 7-12 Pair production in a Coulomb field.

By familiar steps we proceed from the matrix element (7.76) to a differential cross section.

For an unpolarized positron incident on an unpolarized electron at rest in the laboratory frame the result is

$$d\bar{\sigma} = \frac{e^4}{(2\pi)^2}\int \frac{m}{E_+\beta_+}\frac{(-1)}{4}\,\mathrm{Tr}\,\frac{m-\not{p}_+}{2m}\left(\frac{\not{\epsilon}_2 k_1 \not{\epsilon}_1}{2p_-\cdot k_1}+\frac{\not{\epsilon}_1 k_2 \not{\epsilon}_2}{2p_-\cdot k_2}\right)\frac{\not{p}_- + m}{2m}$$
$$\times\left(\frac{\not{\epsilon}_1 k_1 \not{\epsilon}_2}{2p_-\cdot k_1}+\frac{\not{\epsilon}_2 k_2 \not{\epsilon}_1}{2p_-\cdot k_2}\right)\frac{d^3k_1}{2k_1}\frac{d^3k_2}{2k_2}\,\delta^4(k_1+k_2-p_--p_+) \qquad (7.78)$$

where $\beta_+ = p_+/E_+$ is the incident positron velocity, the factor ¼ comes from the spin average over initial states of both the electron and for the laboratory frame of reference and is the same gauge, according to the substitution rule, as was applied to the Compton scattering calculation.

It has the virtue here that we can obtain the trace directly from (7.72) and (7.73), with the substitutions (7.77).

The only task remaining is to reduce the δ functions for laboratory kinematics:

$$\begin{aligned}
&\int \frac{d^3k_1}{2k_1}\frac{d^3k_2}{2k_2}\,\delta^4(k_1+k_2-p_+-p_-) \\
&\quad=\int_0^\infty \frac{1}{2}\,k_1\,dk_1\,d\Omega_{k_1}\,\delta[(p_++p_-)^2-2k_1\cdot(p_++p_-)]\theta(E_++E_--k_1) \\
&\quad=\frac{d\Omega_{k_1}}{2}\int_0^{E_++m} k_1\,dk_1\,\delta[2m^2+2mE_+-2k_1(m+E_+-p_+\cos\theta)] \\
&\quad=\frac{1}{4}\,\frac{m(m+E_+)}{[m+E_+-p_+\cos\theta]^2}\,d\Omega_{k_1}
\end{aligned} \tag{7.79}$$

Collecting the above trace and phase-space results in (7.78), we find the following result for the pair annihilation cross section in terms of laboratory energies and angles:

$$\begin{aligned}
\frac{d\bar{\sigma}}{d\Omega_{k_1}} &= -\frac{\alpha^2(m+E_+)}{8p_+[m+E_+-p_+\cos\theta]^2}\left[-\frac{k_2}{k_1}-\frac{k_1}{k_2}+4(\epsilon_1\cdot\epsilon_2)^2-2\right] \\
&= \frac{\alpha^2(m+E_+)}{8p_+(m+E_+-p_+\cos\theta)^2} \\
&\quad\times\left[\frac{E_+-p_+\cos\theta}{m}+\frac{m}{E_+-p_+\cos\theta}+2-4(\epsilon_1\cdot\epsilon_2)^2\right]
\end{aligned} \tag{7.80}$$

where

and

$$k_1 = \frac{m(m + E_+)}{m + E_+ - p_+ \cos\theta}$$

$$k_2 = m + E_+ - k_1 = \frac{E_+ - p_+ \cos\theta}{m} k_1$$

according to the kinematic constraints in (7.78) and (7.79).

For a total cross section we sum $d\sigma/d\Omega_k$, over final photon polarizations and integrate over the solid angle $d\Omega_k$.

This latter step requires care, since the final state contains two identical particles.

Equation (7.80) tells us that one of the photons emerges in dig; because of their indistinguishability, this can be either of the two photons.

If we were to integrate $d\sigma/d\Omega_k$, over the entire 4π solid angle, we would be counting each distinguishable state exactly twice, that is, we would evidently be counting two photons per scattering event.

We should therefore take one-half of this integral in forming a total cross section

$$\bar{\sigma} = \frac{1}{2} \int \frac{d\bar{\sigma}}{d\Omega_{k_1}} d\Omega_{k_1} \tag{7.81}$$

The low- and high-energy approximations for the total cross section are now readily obtained from (7.81) and (7.80): as $p_+ \to 0$, $\mathbf{k}_1 \to -\mathbf{k}_2$ and the polarization average of $(\epsilon_1 \cdot \epsilon_2)^2 \to \frac{1}{2}$; hence

$$\bar{\sigma} = \frac{\alpha^2 \pi}{\beta_+ m^2}[1 + O(\beta_+^2)] \qquad \beta_+ \ll 1$$

In the extreme relativistic limit we find

$$\bar{\sigma} = \frac{\pi\alpha^2}{mE_+}\left[\ln\frac{2E_+}{m} - 1 + O\left(\frac{m}{E_+}\ln\frac{E_+}{m}\right) + \cdots\right]$$

where the first two terms of (7.80) contribute equally to the leading order contribution and the sum of the last two terms there is smaller by a factor m/E_+.

These results were first obtained in 1930 by Dirac.

7.9 Electron-Electron and Electron-Positron Scattering

Electron-electron scattering is handled in a manner very similar to electron-proton scattering.

However, there is an additional graph which arises because of the identity of the electrons.

The two graphs for this process are shown in Fig. 7.13, which also defines the relevant kinematics.

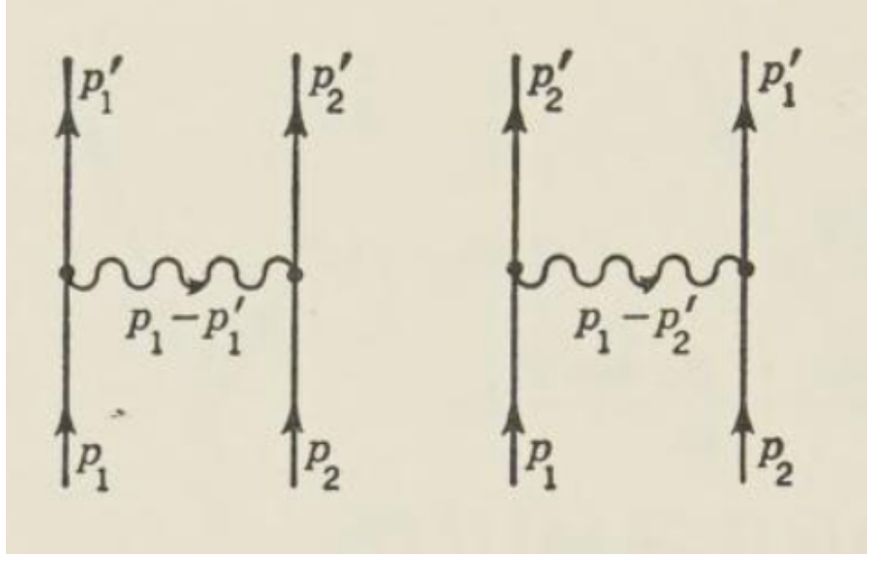

Fig. 7-13 Electron-electron scattering.

The corresponding scattering amplitude is, with spin labels suppressed and with factors as in (7.35) for the electron-proton amplitude,

$$S_{fi}^{M} = \frac{-e^2m^2}{V^2\sqrt{E_1E_2E_1'E_2'}}\left[i\frac{\bar{u}(p_1')(-i\gamma_\mu)u(p_1)\bar{u}(p_2')(-i\gamma^\mu)u(p_2)}{(p_1-p_1')^2} - i\frac{\bar{u}(p_1')(-i\gamma_\mu)u(p_2)\bar{u}(p_2')(-i\gamma^\mu)u(p_1)}{(p_1-p_2')^2}\right] \times (2\pi)^4\delta^4(p_1'+p_2'-p_1-p_2) \tag{7.82}$$

The relative minus sign between the direct and exchange terms is due to the Fermi statistics, which requires the amplitude to be antisymmetric under interchange of the two final electrons.

It is also anti-symmetric under interchange of the two initial electrons as required by the statistics.

By a similar argument the scattering amplitude to or from a state containing two identical Bose particles must be symmetric under their interchange.

We observed this to be the case in the amplitude (7.76) for the pair annihilation process of Fig. 7.11.

No additional normalization factors, such as $1/\sqrt{2}$ or 2, were introduced into (7.82) when the exchange term was added.

The rules for constructing differential cross sections from S_{fi} are not altered by the presence of identical particles in the initial or final states.

We must only take care that the factor ½ of (7.81) is included in integrating for a total cross section when two identical particles appear in the final state.

No special factors appear for identical particles in the initial state, since the incident flux is unchanged.

Electron-electron scattering provides a clear and simple example of the correctness of this rule.

The second, or exchange, term in (7.82) can be neglected for scattering in the forward direction with a small momentum transfer $(p_1 - p'_1)^2$.

In this limit the scattering reduces to the correct Coulomb amplitude, a result which is independent of statistics.

An expression for the differential cross section for scattering of unpolarized electrons can be constructed from (7.82) in the usual way.

In the center-of-mass frame it is

$$
\begin{aligned}
d\bar{\sigma} = \frac{e^4 m^4}{(2\pi)^2 E^4 (2\beta)} \int \frac{1}{4} \Bigg\{ & \frac{1}{[(p'_1 - p_1)^2]^2} \\
& \times \mathrm{Tr}\left(\frac{\not{p}'_1 + m}{2m} \gamma_\mu \frac{\not{p}_1 + m}{2m} \gamma_\nu \right) \mathrm{Tr}\left(\frac{\not{p}'_2 + m}{2m} \gamma^\mu \frac{\not{p}_2 + m}{2m} \gamma^\nu \right) \\
& - \frac{1}{(p'_1 - p_1)^2 (p'_2 - p_1)^2} \\
& \times \mathrm{Tr}\left[\frac{\not{p}'_1 + m}{2m} \gamma_\mu \frac{\not{p}_1 + m}{2m} \gamma_\nu \frac{(\not{p}'_2 + m)}{2m} \gamma^\mu \frac{(\not{p}_2 + m)}{2m} \gamma^\nu \right] \\
& + (p'_1 \leftrightarrow p'_2) \Bigg\} \delta^4(p'_1 + p'_2 - p_1 - p_2)\, d^3p'_1\, d^3p'_2
\end{aligned}
\tag{7.83}
$$

where E is the center-of-mass energy of each electron and β its velocity.

Notice the factor 2β for the relative velocity of the two initial electrons in the center-of-mass system.

For relativistic energies this approaches twice the speed of light.

There is no contradiction here with special relativity, and indeed the velocity of one electron viewed from the other never exceeds that of light.

The symbol $(p'_1 \leftrightarrow p'_2)$ above stands for the two additional terms that are obtained from the first two in $d\bar{\sigma}$ by interchanging p'_1 and p'_2.

The interference term in (7.83) between direct and exchange scattering contains only one very long trace.

A pictorial way of representing the squares of matrix elements in terms of closed loops as in Fig. 7.14 shows the difference of the direct term with two loops and two traces and the interference term with one.

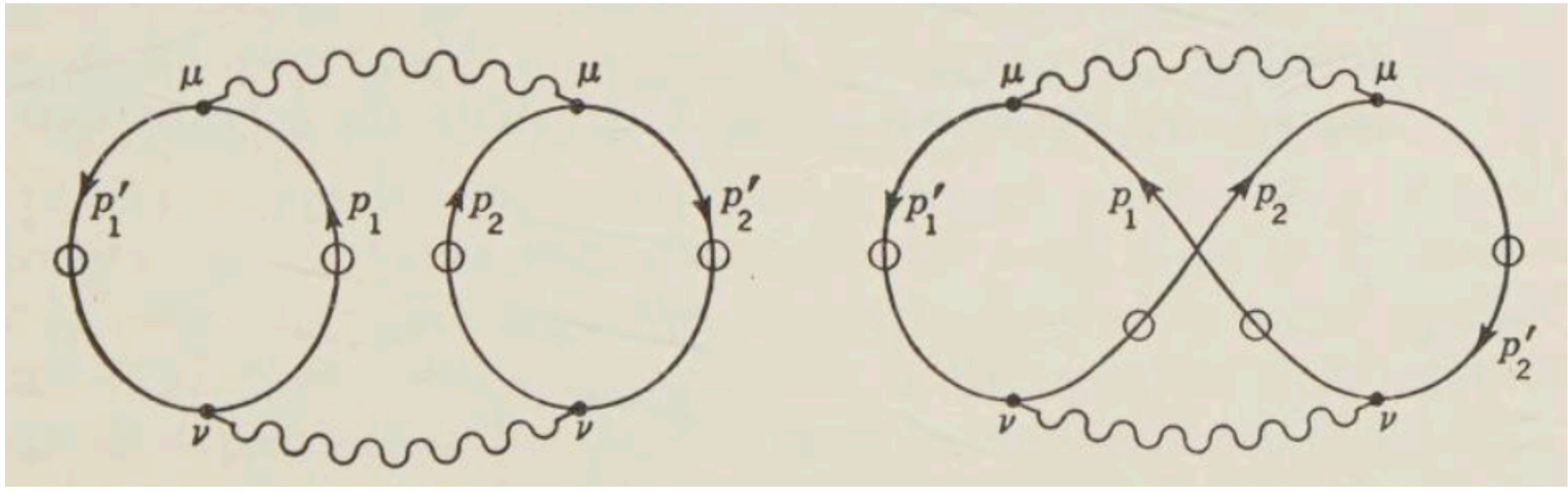

Fig. 7-14 Graphical representation of the squares of matrix elements for electron-electron scattering. The circle on the line indicates that no factor of $(p^2 - m^2)^{-1}$ appears.

These diagrams are useful at times for keeping the order of indices μ, υ and of spinor factors straight.

The circle on the line is a reminder that no denominator $(p^2 - m^2)^{-1}$ appears.

Evaluation of the traces in (7.83) can be carried out by applying the trace theorems following (7.14).

In particular, Theorem 5 is very useful in reducing the trace of eight γ matrices in the interference term.

Simplifying the result to relativistic energies E >> m and neglecting terms proportional to m^2, we find for example

$$\mathrm{Tr}\,(\not p_1'\gamma_\mu \not p_1 \gamma_\nu \not p_2'\gamma^\mu \not p_2 \gamma^\nu) = -2\,\mathrm{Tr}\,(\not p_1'\gamma_\mu \not p_1 \not p_2 \gamma^\mu \not p_2') = -8 p_1\cdot p_2\,\mathrm{Tr}\,\not p_1' \not p_2'$$
$$= -32(p_1\cdot p_2)(p_1'\cdot p_2')$$

In terms of center-of-mass energy E and scattering angle θ the differential cross section is

$$\left(\frac{d\bar\sigma}{d\Omega}\right)_M = \frac{\alpha^2}{8E^2}\left[\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)} + \frac{2}{\sin^2(\theta/2)\cos^2(\theta/2)} + \frac{1+\sin^4(\theta/2)}{\cos^4(\theta/2)}\right] \tag{7.84}$$

where the first and third terms are the squares of the matrix elements for the two graphs of Fig. 7.13, and the second term is the interference contribution.

In obtaining this result we used the kinematical identities $p_1 \cdot p_2 = p'_1 \cdot p'_2 = 2E^2$; $p_1 \cdot p'_2 = p'_1 \cdot p_2 = 2E^2 \cos^2(\theta/2)$; and

$$p_1\cdot p_1' = p_2\cdot p_2' = 2E^2 \sin^2\frac{\theta}{2}$$

which are valid when terms in m^2 are neglected.

Equation (7.84) is the high-energy limit of the Møller formula in the center-of-mass frame.

Turning next to electron-positron scattering, we invoke the substitution rule as in (7.77) to obtain the cross section from the Møller formula.

The Feynman diagrams for this process, known as Bhabha scattering, are shown in Fig. 7.15.

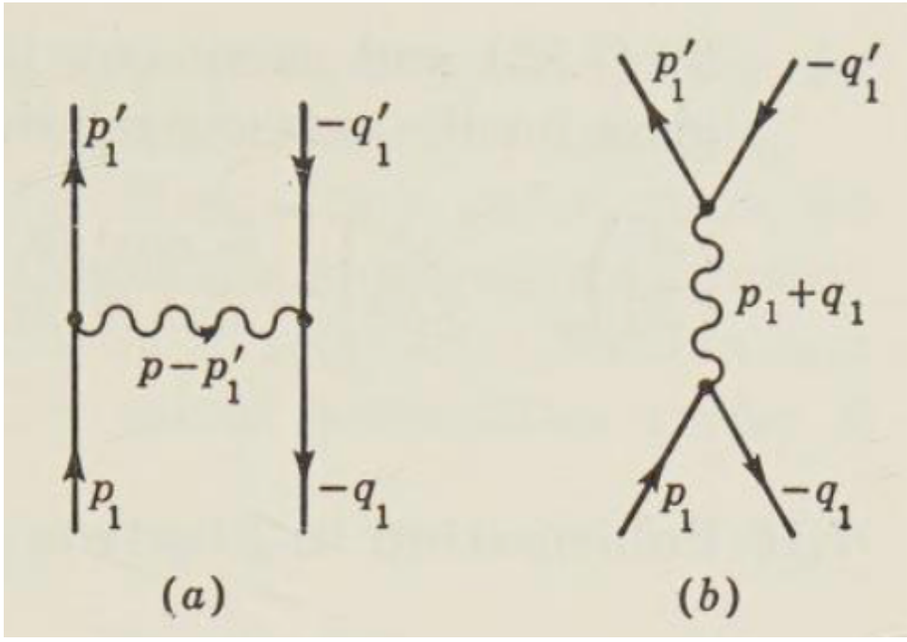

Fig. 7-15 Electron-positron scattering.

With the substitutions

$$\begin{aligned} p_1 &\leftrightarrow p_1 \\ p_1' &\leftrightarrow p_1' \\ p_2 &\leftrightarrow -q_1' \\ p_2' &\leftrightarrow -q_1 \end{aligned} \tag{7.85}$$

and upon changing the overall sign in accordance with (6.56), we find the Bhabha amplitude

$$S_{fi}^{B} = \frac{e^2 m^2}{V^2} \frac{1}{\sqrt{E_{p_1} E_{p_1'} E_{q_1} E_{q_1'}}}$$
$$\times \left[i \frac{\bar{u}(p_1')(-i\gamma_\mu)u(p_1)\bar{v}(q_1)(-i\gamma^\mu)v(q_1')}{(p_1 - p_1')^2} \right.$$
$$\left. - i \frac{\bar{u}(p_1')(-i\gamma_\mu)v(q_1')\bar{v}(q_1)(-i\gamma^\mu)u(p_1)}{(p_1 + q_1)^2} \right]$$
$$\times (2\pi)^4 \delta^4(p_1' + q_1' - p_1 - q_1) \tag{7.86}$$

The first term represents direct electron-positron scattering in analogy with the first scattering term in the electron-electron amplitude (7.82).

The annihilation term corresponds to the second or exchange scattering term there.

The relative minus sign between these two terms comes from applying the substitution rule to (7.82).

The antisymmetry of (7.82) under the interchange of the two final, or initial, electrons becomes in (7.86) an antisymmetry between an incoming positive-energy electron (p_1) and an "incoming" negative-energy electron ($-q'_1$) running backward in time, or between outgoing electrons p'_1 and $-q_1$.

To understand this antisymmetry in the language of hole theory, we note that at a time prior to the interaction the initial state contains an electron p_1 of positive energy and, in addition, a negative-energy sea filled except for the hole in the negative-energy state $-q_1$.

In particular, a negative-energy electron is present in the state $-q'_1$ and therefore, by the Fermi statistics, the initial state must be antisymmetric under the interchange of $p_1 \longleftrightarrow -q'_1$; a similar argument applies to the final state.

Antisymmetrization is also required with respect to all other particles in the sea, but these do not appear in (7.86) and therefore do not change its form.

In order to obtain the cross section for electron-positron scattering in the center-of-mass system, we apply the substitution rules (7.85) to (7.83) and carry out the traces as for Møller scattering.

This gives for the extreme relativistic limit

$$\left(\frac{d\bar{\sigma}}{d\Omega}\right)_B = \frac{\alpha^2}{8E^2}\left[\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)} - \frac{2\cos^4(\theta/2)}{\sin^2(\theta/2)} + \frac{(1+\cos^2\theta)}{2}\right] \tag{7.87}$$

7.10 Polarization in Electron Scattering

As a practical application of the spin projection operators developed in Chap. 3 we return to the Mott cross section in Sec. 7.2 and consider the calculation for an incident beam of polarized electrons.

As will be discussed in Chap. 10, the decay electrons from μ mesons are polarized with their spins pointing antiparallel to their direction of motion.

The Coulomb scattering of an electron incident with momentum p_i and spin s_i, where $s_i \cdot p_i = 0$, and summed over final spin states $\pm s_f$ is given by [see Eq. (7.11)]

$$\frac{d\sigma}{d\Omega} = \frac{4Z^2\alpha^2m^2}{|q|^4}\sum_{\pm s_f}|\bar{u}(p_f,s_f)\gamma^0u(p_i,s_i)|^2 \tag{7.88}$$

In order to take advantage of trace techniques in evaluating (7.88), we introduce the spin projection operator, using (3.19) and (3.22):

$$\Sigma(s_i) = \frac{1 + \gamma_5 \$_i}{2}$$

$$\Sigma(s_i) u(p_i, s_i) = u(p_i, s_i)$$

$$\Sigma(s_i) u(p_i, -s_i) = 0 \tag{7.89}$$

Repeating the development indicated between (7.13) and (7.14), we have

$$\frac{d\sigma}{d\Omega} = \frac{4Z^2\alpha^2 m^2}{|q|^4} \sum_{\pm s_f, s_i} \{\bar{u}(p_f, s_f)\gamma_0 \Sigma(s_i) u(p_i, s_i)\}\{\bar{u}(p_i, s_i)\gamma_0 u(p_f, s_f)\}$$

$$= \frac{4Z^2\alpha^2 m^2}{|q|^4} \operatorname{Tr} \gamma_0 \left(\frac{1 + \gamma_5 \$_i}{2}\right) \frac{(\not{p}_i + m)}{2m} \gamma_0 \frac{(\not{p}_f + m)}{2m} \tag{7.90}$$

According to (7.15) and (7.18), the additional trace involving the spin vector vanishes and we return once again to the Mott formula (7.22).

Our result that the differential cross section is the same for a polarized as for an unpolarized incident beam is a special consequence of the use of lowest order perturbation theory only and is not true generally.

In order to illustrate an observable effect of spin polarization, we again consider an incident electron with spin lined up along its direction of motion and compute the polarization of the scattered electron as a function of the scattering angle.

The initial polarization vector s_i satisfies

$$s_i^2 = -1 = (s_i^0)^2 - \mathbf{s}_i \cdot \mathbf{s}_i \tag{7.91}$$

and

$$s_i \cdot p_i = 0 \qquad \text{or} \qquad s_i^0 = \mathbf{s}_i \cdot \boldsymbol{\beta}_i$$

where

$$\boldsymbol{\beta}_i \equiv \frac{\mathbf{p}_i}{E_i} \tag{7.92}$$

Combining (7.91) and (7.92) gives

$$|\mathbf{s}_i| = \frac{1}{\sqrt{1 - (\boldsymbol{\beta}_i \cdot \hat{\mathbf{s}}_i)^2}} \tag{7.93}$$

where $\hat{\mathbf{s}}_i$ is a unit vector along $\mathbf{s}_i$.

For the electron spin polarization lined up along β_i, denoting a right-handed electron with polarization s_{iR}, we have $\beta_i \cdot \hat{\mathbf{s}}_{iR} = \beta_i$, and

$$|\mathbf{s}_{iR}| = \frac{1}{\sqrt{1 - \beta_i^2}} = \frac{E_i}{m_i} \qquad s_{iR}^0 = \beta_i |\mathbf{s}_{iR}| \tag{7.94}$$

Similarly, for the spin polarization antiparallel to β_i, denoted a left-handed electron with $s_{iL} = - s_{iR}$, we have

$$\boldsymbol{\beta}_i \cdot \hat{\mathbf{s}}_{iL} = -\beta_i$$

and

$$|\mathbf{s}_{iL}| = \frac{E_i}{m_i} \qquad s_{iL}^0 = -\beta_i |\mathbf{s}_{iL}|$$

Similar formulas apply to the scattered electron with the index i replaced everywhere by *f*.

The right- and left-handed vectors $s_{iR} = - s_{iL}$ form a particularly convenient basis for describing electron polarizations to which we shall frequently refer.

The eigenstates of $\sum(s)$ in (7.89) with $s = \pm s_R = \mp s_L$ are known as positive- and negative-helicity eigenstates.

The polarization of the scattered electrons is measured by

$$P = \frac{N_R - N_L}{N_R + N_L} \tag{7.95}$$

where N_R denotes the number emerging with positive helicity (or polarized right-handed) and N_L the number with negative helicity (left-handed).

N_R, N_L, and P are generally functions of the scattering energy and angle.

The polarization for Coulomb scattering of a right-handed electron is given according to (7.11) and (7.95) by

$$P_R = \frac{|\bar{u}(p_f,s_{fR})\gamma_0 u(p_i,s_{iR})|^2 - |\bar{u}(p_f,s_{fL})\gamma_0 u(p_i,s_{iR})|^2}{|\bar{u}(p_f,s_{fR})\gamma_0 u(p_i,s_{iR})|^2 + |\bar{u}(p_f,s_{fL})\gamma_0 u(p_i,s_{iR})|^2}$$

$$= \frac{\left\{\mathrm{Tr}\left[\gamma_0 \frac{(1+\gamma_5 \$_{iR})}{2} \frac{(\not{p}_i + m)}{2m} \gamma_0 \frac{(1+\gamma_5 \$_{fR})}{2} \frac{(\not{p}_f + m)}{2m}\right] - \mathrm{Tr}\left[\gamma_0 \frac{(1+\gamma_5 \$_{iR})}{2} \frac{(\not{p}_i + m)}{2m} \gamma_0 \frac{(1-\gamma_5 \$_{fR})}{2} \frac{(\not{p}_f + m)}{2m}\right]\right\}}{\mathrm{Tr}\left[\gamma_0 \frac{(1+\gamma_5 \$_{iR})}{2} \frac{(\not{p}_i + m)}{2m} \gamma_0 \frac{(\not{p}_f + m)}{2m}\right]}$$

$$= \frac{\mathrm{Tr}\left[\gamma_0\gamma_5 \$_{i\mathrm{R}} \frac{(\not{p}_i + m)}{2m} \gamma_0\gamma_5 \$_{f\mathrm{R}} \frac{(\not{p}_f + m)}{2m}\right]}{\mathrm{Tr}\left[\gamma_0 \frac{(\not{p}_i + m)}{2m} \gamma_0 \frac{(\not{p}_f + m)}{2m}\right]} \tag{7.96}$$

The subscript appended to P_R denotes the polarization for an incident beam that is completely right-handed.

All terms linear in sir or s_{iR} vanish as in (7.90).

The denominator trace in (7.96) has already been computed in (7.21) and the numerator is reduced by anticommuting the two γ_5 matrices together and applying (7.17).

For an incident electron beam that is not completely but only partially polarized along its direction of motion, we expect that (7.97) is modified to

The result after a short calculation and insertion of (7.94) is

$$P_{\mathrm{R}} = 1 - \left[\frac{2m^2 \sin^2(\theta/2)}{E^2 \cos^2(\theta/2) + m^2 \sin^2(\theta/2)}\right] \tag{7.97}$$

In the relativistic limit m/E $\rightarrow$ 0, or $\beta \rightarrow 1$, we find $P_R \rightarrow 1$, indicating no depolarization of the incident electrons in the high-energy limit of Coulomb scattering.

For an incident electron beam that is not completely but only partially polarized along its direction of motion, we expect that (7.97) is modified to

$$P = pP_{\mathrm{R}} \tag{7.98}$$

Here p denotes the polarization of the incident electrons, that is,

$$p \equiv p_R - p_L$$

where p_R is the fraction with positive helicity and $p_L = 1 - p_R$ is the fraction with negative helicity.

To verify (7.98), it is only necessary to return to (7.96) and observe that the expression for the polarization is linear in the initial spin projection operator.

Therefore, with the aid of the identity

$$p_R \frac{1 + \gamma_5 \$_{iR}}{2} + p_L \frac{1 - \gamma_5 \$_{iR}}{2} = \frac{1 + p\gamma_5 \$_{iR}}{2} \tag{7.99}$$

A special case of (7.99) for p = 0 shows that an initially unpolarized beam of electrons remains unpolarized in Coulomb scattering.

For a geometric picture to associate with these polarization results we define the angle between the spin of a moving electron with spinor wave function u(p,s) and an arbitrary direction along the unit vector $n^\mu = (0, \mathbf{n})$ by

$$\begin{aligned} \cos\alpha \equiv \langle \boldsymbol{\sigma} \cdot \mathbf{n} \rangle &= \frac{u^\dagger(p,s)\boldsymbol{\sigma} \cdot \mathbf{n} u(p,s)}{u^\dagger(p,s)u(p,s)} \\ &= \sqrt{1 - \beta^2}\, \bar{u}(p,s)\gamma_5 \not{n} u(p,s) \end{aligned} \tag{7.100}$$

where $\beta = \mathbf{p}/E$.

Again introducing projection operators and resorting to trace techniques to evaluate the matrix element in (7.100) we find, with the aid of (7.93),

$$\begin{aligned}\cos\alpha &= \sqrt{1-\beta^2}\,\mathrm{Tr}\left(\frac{\not p + m}{2m}\right)\left(\frac{1+\gamma_5 \not s}{2}\right)\gamma_5 \not n \\ &= \sqrt{1-\beta^2}\,\mathbf{s}\cdot\mathbf{n} \\ &= \sqrt{\frac{1-\beta^2}{1-(\boldsymbol{\beta}\cdot\hat{\mathbf{s}})^2}}\,\hat{\mathbf{s}}\cdot\mathbf{n}\end{aligned} \tag{7.101}$$

According to this, $|\cos\alpha| \leq \sqrt{1-\beta^2}$ for $\hat{\mathbf{s}}$ perpendicular to $\boldsymbol{\beta}$ and the expectation value of spin given in (7.100) for a state with its spin axis perpendicular to the velocity direction vanishes as $\beta \to 1$.

On the other hand, if the spin s is taken along the velocity axis, that is, for helicity states,

$$\cos\alpha = \hat{\mathbf{s}}\cdot\mathbf{n} \tag{7.102}$$

and the spin projection is ± 1 along a direction $\mathbf{n}$ parallel or antiparallel to $\mathbf{s}$.

In this case we call $\alpha = \delta$ and

$$\cos\delta = \pm 1 \tag{7.103}$$

respectively.

The average value of $\cos\alpha$ for a beam of scattered electrons is given by

$$\langle\cos\alpha\rangle = \sum_{\pm s} \omega(s,p)\cos\alpha \tag{7.104}$$

where ω(s,p) is the transition probability to a given final state with momentum p and spin s.

The sum in (7.104) is most conveniently taken over the two helicity states.

For the spin projection along the direction of motion we find from (7.103), (7.104), and (7.95)

$$\langle \cos \delta \rangle = w(s_R,p) - w(s_L,p) = P \tag{7.105}$$

that is, the polarization represents the cosine of the angle between spin and momentum vectors.

For Coulomb scattering of an initially polarized beam of electrons with p = 1, (7.105) and (7.99) tell us that at high energies E >> m or small scattering angles θ << 1,

$$\langle \delta \rangle \sim \frac{m}{E}\theta \tag{7.106}$$

that is, the angle between the spin and momentum vectors of the emerging electrons is m/E times the scattering angle.

The relativistic limit of polarization calculations is most simply achieved by directly reducing the polarization projection operators for m/E $\rightarrow$ 0.

In this limit the spin projection operators for longitudinally polarized electrons with **s** parallel to p can be further reduced.

By (7.93) and (7.94) we write

$$s^{\mu}_{..} = \frac{1}{m\beta} p^{\mu} - \frac{\sqrt{1-\beta^2}}{\beta} g^{\mu 0}$$

$$\rightarrow \frac{p^{\mu}}{m} \qquad \text{as } \beta \rightarrow 1$$

and find in this limit

$$\left(\frac{1 \pm \gamma_5 \$_R}{2}\right)\left(\frac{\not{p} + m}{2m}\right) \rightarrow \left(\frac{1 \pm \gamma_5}{2}\right)\left(\frac{\not{p} + m}{2m}\right)$$

Similarly,

$$\left(\frac{1 \pm \gamma_5 \$_R}{2}\right)\left(\frac{m - \not{p}}{2m}\right) \rightarrow \left(\frac{1 \mp \gamma_5}{2}\right)\left(\frac{m - \not{p}}{2m}\right) \qquad (7.107)$$

Since the spin projection operators stand next to energy projection operators in the cross-section calculations, the simplifications in (7.107) can be made in the relativistic limit.

The result of (7.99), that Coulomb-scattered electrons are not depolarized for m/E $\rightarrow$ 0, is seen immediately in this limit.

The matrix element for a right-handed relativistic electron

$$u(p_i, s_i) = \frac{(1 + \gamma_5)}{2} u(p_i, s_i)$$

to scatter to a left-handed one

$$u(p_f,s_f) = \frac{1-\gamma_5}{2}\, u(p_f,s_f)$$

with interaction γ^μ is proportional to

$$\begin{aligned}
\bar{u}(p_f,s_f)\gamma_\mu u(p_i,s_i) &= \bar{u}(p_f,s_f)\left(\frac{1+\gamma_5}{2}\right)\gamma_\mu\left(\frac{1+\gamma_5}{2}\right)u(p_i,s_i) \\
&= \bar{u}(p_f,s_f)\gamma_\mu\left(\frac{1-\gamma_5}{2}\right)\left(\frac{1+\gamma_5}{2}\right)u(p_i,s_i) \\
&= 0
\end{aligned}$$

the the the the the the

8 Higher-order Corrections to the Scattering Matrix

8.1 Electron-Positron Scattering in Fourth Order

The rules for writing *S*-matrix elements which have been developed in the preceding examples can be extended to higher orders in the coupling constant, although there are new problems to be faced.

Consider, for example, the et contributions to electron-positron scattering.

In order to construct such an amplitude, we draw all possible Feynman graphs with four electromagnetic vertices which correspond to this scattering process.

Then, following the rules given by the examples discussed so far, we write down the desired matrix elements.

Several of the graphs contributing to this calculation (there are 18 altogether) are shown in Fig. 8.1.

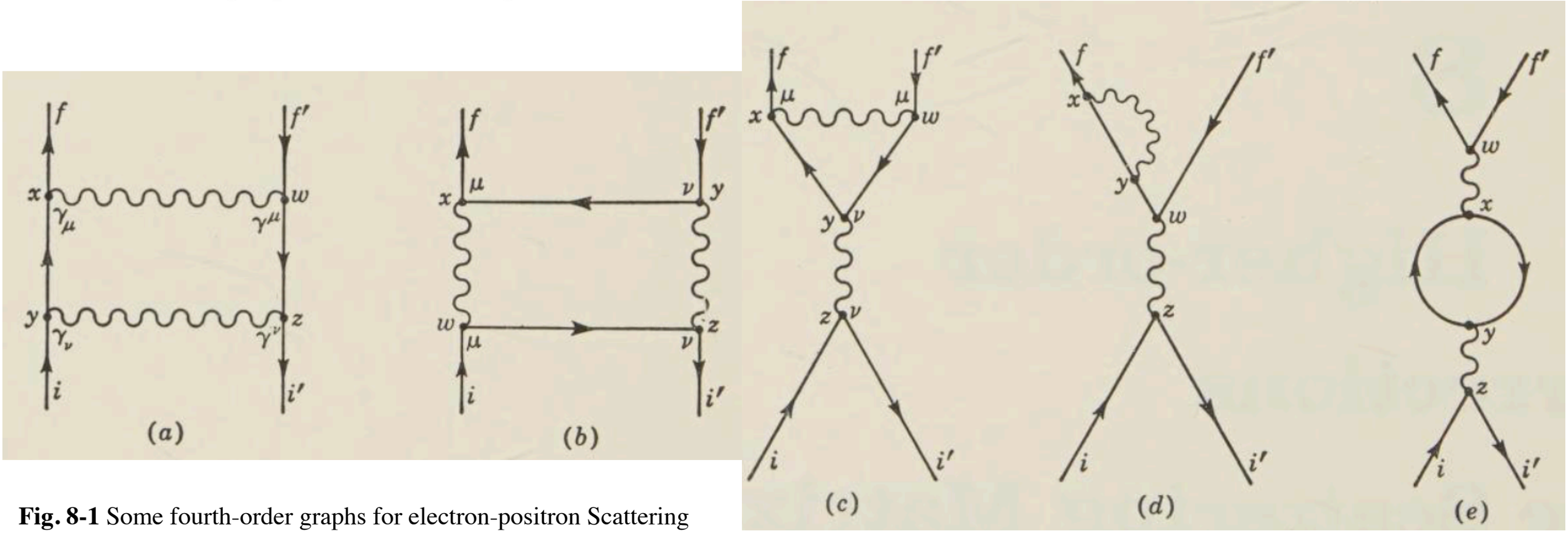

Fig. 8-1 Some fourth-order graphs for electron-positron Scattering

Graph (a) shows a two-photon exchange between electron and positron and contributes an amplitude analogous to (7.47) and (7.48) for electron-proton scattering:

$$S_{fi}^{(4a)} = -(-ie)^4 \int d^4w\, d^4x\, d^4y\, d^4z\, [\bar{\psi}_f^{(+)}(x)\gamma_\mu i S_F(x-y)\gamma_\nu \psi_i^{(+)}(y)] \times iD_F(x-w) iD_F(y-z) [\bar{\psi}_{i'}^{(-)}(z)\gamma^\nu i S_F(z-w)\gamma^\mu \psi_{f'}^{(-)}(w)] \tag{8.1}$$

where i and f, i' and f', are the quantum numbers of the electron and positron, respectively.

Graph (b) is an annihilation term and contributes a minus sign relative to (8.1)

$$S_{fi}^{(4b)} = +(-ie)^4 \int d^4w\, d^4x\, d^4y\, d^4z\, [\bar{\psi}_f^{(+)}(x)\gamma_\mu i S_F(x-y)\gamma_\nu \psi_{f'}^{(-)}(y)] \times iD_F(x-w) iD_F(y-z) [\bar{\psi}_{i'}^{(-)}(z)\gamma^\nu i S_F(z-w)\gamma^\mu \psi_i^{(+)}(w)] \tag{8.2}$$

The origin of the relative minus sign between the two amplitudes is the same as in the lower-order calculation (7.82); it comes from anti-symmetrization of the wave functions for the initial positive- and negative-energy electron state.

The amplitude for graph (c) corresponds to a process in which the pair produced from the annihilation photon scatters before emerging into the final state.

According to our by-now-familiar rules of writing $-ie\gamma_\mu$ and the invariant volume integral $\int d^4x$ at each vertex, a Feynman propagator $iS_F(x - y)$ for each internal line, and wave functions for free incident and emerging particles, the amplitude for this process is

$$S_{fi}^{(4c)} = +(-ie)^4 \int d^4w\, d^4x\, d^4y\, d^4z\, iD_F(x-w) iD_F(y-z)$$
$$\times [\bar{\psi}_f^{(+)}(x)\gamma_\mu iS_F(x-y)\gamma_\nu iS_F(y-w)\gamma^\mu \psi_{f'}^{(-)}(w)][\bar{\psi}_{i'}^{(-)}(z)\gamma^\nu \psi_i^{(+)}(z)] \tag{8.3}$$

Only the choice of overall sign in (8.3) requires comment.

It comes from the requirement of Fermi-Dirac statistics that two electron states must be antisymmetric under interchange of the electrons.

One of the possible time orderings of the four vertices in (a) is drawn in Fig. 8.2, together with a corresponding one from diagram (c).

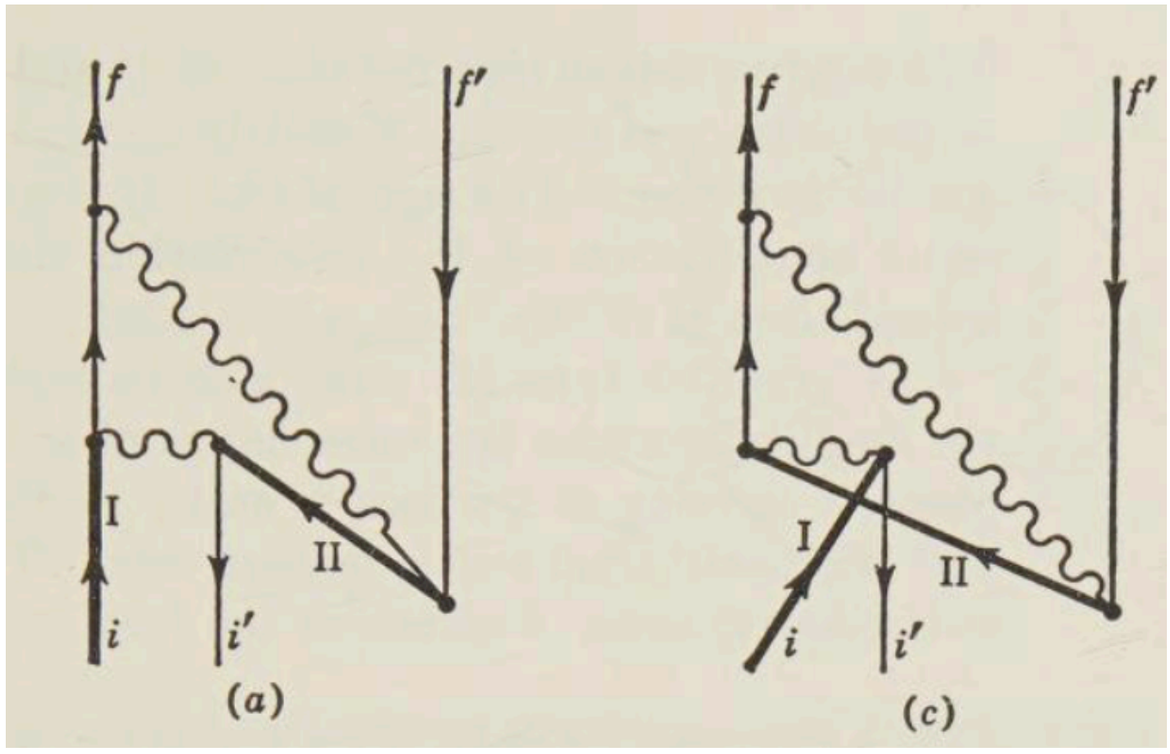

Fig. 8-2 Origin of the relative minus sign between graphs (a) and (c) due to Fermi statistics.

These two graphs differ by the exchange of the two electron lines labeled I and II.

The relative minus sign between (8.1) and (8.3) assures the required antisymmetry of the total S-matrix element under the exchange of two similar fermions.

The sign of (8.3) is the same as that of the lowest order contribution of the annihilation diagram, Fig. 7.15b, to the second term of (7.86).

A graph of type (d) must also be included when we symmetrize the amplitude under the interchange of the two photons; that is, the photons arriving at vertices w and z in Fig. 8.1c may have equally well originated from z or x, respectively.

This leads to a contribution to the fourth-order S matrix of the form

$$S_{fi}^{(4d)} = +(-ie)^4 \int d^4w\, d^4x\, d^4y\, d^4z\, iD_F(x-y) iD_F(w-z) \times [\bar{\psi}_f^{(+)}(x)\gamma_\mu iS_F(x-y)\gamma^\mu iS_F(y-w)\gamma_\nu \psi_{f'}^{(-)}(w)][\bar{\psi}_{i'}^{(-)}(z)\gamma^\nu \psi_i^{(+)}(z)] \tag{8.4}$$

and with the same sign as (8.3).

The amplitude for graph (e) of Fig. 8.1 corresponds to a process in which the pair produced from the annihilation photon interacts, again via annihilation, before emerging into the final state.

Above the vertex at y, graphs (c) and (e) of Fig. 8.1 are related in the same way as the two second-order processes of electron-positron scattering in Fig. 7.15, and we expect to find, as in (7.86), that their contributions to the S matrix are of opposite sign.

This leads to the result

$$S_{fi}^{(4e)} = -(-ie)^4 \int d^4w\, d^4x\, d^4y\, d^4z\, [\bar{\psi}_f^{(+)}(w)\gamma_\mu \psi_{f'}^{(-)}(w)] iD_F(w-x) \times [\gamma^\nu_{\alpha\beta} iS_F(y-x)_{\beta\lambda} \gamma^\mu_{\lambda\tau} iS_F(x-y)_{\tau\alpha}] iD_F(y-z) \times [\bar{\psi}_{i'}^{(-)}(z)\gamma_\nu \psi_i^{(+)}(z)] \tag{8.5}$$

The overall sign in (8.5) can be independently verified by constructing an appropriate time-ordered sequence as in Fig. 8.2.

The symmetry and antisymmetry requirements that have carried us this far lead to one additional class of graphs as illustrated by Fig. 8.3.

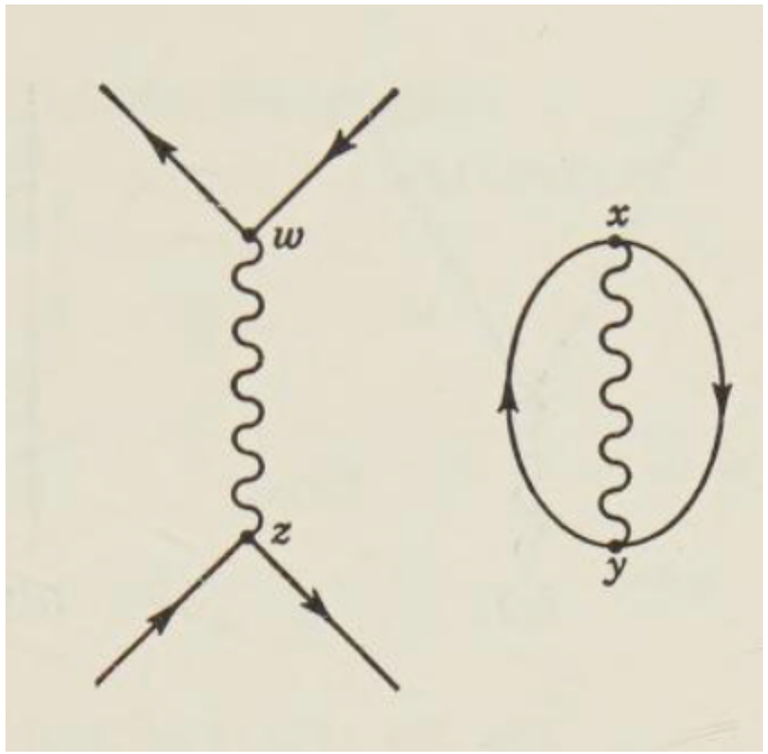

Fig. 8-3 Example of a disconnected graph.

This arises from symmetrizing the two photons in Fig. 8.le; the photon arriving at w can equally well originate from z or x.

All such disconnected graphs, that is, graphs containing a completely isolated part into or out of which none of the initial or final particles emerge, are properly ignored in all calculations.

Figure 8.3 shows an electron propagating to x; there it emits a photon and scatters backward to y, where it destroys itself and the photon.

In the language of hole theory this is a fluctuation effect in which an electron jumps out of the negative-energy sea into an empty positive-energy state with virtual photon emission and then drops back into the negative-energy sea upon reabsorbing the photon.

Such fluctuations are taking place all the time; and to find the scattering amplitude relative to what happens in the vacuum, we divide out the contribution of all disconnected bubbles, which evidently provides a multiplicative factor on the connected graphs of interest.

We may summarize the rules for constructing the amplitudes for higher-order processes as follows:

1. Draw all connected graphs.
2. Associate with each graph an amplitude with a factor

$$-ie\gamma_\mu \int d^4x$$

at each vertex.

3. Include a propagator $iS_F(x - y)$ or $iD_F(x - y)$ for each line representing a fermion or photon which terminates at vertices x and y —this is an internal line. For photons insert an additional factor $g_{\mu\upsilon}$ to tie together the γ^μ and γ^υ the vertices connected by the photon line.

4. Introduce a wave function for each external line, that is, a line representing an incident or scattered particle.

These rules are as developed in the low-order examples, with the new condition that only connected graphs are to be calculated.

Finally, we add the sign conditions:

5. There must be a relative minus sign between two terms which differ by the exchange of identical fermions, as in (7.82) and Fig. 7.13 for two positive-energy electrons and in (7.86) and Fig. 7.15 for one positive- and one negative-energy electron. In the amplitude (8.5), for Fig. 8.le, this led to the introduction of an additional minus sign with the closed electron loop.

As a general rule a factor of (-1) is included with each closed Fermion loop, as in Fig. 8.le, in constructing the amplitude for a given Feynman diagram.

6. In accordance with (6.56) there is an overall factor $(-)^{\bar{n}}$, where $\bar{n}$ is the number of positrons appearing in the initial state.

The big question remaining is how to compute the integrals, in particular for the fourth-order interactions, and obtain numbers for comparison with experiments.

Diagrams (a) and (b) of Fig. 8.1, along with the two crisscrossing photon lines obtained by interchanging x and y, present a formidable four-dimensional integral of the type given by (7.51) for the similar contribution to electron-proton scattering, and they will not be computed here.

It is convenient in discussing diagrams (c) to (e) of Fig. 8.1 to go into momentum space and relate them to the similar lowest order graph, Fig. 7.15b, which contributes the second term of (7.86).

Making the by now familiar-expansions in momentum space, we find that $S^{(4c)}_{fi}$ differs from the second term of (7.86) by the replacement of the current

$$\bar{u}(p_1')e\gamma_\mu v(q_1') \to \bar{u}(p_1')\int \frac{d^4k}{(2\pi)^4}\frac{(-i)}{k^2+i\epsilon}(-ie\gamma_\nu)\frac{i}{\not{p}_1'-k-m+i\epsilon} \times e\gamma_\mu \frac{i}{-\not{q}_1'-k-m+i\epsilon}(-ie\gamma^\nu)v(q_1') \tag{8.6}$$

$S^{(4d)}_{fi}$ differs by the insertion for the final electron wave function

$$\bar{u}(p_1') \to \bar{u}(p_1') \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2 + i\epsilon} (-ie\gamma_\nu) \times \frac{i}{\not{p}_1' - k - m + i\epsilon} (-ie\gamma^\nu) \frac{i}{\not{p}_1' - m + i\epsilon} \tag{8.7}$$

and $S_{fi}^{(4e)}$ differs by the insertion into the photon propagator

$$\frac{-ig_{\mu\nu}}{(p_1 + q_1)^2 + i\epsilon} \to (-1) \left[\frac{(-i)}{(p_1 + q_1)^2 + i\epsilon} \right]^2 \times \int \frac{d^4k}{(2\pi)^4} \operatorname{Tr} (-ie\gamma_\mu) \frac{i}{k - m + i\epsilon} (-ie\gamma_\nu) \frac{i}{k - \not{p}_1 - \not{q}_1 - m + i\epsilon} \tag{8.8}$$

$$\equiv \frac{(-i)}{(p_1 + q_1)^2 + i\epsilon} I_{\mu\nu}(p_1 + q_1) \frac{(-i)}{(p_1 + q_1)^2 + i\epsilon}$$

The portions of the graphs contributing here are shown in Fig. 8.4.

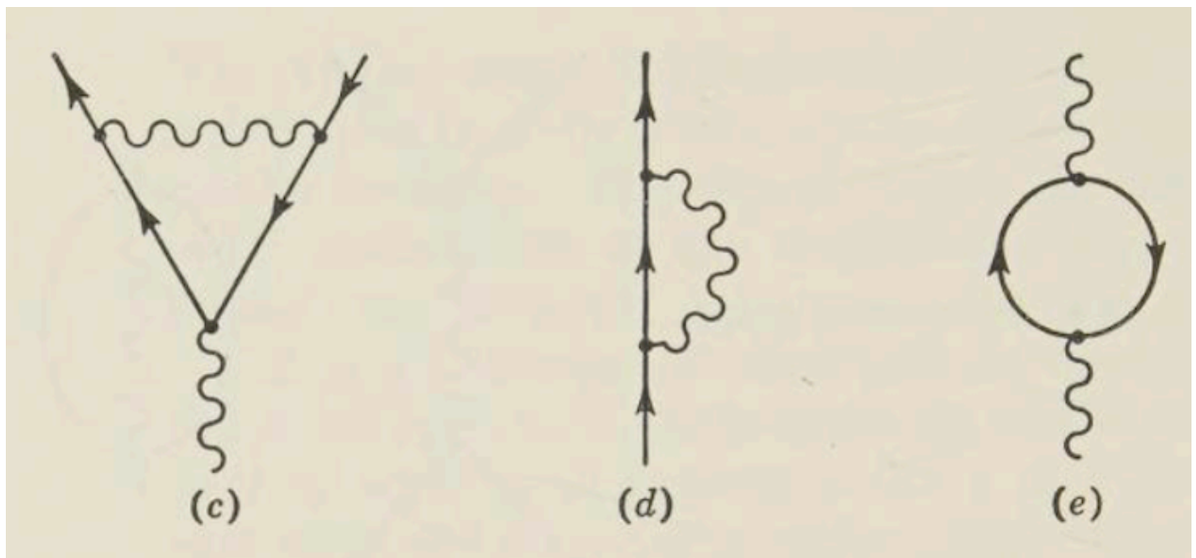

Fig. 8-4 Feynman graphs illustrating (c) vertex corrections, (d) electron self-mass, and (e) vacuum polarization.

All of the remaining fourth-order graphs give rise to insertions of these same three types.

It is unfortunate that these closed-loop contributions diverge for $k \to \infty$.

We discuss and compute them in order.

8.2 Vacuum Polarization

The most severe divergence is that in (8.8) corresponding to the closed electron loop of Fig. 8.4e.

We refer to this contribution as the second-order photon self-energy part.

The integral contains two electron propagators and therefore, with only two powers of k in the denominator, diverges quadratically.

The quadratic divergence may be argued away with a certain amount of plausibility by appealing to the condition of gauge invariance as discussed above (7.60).

A gauge change $A_\mu(q) \to A_\mu(q) + q_\mu \Delta(q)$ must not alter final results of a calculation of physical amplitudes.

This requirement has the following significance for (8.8).

Let the photon in Fig. 8.4e be a real physical photon with $q^2 = 0$ according to the Einstein condition, such as occurs in the bremsstrahlung or Compton process.

As illustrated in Fig. 8.5, the electron loop gives an e^2 correction to the current flowing through the question box with which $A_\mu(g)$ interacts.

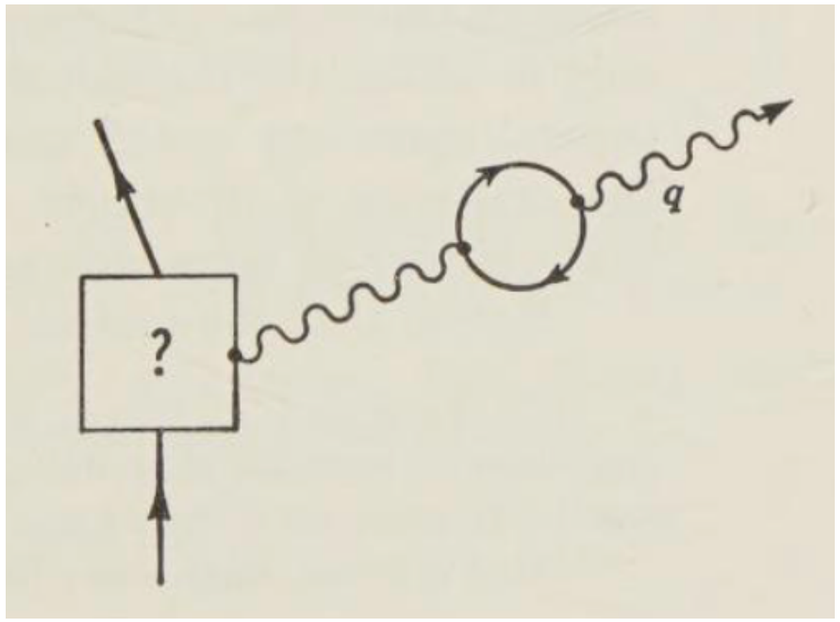

Fig. 8-5 Vacuum polarization correction to an electromagnetic process.

The gauge requirement is that q_μ times the current vanish, which in terms of (8.8) means

$$q^\mu I_{\mu\nu}(q) = 0 \tag{8.9}$$

when

$$q^2 = 0$$

This may be rewritten as

$$\begin{aligned} q^\mu I_{\mu\nu}(q) &= -e^2 \,\mathrm{Tr} \int \frac{d^4k}{(2\pi)^4} \not{q} \frac{1}{k - m + i\epsilon} \gamma_\nu \frac{1}{k - \not{q} - m + i\epsilon} \\ &= -e^2 \,\mathrm{Tr} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k - \not{q} - m + i\epsilon} [(k - m + i\epsilon) \\ &\qquad - (k - \not{q} - m + i\epsilon)] \frac{1}{k - m + i\epsilon} \gamma_\nu \\ &= -e^2 \int \frac{d^4k}{(2\pi)^4} \,\mathrm{Tr} \left(\frac{1}{k - \not{q} - m + i\epsilon} - \frac{1}{k - m + i\epsilon} \right) \gamma_\nu \end{aligned} \tag{8.10}$$

If the integral were finite, we could let k' = k - q in the first term and thus obtain zero.

The fact that the integral is not finite is unavoidable, and consequently (8.10) remains ambiguous.

In order to proceed, we cut off (8.8) at high frequencies, making the replacement

$$I_{\mu\nu}(q,m^2) \to \bar{I}_{\mu\nu}(q) = I_{\mu\nu}(q,m^2) + \sum_i C_i(M_i^2) I_{\mu\nu}(q,M_i^2)$$
$$\equiv \sum_i c_i I_{\mu\nu}(q,m_i^2) \qquad (8.11)$$

where the M_i^2 are large masses and the C_i are chosen such that the integrals converge.

This cutoff procedure is not unique and is adopted only to define the mathematics.

If physically measurable quantities depend upon any cutoff parameters, the theory fails.

In any case the existence of divergent quantities leads one to suspect trouble in the theory at large momenta or, equivalently, small distances.

Notice that the method of cutoff in (8.11) has the virtue of pre-serving the gauge condition (8.9).

Were the individual propagators to be cut off, we would not be able to maintain gauge invariance.

The calculation of $I_\mu(q)$ subject to the condition (8.9) is most readily done by elevating the propagator denominators into exponential factors by the identity

$$\frac{i}{k-m+i\epsilon} = \frac{i(k+m)}{k^2-m^2+i\epsilon} = (k+m)\int_0^\infty dz\, e^{iz(k^2-m^2+i\epsilon)} \tag{8.12}$$

This gives

$$\begin{aligned} I_{\mu\nu}(q) = -4(-ie)^2 \int_0^\infty dz_1 \int_0^\infty dz_2 \int \frac{d^4k}{(2\pi)^4} \\ \times [k_\mu(k-q)_\nu + k_\nu(k-q)_\mu - g_{\mu\nu}(k^2 - k\cdot q - m^2)] \\ \times \exp\{iz_1[k^2-m^2+i\epsilon] + iz_2[(k-q)^2-m^2+i\epsilon]\} \end{aligned} \tag{8.13}$$

where the trace has been carried out and orders of integration inverted.

Completing the square in the exponential by changing the integration variable to

$$l = k - \frac{qz_2}{z_1+z_2} = k - q + \frac{qz_1}{z_1+z_2} \tag{8.14}$$

we perform the momentum integrals, using the identities

$$\int \frac{d^4l}{(2\pi)^4}[1, l_\mu, l_\mu l_\nu]\, e^{il^2(z_1+z_2)} = \frac{1}{16\pi^2 i(z_1+z_2)^2}\left[1, 0, \frac{ig_{\mu\nu}}{2(z_1+z_2)}\right] \tag{8.15}$$

NOTE: These are best evaluated in rectangular coordinates.

Each integral, with a rotation of contour of 45°, becomes a gaussian integral, for example,

$$\int_{-\infty}^{\infty} \frac{dl_0}{2\pi} e^{il_0^2(a+i\epsilon)} = \frac{e^{i\pi/4}}{2\sqrt{\pi a}}$$

The result is

$$\bar{I}_{\mu\nu} = -i \sum_i c_i \frac{\alpha}{\pi} \int_0^\infty dz_1 \int_0^\infty \frac{dz_2}{(z_1+z_2)^2} \times \left(\exp\left\{ i \left[q^2 \frac{z_1 z_2}{z_1+z_2} - (m_i^2 - i\epsilon)(z_1+z_2) \right] \right\} \right) \times \left\{ 2(g_{\mu\nu}q^2 - q_\mu q_\nu) \frac{z_1 z_2}{(z_1+z_2)^2} + g_{\mu\nu} \left[\frac{-i}{(z_1+z_2)} - \frac{q^2 z_1 z_2}{(z_1+z_2)^2} + m_i^2 \right] \right\} \tag{8.16}$$

The terms proportional to ($g_{\mu\upsilon}q2$ - $q_\mu q_\upsilon$) automatically satisfy the gauge condition (8.9), whereas the last three terms proportional to $g_{\mu\upsilon}$ do not.

However, we may show that these vanish, that is,

$$\int_0^\infty \int_0^\infty \frac{dz_1\, dz_2}{(z_1+z_2)^2} \sum_i c_i \left[m_i^2 - \frac{i}{(z_1+z_2)} - \frac{q^2 z_1 z_2}{(z_1+z_2)^2} \right] \times \left\{ \exp i \left[q^2 \frac{z_1 z_2}{z_1+z_2} - (m_i^2 - i\epsilon)(z_1+z_2) \right] \right\}$$

$$\times \left\{ \exp i \left[q^2 \frac{z_1 z_2}{z_1 + z_2} - (m_i^2 - i\epsilon)(z_1 + z_2) \right] \right\}$$

$$= \int_0^\infty \int_0^\infty \frac{dz_1\, dz_2}{(z_1 + z_2)^2} \sum_i c_i \left[m_i^2 - \frac{i}{\lambda(z_1 + z_2)} - \frac{q^2 z_1 z_2}{(z_1 + z_2)^2} \right]$$

$$\times \left\{ \exp i\lambda \left[q^2 \frac{z_1 z_2}{z_1 + z_2} - (m_i^2 - i\epsilon)(z_1 + z_2) \right] \right\}$$

$$= i\lambda \frac{\partial}{\partial \lambda} \int_0^\infty \int_0^\infty \frac{dz_1\, dz_2}{\lambda(z_1 + z_2)^3}$$

$$\times \sum_i c_i \exp \left\{ i\lambda \left[\frac{q^2 z_1 z_2}{z_1 + z_2} - (m_i^2 - i\epsilon)(z_1 + z_2) \right] \right\} \tag{8.17}$$

where we have let $z_i \rightarrow \lambda z_i$ in the second step.

Upon letting $\lambda z_i \rightarrow z_i$ in the integrand, we see that the integral is independent of y; hence (8.17) is identically zero.

The remaining contribution to $\bar{I}_{\mu\nu}$, is evaluated with the aid of the same scaling trick.

Using the identity

$$1 = \int_0^\infty \frac{d\lambda}{\lambda} \delta\left(1 - \frac{z_1 + z_2}{\lambda}\right) \tag{8.18}$$

$$\bar{I}_{\mu\nu}(q) = \frac{2i\alpha}{\pi}(q_\mu q_\nu - g_{\mu\nu}q^2)\int_0^\infty\int_0^\infty\int_0^\infty \frac{d\lambda\, dz_1\, dz_2\, z_1 z_2}{\lambda(z_1+z_2)^4}$$

$$\times\delta\left(1-\frac{z_1+z_2}{\lambda}\right)\times\sum_i c_i \exp\left\{i\left[\frac{q^2 z_1 z_2}{z_1+z_2}-(m_i^2-i\epsilon)(z_1+z_2)\right]\right\}$$

$$= \frac{2i\alpha}{\pi}(q_\mu q_\nu - g_\mu q^2)\int_0^\infty\int_0^\infty dz_1\, dz_2\, z_1 z_2\delta(1-z_1-z_2)\int_0^\infty\frac{d\lambda}{\lambda}$$

$$\times\sum_i c_i \exp\left[i\lambda(q^2 z_1 z_2 - m_i^2 + i\epsilon)\right] \qquad (8.19)$$

where we have again scaled $z_i \to \lambda z_i$.

The integral over λ, unhappily, diverges logarithmically, and we evaluate it with the aid of the cutoff procedure.

Choosing, in (8.11), $C_1 = -1$, $C_i = 0$ $(i > 1)$, we find

$$\bar{I}_{\mu\nu} = I_{\mu\nu}(m^2) - I_{\mu\nu}(M^2)$$

$$\cong \frac{2i\alpha}{\pi}(q_\mu q_\nu - g_{\mu\nu}q^2)\int_0^1 dz\, z(1-z)\log\frac{M^2}{m^2 - q^2 z(1-z)}$$

$$= \frac{i\alpha}{3\pi}(q_\mu q_\nu - g_{\mu\nu}q^2)$$

$$\times\left[\log\frac{M^2}{m^2} - 6\int_0^1 dz\, z(1-z)\log\left(1-\frac{q^2}{m^2}z(1-z)\right)\right] \qquad (8.20)$$

To understand the physical significance of (8.20), we consider, as in Fig. 8.le, the contribution of the closed loop to the scattering.

Adding it to the second-order amplitude in (7.86) gives, according to (8.8), a photon propagator which can be written as the sum of two terms

$$\frac{-ig_{\mu\nu}}{q^2} + \frac{(-i)}{q^2}\,\bar{I}_{\mu\nu}(q)\,\frac{(-i)}{q^2} \tag{8.21}$$

Inserting (8.20) and dropping the terms proportional to q_μ and q_υ which vanish by current conservation at the electron vertices, we find

$$\frac{-ig_{\mu\nu}}{q^2}\left[1 - \frac{\alpha}{3\pi}\log\frac{M^2}{m^2} + \frac{2\alpha}{\pi}\int_0^1 dz\, z(1-z)\log\left(1 - \frac{q^2 z(1-z)}{m^2 - i\epsilon}\right)\right] \tag{8.22}$$

This is the photon propagator including corrections of order α.

In any Feynman graph the effect of an electron loop on the amplitude for the exchange of a photon between two conserved currents is given by (8.22).

In the limit $q^2 \to 0$, the propagator is changed only by the multiplicative factor Z_3, defined by

$$Z_3 \cong 1 - \frac{\alpha}{3\pi}\log\frac{M^2}{m^2} \tag{8.23}$$

and therefore, for example, the Coulomb scattering amplitude for small momentum transfers becomes

$$\frac{ie^2\bar{u}\gamma_0 u}{q^2} \to ie^2\frac{\bar{u}\gamma_0 u}{q^2}\left(1-\frac{\alpha}{3\pi}\log\frac{M^2}{m^2}\right)=\frac{ie_R^2\bar{u}\gamma_0 u}{q^2} \tag{8.24}$$

We conclude that the parameter e appearing in the Dirac equation is not $4\pi/137$ but something larger, since it is e_R^2 which is *measured* to be $4\pi/137$.

e_R is called the renormalized charge and e the bare charge.

In any process where a photon is exchanged, this multiplicative factor will be present

$$e_R^2 \equiv Z_3 e^2 \cong e^2\left(1-\frac{\alpha}{3\pi}\log\frac{M^2}{m^2}\right) \tag{8.25}$$

independently of momentum transfer.

Consequently, there will also be the same renormalization of the electron charge arising from the static polarizability of the vacuum.

Therefore, the divergence in the calculation, to order e^2, disappears if we rewrite cross sections in terms of the observed charges e_R.

The observable, momentum-dependent corrections come from the second term of (8.22), which vanishes in the static limit $q^2 \to 0$.

Its contribution is finite and independent of the cutoff procedure adopted in the calculation.

Only the relation between bare and physical charge is cutoff-dependent.

In the limit of low momentum transfer $|q^2/m^2| \ll 1$, it alters, for instance, the Coulomb scattering amplitude (8.24) by the factor

$$ie^2 \frac{\bar{u}\gamma_0 u}{q^2}\left(1 - \frac{\alpha}{3\pi}\log\frac{M^2}{m^2} - \frac{\alpha}{15\pi}\frac{q^2}{m^2}\right) \cong ie_R^2 \frac{\bar{u}\gamma_0 u}{q^2}\left[1 - \frac{\alpha_R}{15\pi}\frac{q^2}{m^2} + 0(\alpha_R^2)\right] \tag{8.26}$$

This can be expressed as an additional interaction in coordinate space of the form

$$\left(1 - \frac{\alpha_R}{15\pi m^2}\nabla_x^2\right)\frac{e_R^2}{4\pi r} = \frac{e_R^2}{4\pi r} + \frac{\alpha_R e_R^2}{15\pi m^2}\delta^{(3)}(\mathbf{x}) \tag{8.27}$$

It leads to a first-order change ΔE_{nl} in the atomic energy levels in hydrogen-like atoms of charge Z

$$\Delta E_{nl} = \frac{-Ze_R^2\alpha_R}{15\pi m^2}|\psi_{nl}(0)|^2 = -(\tfrac{1}{2}Z^2\alpha^2 m)\frac{8Z^2\alpha^3}{15\pi n^3}\delta_{l0} \tag{8.28}$$

For n = 2, l = 0, and Z = 1

$$\nu = \frac{\Delta E}{h} = -27 \text{ megacycles per second}$$

The signs of the contribution (8.26) and (8.27) are what the discussion in Chap. 5 leads us to expect.

For an electron scattering with low momentum transfer $|q^2| < m^2$ corresponding to a large impact parameter, the interaction is proportional to the total charge.

For scatterings with small impact parameters and large momentum transfers $q^2 = -|\mathbf{q}^2|$, the electron penetrates the polarization cloud and the interaction strength increases.

The resulting modification of Coulomb's law, as first calculated by Uehling in 1935, was the object of the original measurements of the $2S_{½} - 2P_{½}$ Lamb shift splitting in the hydrogen atom.

Instead of 0.2 mc/sec, a shift of ~+1,000 mc/ sec was discovered in 1947; this was due primarily to the vacuum fluctuations of the radiation field as discussed in Chap. 4.

Very precise measurements and calculations during the last decade agree to within 0.2 mc/sec for the Lamb shift of the n = 2 levels in hydrogen and therefore confirm to high accuracy the presence of the - 27 mc/sec of vacuum polarization.

This is an impressive vote of support both for the hole theory of the Dirac equation, which has given rise to the closed-loop contributions, and for the simple form of point coupling used in describing the interaction of electrons with photons.

It still remains for experiments probing the theory for large q^2, corresponding to interactions at small distances, to demonstrate the need for any modifications.

For large momentum transfer scattering with

$$|\mathbf{q}^2| = -q^2 \gg m^2$$

the correction in (8.22) increases logarithmically and the photon propagator, to first order in the unrenormalized charge α, is

$$\frac{-ig_{\mu\nu}}{q^2}\left(1+\frac{\alpha}{3\pi}\log\frac{|\mathbf{q}|^2}{m^2}\right)\left(1-\frac{\alpha}{3\pi}\log\frac{M^2}{m^2}\right) \tag{8.29}$$

When the momentum transfer reaches the cutoff value M^2, the correction compensates the charge renormalization factor and suggests that in the limit of infinite energies the interaction is measured by the strength of the bare point charge of Fig. 5.3.

This is an interesting but unproved conjecture.

Whenever the virtual photon momentum q is time-like and q^2 exceeds $4m^2$, as in the pair production diagram of Fig. 8.1e, the correction to the propagator in (8.22) becomes complex, with an imaginary part given by

$$\frac{+ig_{\mu\nu}}{q^2}\left(\frac{2\alpha}{\pi}\right)\int_0^1 dz\, z(1-z)i\pi\theta\left[z(1-z)-\frac{m^2}{q^2}\right]$$
$$= +\frac{ig_{\mu\nu}}{q^2}\frac{i\alpha}{3}\left(1+\frac{2m^2}{q^2}\right)\sqrt{1-\frac{4m^2}{q^2}}\,\theta\left(1-\frac{4m^2}{q^2}\right) \tag{8.30}$$

To understand the origin of this imaginary part, we recall from Chap. 6 and the discussion of scattering in the nonrelativistic propagator theory that the S matrix in the Schrödinger theory is unitary.

The condition of unitarity

$$S^\dagger S = 1$$

that is,

$$\sum_n S^*_{nf} S_{ni} = \delta_{fi} \tag{8.31}$$

ensures the probability interpretation of the scattering solutions according to which the sum of all transition probabilities for a given initial state must add up to unity.

In the positron theory, particles are produced and destroyed and the sum over states *n* must include all electron, positron, and photon final states to which a given initial state can scatter.

One finds that (8.31) retains its interpretation as a statement of probability conservation.

Since it is an identity in e, each order in an expansion of S in powers of the interaction constant must satisfy (8.31).

If we expand

$$S_{fi} = \delta_{fi} + S_{fi}^{(1)} + S_{fi}^{(2)} + \cdots \tag{8.32}$$

the unitarity condition becomes

$$S_{fi}^{(1)} + S_{if}^{(1)*} = 0 \tag{8.33a}$$

$$S_{fi}^{(2)} + S_{if}^{(2)*} = -\sum_n S_{nf}^{(1)*} S_{ni}^{(1)} \tag{8.33b}$$

$$S_{fi}^{(3)} + S_{if}^{(3)*} = -\sum_n [S_{nf}^{(1)*} S_{ni}^{(2)} + S_{nf}^{(2)*} S_{ni}^{(1)}] \tag{8.33c}$$

$$S_{fi}^{(4)} + S_{if}^{(4)*} = -\sum_n [S_{nf}^{(1)*} S_{ni}^{(3)} + S_{nf}^{(2)*} S_{ni}^{(2)} + S_{nf}^{(3)*} S_{ni}^{(1)}] \tag{8.33d}$$

$$\cdots\cdots\cdots$$

For i representing an initial free electron-positron state, $S_{fi}^{(1)} = 0$; the reaction $e^- + e^+ \rightarrow 1\gamma$ is forbidden by energy-momentum conservation.

Relation (8.33b) is satisfied by (7.86), which is anti-hermitian as required.

Relation (8.33d) gives a nonvanishing hermitian part of the fourth-order amplitude in terms of the second-order contributions.

Equation (8.30) represents just this fourth-order contribution; it is real and therefore gives a hermitian contribution to the S matrix (8.5) as required.

The threshold function $\theta(1 - 4m^2/q^2)$ indicates that (8.30) is present only for momenta which could lead to a final real pair state in addition to the virtual pairs in the closed electron loop. ry.?

The proof that the S matrix is unitary in any order is best carried out within the framework of field theory.

8.3 Renormalization of External Photon Lines

The contribution of (8.8) has been discussed so far for the propagator of a virtual photon.

Closed electron loops will also correct the contributions of external photon lines.

Here the photon may be visualized as in Fig. 8.6 as having been produced by some distant source.

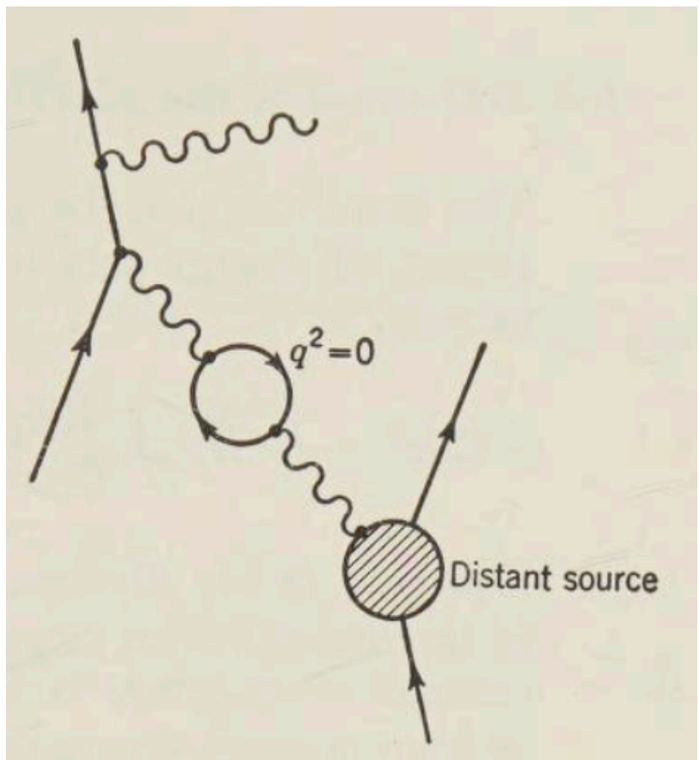

Fig. 8-6 Exchange of a "real" photon between two currents macroscopically separated in space.

If the vacuum polarization bubble is incorporated into the system of interest, it provides a multiplicative factor of Z_3 to the uncorrected matrix element, according to (8.23) and (8.24).

However, the source current then remains unrenormalized.

If a factor $\sqrt{Z_3}$ is associated with the source and the other $\sqrt{Z_3}$ with the system of interest, the bare charge e at each vertex will be replaced by $\sqrt{Z_3}$ $e = e_R$.

Thus a working rule for dealing with real external photons is to ignore corrections to all external lines and replace e by e_R at each external vertex.

This is equivalent to calculating all graphs, including vacuum polarization bubbles on external lines, and then dividing by $\sqrt{Z_3}$, for each external photon line.

Hereafter we shall assume, when writing equations, that charge renormalization has been carried out.

$e^2/4\pi$ denotes 1/137, and the bare charge, whenever needed, will be denoted by e_0.

8.4 Self-mass of the Electron

The amplitude for the graph of Fig. 8.4d—known as the electron proper self energy part to order e^2 -is given by the integral in (8.7), namely,

$$-i\Sigma(p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{(-i)}{k^2 - \lambda^2 + i\epsilon} \gamma_\nu \frac{i}{\not p - \not k - m + i\epsilon} \gamma^\nu \tag{8.34}$$

Equation (8.34) diverges, since there are only three powers of k in the denominator, two coming from the photon and one from the electron propagator.

λ is a small photon mass inserted to protect us from infrared divergences which will appear.

Introducing (8.12) and taking the analogous steps to (8.16), we come to

$$\Sigma(p) = \frac{\alpha}{2\pi}\int_0^\infty\int_0^\infty \frac{dz_1\,dz_2}{(z_1+z_2)^2}\left[2m - \frac{\not p z_1}{z_1+z_2}\right] \times \exp\left[i\left(\frac{p^2 z_1 z_2}{z_1+z_2} - m^2 z_2 - \lambda^2 z_1\right)\right] \tag{8.35}$$

Σ(p) in (8.35) applies both for internal electron lines with arbitrary p^2 and $\not p$ in a Feynman graph and for external lines.

In the latter case $p^2 = m^2$ and $\not p$ stands next to a free-particle spinor as in (8.7).

The Dirac equation may then be used to set $\not p = m$.

As in vacuum polarization, we use (8.18) and let $z_i \to \gamma z_i$, obtaining

$$\Sigma(p) = \frac{\alpha}{2\pi}\int_0^1 dz\,[2m - \not p(1-z)]\int_0^\infty \frac{d\gamma}{\gamma}\exp\{i\gamma[p^2 z(1-z) - m^2 z - \lambda^2(1-z) + i\epsilon]\} \tag{8.36}$$

The integral

$$J(p,m,\lambda) = \int_0^\infty \frac{d\gamma}{\gamma}\exp\{i\gamma[p^2 z(1-z) - m^2 z - \lambda^2(1-z) + i\epsilon]\}$$

diverges logarithmically; we cut it off by subtracting off J (p, m, Λ) with Λ a large mass.

Using the identity

$$\int_0^\infty \frac{dx}{x}\,(e^{iax} - e^{ibx}) = \left(\log \frac{b}{a}\right) \tag{8.37}$$

we find for the propagator, after cutoff,

$$\begin{aligned}
\bar{\Sigma}(p) &= \frac{\alpha}{2\pi}\int_0^1 dz\,[2m - \not{p}(1-z)] \\
&\qquad \times \log \frac{\Lambda^2(1-z)}{m^2 z + \lambda^2(1-z) - p^2 z(1-z) - i\epsilon} \\
&= \frac{\alpha}{2\pi}\int_0^1 dz\,[2m - \not{p}(1-z)] \log \frac{\Lambda^2(1-z)}{m^2 z^2} \\
&\qquad + \frac{\alpha}{2\pi}\int_0^1 dz\,[2m - \not{p}(1-z)] \\
&\qquad\qquad \times \log \frac{m^2 z^2 + \lambda^2(1-z)}{m^2 z + \lambda^2(1-z) - p^2 z(1-z)} \\
&\cong \frac{3\alpha m}{4\pi}\log\frac{\Lambda^2}{m^2} - \frac{\alpha}{4\pi}(\not{p} - m)\log\frac{\Lambda^2}{m^2} \\
&\qquad + \frac{\alpha}{2\pi}\int_0^1 dz\,[2m - \not{p}(1-z)] \\
&\qquad\qquad \times \log \frac{m^2 z^2 + \lambda^2(1-z)}{m^2 z + \lambda^2(1-z) - p^2 z(1-z)}
\end{aligned} \tag{8.38}$$

All the cutoff dependence lies in the first two terms, which will be disposed of by the renormalization procedure.

The finite term separated off in (8.38) is fixed uniquely by requiring it to vanish identically for the electron on the mass shell, $p^2 = m^2$.

The integral is readily evaluated for $p^2 - m^2 \gg m\lambda$); one obtains

$$\frac{\alpha}{2\pi}\int_0^1 dz\,[2m - \not{p}(1-z)]\log\frac{m^2 z}{m^2 - p^2(1-z)}$$
$$= \frac{\alpha m}{\pi}\left(\frac{m^2-p^2}{p^2}\right)\log\frac{m^2-p^2}{m^2}$$
$$- \frac{\alpha}{4\pi}\not{p}\left(\frac{m^2-p^2}{p^2}\right)\left[1 + \left(\frac{m^2+p^2}{p^2}\right)\log\frac{m^2-p^2}{m^2}\right] \qquad (8.39)$$

Near the "mass shell," that is, when $p^2 \sim m^2$ (but $p^2 - m^2 \gg m\lambda$), and when Σ stands next to a free-particle spinor ($\not{p} = m$),

$$\bar{\Sigma}(p) \cong \frac{3\alpha}{4\pi} m \log\frac{\Lambda^2}{m^2} - \frac{\alpha}{4\pi}(\not{p} - m)\left(\log\frac{\Lambda^2}{m^2} + 4\log\frac{m^2-p^2}{m^2}\right) \qquad (8.40)$$

Notice the logarithmic singularity as $p^2 \to m^2$

For $p^2 > m^2$, Σ becomes complex, corresponding to the existence of the process of virtual electron decaying into electron and photon, in analogy to what happened to the photon propagator,

For $p^2 - m^2 \ll m\lambda$ the last logarithm in (8.40) is replaced by $\log(\lambda/m)$.

This may be verified by a direct calculation of the integral in (8.38) in the limit $p^2 \to m^2$.

8.5 Renormalization of the Electron Propagator

The modification of the electron propagator thus far has been the replacement, according to (8.34),

$$\frac{i}{\not p - m} \to \frac{i}{\not p - m} + \frac{i}{\not p - m}(-i\Sigma(p))\frac{i}{\not p - m} = \frac{i}{\not p - m - \Sigma(p)} + 0(\alpha^2) \tag{8.41}$$

From (8.40), we write

$$\Sigma(p) = \delta m - [Z_2^{-1} - 1 + C(p)](\not p - m) \tag{8.42}$$

with

$$\delta m = \frac{3\alpha m}{4\pi}\log\frac{\Lambda^2}{m^2}$$

and

$$Z_2^{-1} - 1 + C(p) \cong \frac{\alpha}{4\pi}\left(\log\frac{\Lambda^2}{m^2} + 4\log\frac{m^2 - p^2}{m^2}\right) \qquad m\lambda \ll p^2 - m^2 \ll m^2$$

C(p) is chosen such that at $\not p$ = m, C(p) = 0; it thus contains no dependence upon the cutoff Λ.

At p = m this becomes

$$Z_2^{-1} - 1 = \frac{\alpha}{4\pi}\left(\log\frac{\Lambda^2}{m^2} - 2\log\frac{m^2}{\lambda^2}\right) \tag{8.43}$$

Using (8.42) we may now rewrite (8.41) as follows:

$$\frac{i}{\not p - m - \Sigma(p)} = \frac{iZ_2}{(\not p - m)[1 + Z_2 C(p)] - Z_2\,\delta m} = \frac{iZ_2}{(\not p - m - \delta m)[1 + C(p)]} + 0(\alpha^2) \tag{8.44}$$

We identify $m_{ph} = m + \delta m$ as the physical mass of the electron; the parameter *m* in the Dirac equation is, like the bare charge, another unmeasured number.

The necessity of mass renormalization already occurs in classical electrodynamics; experiments on a free electron measure *m*, the parameter in the Lorentz force law, plus the inertia of the electron's self-field.

For a classical electron of radius ~ a, the electromagnetic self-energy is $\sim\alpha/a$ and the observed mass is $\sim(m + \alpha/a) = m_{ph}$.

For a point charge, $a \to 0$ and the correction to the mass becomes infinite.

This is true also in Dirac theory; however, here it diverges logarithmically with the cutoff in contrast to the classical self-energy correction, which is linearly divergent as the charge radius $a \to 0$.

This weakening of the divergence is a consequence of hole theory.

As first studied by Weisskopf, the virtual pairs in the time-ordered graph of Fig. 8.7a cancel the leading divergence in Fig. 8.7b.

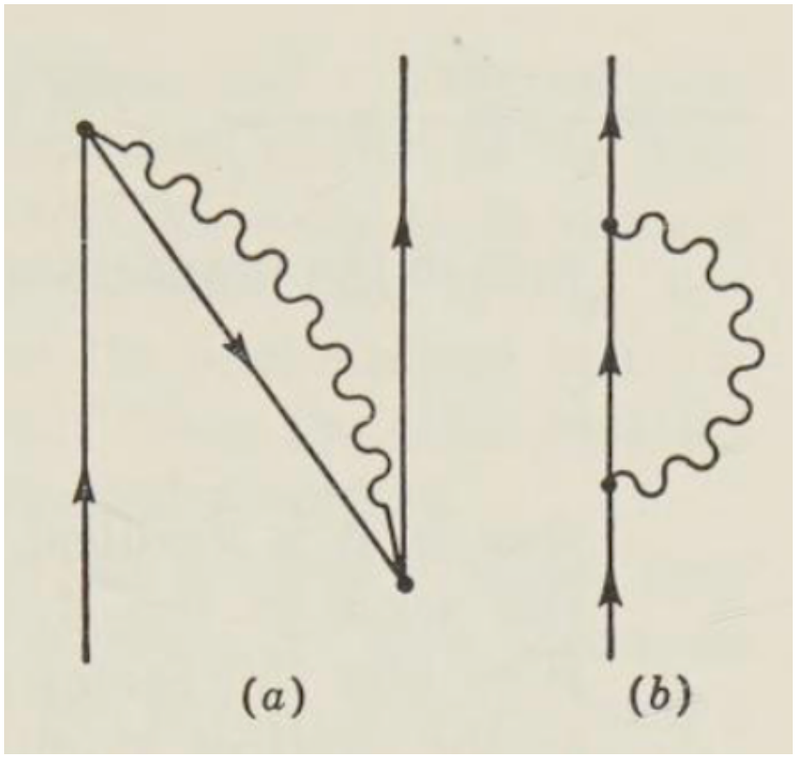

Fig. 8-7 Time orderings for the second-order self-mass correction.

Although formally infinite, the mass correction is small for cutoff masses $\Lambda < me^{2\pi/3\alpha} \sim 10^{100}m$.

On the other hand, the mass of the universe is estimated to be $\sim 10^{80}m$.

A systematic way of carrying out mass renormalization is to rewrite the Dirac equation in terms of the physical mass and treat the difference as an additional interaction term.

That is, we write

$$(i\not{\nabla} - m_{ph})\psi = e\not{A}\psi + (m - m_{ph})\psi = e\not{A}\psi - \delta m\psi \tag{8.45}$$

The additional interaction term is represented by the graph in Fig. 8.8.

This term just cancels out the first term in (8.40), and the propagator reduces to a multiple of the free propagator as $\not{p} \rightarrow m_{ph}$.

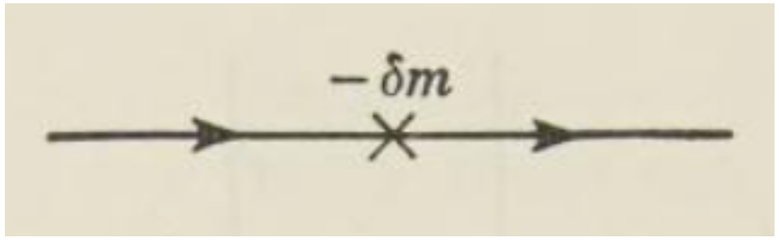

Fig. 8-8 Mass renormalization counterterm.

We shall hereafter suppose mass renormalization to be carried out, that is, the graphs of Fig. 8.8 to be included; we shall let *m* denote the *physical* mass of the electron.

The rest of the correction to the propagator lies in Z_2 and the function C(p), chosen such that at $\not{p} = m$, $C(p) = 0$.

Thus, for $\not{p} \simeq m$ the propagator is given by

$$\frac{i}{\not{p} - m} \rightarrow \frac{iZ_2}{\not{p} - m} \tag{8.46}$$

that is, it is modified by a multiplicative factor. Z_2 is analogous in this sense to the Z_3 factor encountered in the photon propagator.

Here also this factor may be absorbed into the charge e_0 appearing at the vertices at either end of the electron line; however, this is unnecessary, since we shall see that the correction to the vertex will cancel the Z_2.

We cannot expect Z_2 to contain much physics, since it depends upon the photon mass according to (8.43).

One must be careful not to correct external lines twice; the situation here is similar to that encountered for the photons.

The propagator is an expression bilinear in the field amplitudes, as seen, for example, in (6.48).

However, an external line represents a field amplitude; hence it is renormalized by the factor $\sqrt{Z_2}$.

Thus if all graphs giving corrections to external lines are included in the calculation, the result must be divided by $\sqrt{Z_2}$ for each external electron line.

A familiar example of this effect is found in nonrelativistic perturbation theory, where

$$\psi_n = \sqrt{Z_n}\,\varphi_n + \sum_{m\neq n} \frac{(\varphi_m, V\psi_n)}{E_n - E_m^0}\,\varphi_m \tag{8.47}$$

with

$$Z_n = 1 - \sum_{m\neq n} \frac{|(\varphi_m, V\psi_n)|^2}{(E_n - E_m^0)^2} \tag{8.48}$$

Again, the Z factor is computed essentially from the Green's function and the wave function is renormalized by $\sqrt{Z_n}$.

8.6 The Vertex Correction

There remains only the graph of Fig. 8.4c, which shows the correction due to a photon bridging the vertex γ_μ.

This contribution is referred to as the second-order vertex part.

In order to compute its contribution to physical processes, we study the integral

$$\Lambda_\mu(p',p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{(-i)}{k^2 - \lambda^2 + i\epsilon} \gamma_\nu \frac{i}{\not{p}' - \not{k} - m + i\epsilon} \times \gamma_\mu \frac{i}{\not{p} - \not{k} - m + i\epsilon} \gamma^\nu \tag{8.49}$$

where p' denotes the momentum of the electron and -p the momentum of the physical positron created by the virtual photon in Fig. 8.4c.

Equally well, (8.49) represents a radiative correction due to an electron scattering from some external potential, as illustrated in Fig. 8.9.

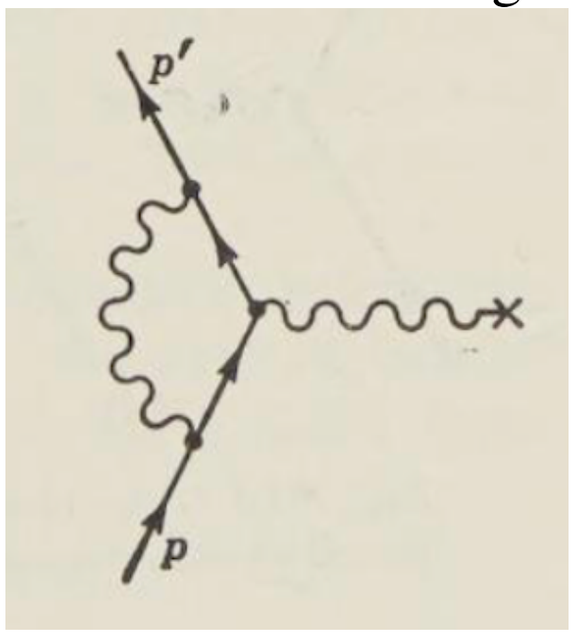

Fig. 8-9 Vertex correction to scattering in an external electromagnetic potential.

In this case, p' is again the momentum of the final electron but p is now the momentum of the initial electron.

Thus the same function (8.49) describes corrections to different physical processes.

The amplitude (8.49) diverges, since the integral contains only four powers of k in the denominator.

In addition, we shall encounter an infrared divergence and again assign the photon a small mass λ to cut off the contribution of very soft photons.

We identify the infinite part to be separated out by considering Λ^{ν}(p', p) for

$$q = p' - p \to 0$$

and for free-particle momenta for the initial and final electron, that is, $\not{p} = m, \not{p}' = m.$

In this case

$$\bar{u}(p)\Lambda_\mu(p,p)u(p) = (Z_1^{-1} - 1)\bar{u}(p)\gamma_\mu u(p) \tag{8.50}$$

where Z_1 is a constant depending upon the masses $m^2 = p^2$, λ^2, and the cutoff needed to make it finite.

Equation (8.50) is general, since the only other four-vector, p_υ, is the same as $m\gamma$, when sandwiched between spinors $\bar{u}(p)$ and u(p) in (8.50).

It is not necessary to calculate Z_1, because a direct comparison of (8.49) for p' = p and of the propagator Σ(p) in (8.34) shows that

$$\Lambda_\mu(p,p) = -\frac{\partial\Sigma(p)}{\partial p^\mu} \tag{8.51}$$

Here the important identity

$$\frac{\partial}{\partial p^\mu}\frac{1}{\not{p} - m} = -\frac{1}{\not{p} - m}\gamma_\mu\frac{1}{\not{p} - m} \tag{8.52}$$

is used; it says that differentiation of a free propagator with respect to momentum is equivalent to the insertion of a zero-energy photon in the line.

$\partial\Sigma$(p)/∂p^μ may be computed directly from (8.42), and we find

$$\bar{u}(p)\Lambda_\mu(p,p)u(p) = (Z_2^{-1} - 1)\bar{u}(p)\gamma_\mu u(p) \tag{8.53}$$

or by (8.50)

$$Z_1 = Z_2 \tag{8.54}$$

to order e^2.

To this order, the vertex correction is then

$$\Lambda_\mu(p',p) = (Z_1^{-1} - 1)\gamma_\mu + \Lambda_\mu^c(p',p) \tag{8.55}$$

All the cutoff dependence is contained in Z_1.

$\Lambda^c{}_\mu(p',p)$ is finite provided we avoid the infrared catastrophe by keeping the photon mass $\lambda > 0$.

It is also unique, satisfying the condition

$$\bar{u}(p)\Lambda_\mu^c(p,p)u(p) = 0 \tag{8.56}$$

We now may either regard Z_1 as renormalizing the charge e at the vertex or observe that it just cancels the $\sqrt{Z_2}$ wave function renormalization of the external lines.

This is best seen by looking at all the graphs to order e for the forward scattering of an electron from a potential.

These are shown in Fig. 8.10.

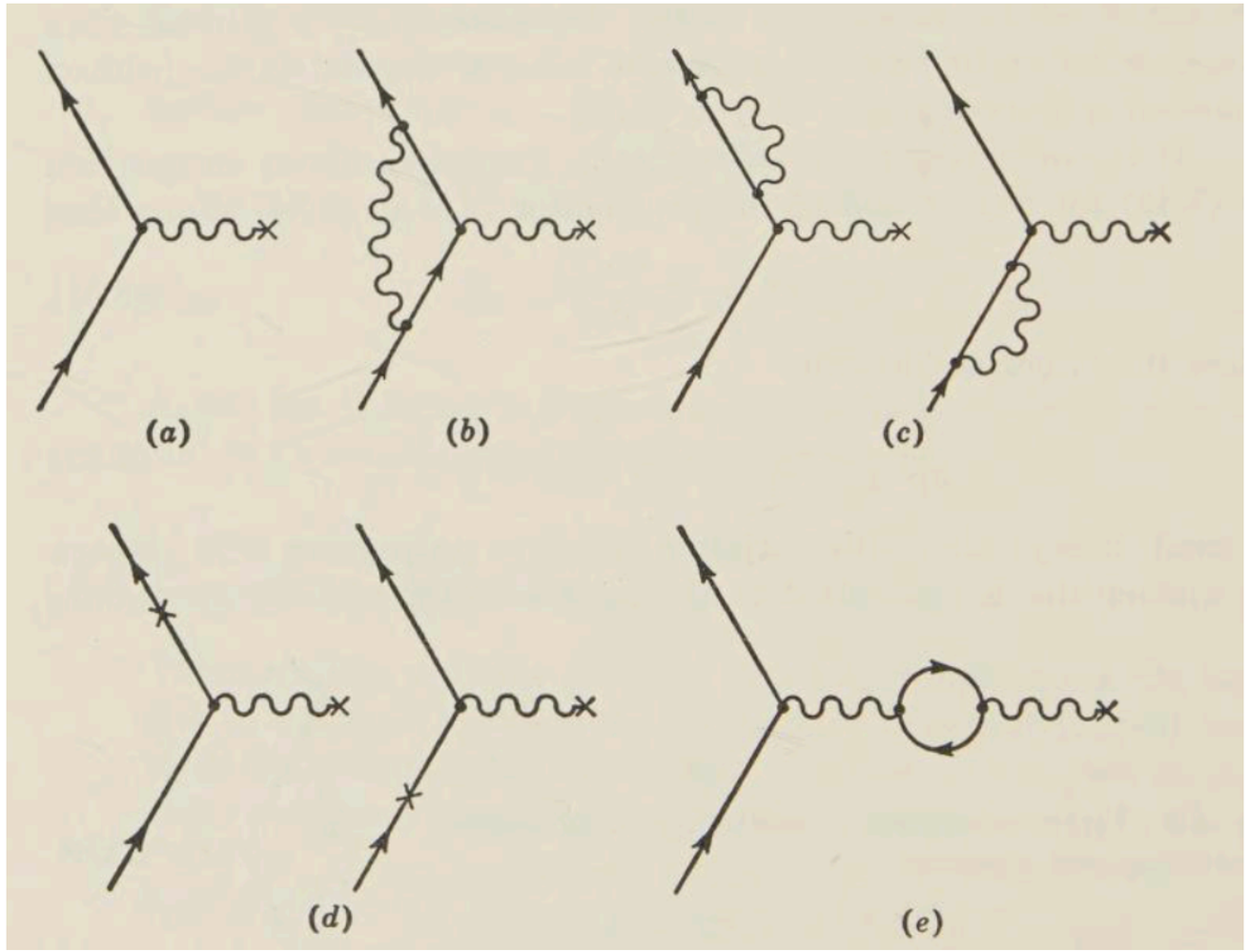

Fig. 8-10 Second-order radiative corrections to scattering in an external electromagnetic potential.

The contributions of these graphs in the limit q→ 0 are listed below:

$$
\begin{aligned}
&(a) \qquad -ie\gamma_\mu \\
&(b) \qquad -ie\gamma_\mu(Z_1^{-1} - 1) \\
&(c) \qquad +\delta m \frac{1}{\not p - m}(-ie\gamma_\mu) - (Z_2^{-1} - 1)(-ie\gamma_\mu) \\
&(d) \qquad -\delta m \frac{1}{\not p - m}(-ie\gamma_\mu) \\
&(e) \qquad -(-ie\gamma_\mu)\frac{\alpha}{3\pi}\log\frac{\Lambda^2}{m^2} = -ie\gamma_\mu(Z_3 - 1)
\end{aligned}
\tag{8.57}
$$

According to our previous discussion, we also divide by $\sqrt{Z_2}$ for each external electron line and $\sqrt{Z_2}$ for the photon line; the sum of all these contributions is, to order e^2,

$$\begin{aligned}
&\frac{1}{Z_2\sqrt{Z_3}}(-ie\gamma_\mu)[1+(Z_1^{-1}-1)-2(Z_2^{-1}-1)+(Z_3-1)] \\
&\qquad\approx \frac{1}{Z_2\sqrt{Z_3}}(-ie\gamma_\mu)\frac{[1+(Z_1^{-1}-1)][1+(Z_3-1)]}{[1+(Z_2^{-1}-1)]^2} \\
&\qquad= -ieZ_1^{-1}Z_2\sqrt{Z_3}\,\gamma_\mu \\
&\qquad= -ie_R\gamma_\mu
\end{aligned} \tag{8.58}$$

where (8.25) and (8.54) are used in the last step.

Between the vertex part and the propagator the Z_2 renormalization is completely removed.

The vacuum polarization is entirely responsible for the charge renormalization.

The rather elaborate notation employed in arriving at (8.58) is used with an eye to dealing with higher orders.

In particular, (8.51) and the relation $Z_1 = Z_2$ in (8.54) are true to all orders (Ward's identity) as is the result that all divergent integrals can be absorbed into the renormalization constants Z_1, Z_2, and Z_3.

We have already found a physically observable effect in the finite part of the vacuum polarization graph.

Looking into the finite part of the vertex and electron self-energy contributions, we also uncover predictions of great physical interest.

Turning to the vertex $\Lambda_\mu(p',p)$, a somewhat lengthy calculation is required to reduce the integrals in (8.49).

We first rationalize electron propagators and combine denominators, using either exponentiation of the propagator denominators (8.12) followed by the scaling trick (8.18) or, more directly, the formula

$$\frac{1}{a_1 \cdots a_n} = (n-1)! \int_0^\infty \frac{dz_1 \cdots dz_n \, \delta\left(1 - \sum_i z_i\right)}{\left(\sum_i a_i z_i\right)^n} \tag{8.59}$$

So you can see some of the algebra, I copy the Feynman Integral derivation from a paper [Phys. Rev., 76, 679 (1949)]

$$\int_{-\infty}^{\infty} d^4k\, f(k) \frac{1}{k^2 - \lambda^2 + i\epsilon} \frac{1}{(p'-k)^2 - m^2 + i\epsilon} \frac{1}{(p-k)^2 - m^2 + i\epsilon}$$

$$= 2\int_0^\infty \int_0^\infty \int_0^\infty dz_1\, dz_2\, dz_3\, \delta(1 - z_1 - z_2 - z_3)$$

$$\times \int_{-\infty}^{\infty} \frac{d^4k\, f(k)}{[k^2 - 2k\cdot p' z_2 - 2k\cdot p z_3 - \lambda^2 z_1 + (p'^2 - m^2)z_2 + (p^2 - m^2)z_3 + i\epsilon]^3}$$

$$= 2\int_0^\infty \int_0^\infty \int_0^\infty dz_1\, dz_2\, dz_3\, \delta(1 - z_1 - z_2 - z_3) \int_{-\infty}^{\infty} \frac{d^4k\, f(k + p'z_2 + pz_3)}{(k^2 - c + i\epsilon)^3}$$

with $c \equiv (p'z_2 + pz_3)\cdot(p'z_2 + pz_3) + \lambda^2 z_1 - (p'^2 - m^2)z_2 - (p^2 - m^2)z_3$

$$= -p'^2 z_2(1 - z_2) - p^2 z_3(1 - z_3) + 2p\cdot p' z_2 z_3 + m^2(1 - z_1) + \lambda^2 z_1$$

$$= -(p'^2 - m^2)z_2(1 - z_2) - (p^2 - m^2)z_3(1 - z_3) - q^2 z_2 z_3 + m^2(1 - z_1)^2 + \lambda^2 z_1$$

We now perform the $\int d^4k$ by contour methods, carrying out the $\int_{-\infty}^{\infty} dk_0$ with the contour displaced from the poles at $\pm\sqrt{|\mathbf{k}|^2 + c}$ by $\mp i\epsilon$ as indicated: $k^2 - c + i\epsilon = (k_0 - \sqrt{|\mathbf{k}|^2 + c} + i\epsilon)(k_0 + \sqrt{|\mathbf{k}| + c} - i\epsilon)$. This gives

$$\int_{-\infty}^{\infty} d^4k \frac{1}{(k^2 - c + i\epsilon)^3} = \frac{\pi^2}{2ic}$$

The results for arbitrary powers of the denominator, $n > 3$, are obtained by differentiating with respect to c. Because of the symmetry of the denominator, which is a function of k^2 only in the above expression, numerator factors are also easily reduced: odd powers of $k_\mu \to 0$

$$k_\mu k_\nu \to \tfrac{1}{4} g_{\mu\nu} k^2 \qquad \text{etc.}$$

One finds, after the four-dimensional k integration (and using a cutoff Λ^2 on the divergent integral)

$$\Lambda_\mu(p',p) = \frac{\alpha}{4\pi}\gamma_\mu\left[\log\frac{\Lambda^2}{m^2} + 0(1)\right]$$

$$+ \frac{\alpha}{2\pi}\gamma_\mu\int_0^\infty\int_0^\infty\int_0^\infty dz_1\,dz_2\,dz_3\,\delta\left(1-\sum_{i=1}^{3}z_i\right)$$

$$\times\log\frac{m^2(1-z_1)^2+\lambda^2 z_1}{m^2(1-z_1)^2+\lambda^2 z_1 - q^2 z_2 z_3 - i\epsilon}$$

$$-\frac{\alpha}{4\pi}\int_0^\infty\int_0^\infty\int_0^\infty dz_1\,dz_2\,dz_3\,\delta\left(1-\sum_{i=1}^{3}z_i\right)$$

$$\times\frac{\gamma_\nu[\not p'(1-z_2)-\not q z_3+m]\gamma_\mu[\not p(1-z_3)-\not p' z_2+m]\gamma^\nu}{m^2(1-z_1)^2+\lambda^2 z_1 - q^2 z_2 z_3 - i\epsilon} \qquad (8.60)$$

At this stage it is convenient to reduce the numerator of the last term by anticommuting $\not p$ and $\not p'$ to the sides, where they may act upon electron spinors between which we tacitly assume Λ_μ to be sandwiched.

The Gordon reduction (3.26) helps here; the numerator of the last term becomes

$$-\gamma_\mu[2m^2(1-4z_1+z_1^2)+2q^2(1-z_2)(1-z_3)] - 2mz_1z_2[q,\gamma_\mu] \qquad (8.61)$$

The integration over the z's is in general a mess, although an analytic result has been obtained and is quoted in many papers.

We restrict ourselves here to the two limits $|q|^2 \ll m^2$ and $|q^2| \gg m^2$.

In the first case the integrations are straightforward and yield, to order q^2,

$$\gamma_\mu + \Lambda_\mu^c(p',p) \cong \gamma_\mu \left[1 + \frac{\alpha}{3\pi}\frac{q^2}{m^2}\left(\log\frac{m}{\lambda} - \frac{3}{8}\right)\right] + \frac{\alpha}{8\pi m}[q,\gamma_\mu] \tag{8.62}$$

For $|q^2| > m^2$ we obtain the λ-dependent terms only and find

$$\gamma_\mu + \Lambda_\mu^c(p',p) \cong \gamma_\mu \left\{1 - \frac{\alpha}{\pi}\log\frac{m}{\lambda}\left[\log\frac{(-q^2)}{m^2} - 1 + 0\left(\frac{m^2}{q^2}\right)\right]\right\} \tag{8.63}$$

Adding these results to the contribution from the vacuum polarization (8.26) gives the radiative correction to order a for an electron scattering in an external field which supplies a virtual photon q.

From (8.26) we find that the vacuum polarization adds a constant -1/5 to to the -3/8 in (8.62) in the low momentum transfer limit and has no effect on the infrared or magnetic moment terms in (8.62) and (8.63).

The last term of (8.62) adds a magnetic moment of $\alpha/2\pi$ to an electron, since it modifies the static limit of the interaction of an electron with an external field to

$$\begin{aligned} &-ie\bar{u}(p')\left(\gamma_\mu + \frac{i\alpha}{2\pi}\frac{\sigma_{\mu\nu}q^\nu}{2m}\right)u(p)A^\mu(q) \\ &\quad = -ie\bar{u}(p')\left[\frac{(p+p')_\mu}{2m} + \left(1 + \frac{\alpha}{2\pi}\right)\frac{i\sigma_{\mu\nu}q^\nu}{2m}\right]u(p)A^\mu(q) \end{aligned} \tag{8.64}$$

This correction factor of $(1 + \alpha/2\pi)$ for the electron magnetic moment was first derived by Schwinger in 1948 and has since been confirmed experimentally.

The experiments have become sufficiently accurate to study the a correction to the magnetic moment.

This has been calculated by Sommerfeld and Petermann; their result of $-(\alpha^2/\pi^2)$ (0.328) is in agreement with present experimental limits.

The result is obtained by considering all vertex graphs involving the exchange of two virtual photons.

The other terms of (8.62) and (8.63) lead to infrared divergent contributions to electron scattering.

These, however, disappear when the contribution from bremsstrahlung of soft photons is included in the cross section.

Any experimental apparatus has finite resolution; if electrons are detected with a given energy resolution ΔE, the number of observed events corresponds to the elastic cross section plus the bremsstrahlung cross section leading to electrons whose energy is within ΔE of the elastic value.

We verify to order e that this sum of elastic plus inelastic cross sections is finite and free of the infrared difficulty by comparing (7.64) and (8.62) and (8.63).

The infrared part of the elastic cross section to order e^2 is

$$\left(\frac{d\sigma}{d\Omega}\right)_\lambda = \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1 - \frac{2\alpha}{\pi} \log \frac{m}{\lambda} \chi(q^2)\right] \tag{8.65}$$

with

$$\chi(q^2) = \begin{cases} -\frac{1}{3}\frac{q^2}{m^2} & -\frac{q^2}{m^2} \ll 1 \\ \log\frac{(-q^2)}{m^2} - 1 & -\frac{q^2}{m^2} \gg 1 \end{cases} \tag{8.66}$$

$(d\sigma/d\Omega)_0$ is the lowest order contribution to the elastic scattering cross section.

The bremsstrahlung cross section is given by (7.64)

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{brem}} = \left(\frac{d\sigma}{d\Omega}\right)_0 \frac{2\alpha}{\pi} \log\frac{k_{\max}}{k_{\min}} \chi(q^2) \tag{8.67}$$

We cannot directly add these last two equations together because we have cut off the low-frequency photons differently in the two cases.

To overcome this problem, we have the choice of rederiving the bremsstrahlung cross section with photons of finite mass or of recomputing the vertex corrections assuming that emission of photons of energy less than k_{min} is suppressed.

We choose the latter alternative in order to avoid the intricacies due to real longitudinal quanta, absent when $\lambda = 0$.

Since the introduction of the photon cutoff k_{min} is a noncovariant procedure, identification of the renormalization parts in the vertex correction becomes very delicate.

This is why the development was given first in terms of the invariant photon mass λ.

Furthermore, we choose $k_{min} >> \lambda$ to ease the mathematics.

Therefore, we must now return to (8.49) and modify the photon propagator by suppressing the $k < k_{min}$ amplitude.

This means that $D_F(x - y, \lambda)$ in the photon propagator (7.30) is modified from

$$D_F(x-y,\lambda) = i\int \frac{d^3q}{(2\pi)^3 2|q_0|} e^{iq\cdot(x-y)-i|q_0||x_0-y_0|}$$
$$= -\int \frac{d^4q}{(2\pi)^4} \frac{e^{-iq\cdot(x-y)}}{(q^2-\lambda^2+i\epsilon)} \qquad (8.68)$$

to

$$D_F(x-y,k_{min}) = i\int_{|q|>k_{min}} \frac{d^3q}{(2\pi)^3 2|q|} e^{iq\cdot(x-y)-i|q||x_0-y_0|}$$
$$\cong D_F(x-y,\lambda) - i\int_{|q|<k_{min}} \frac{d^3q}{(2\pi)^3 2|q_0|} e^{iq\cdot(x-y)-i|q_0||x_0-y_0|}$$
$$= D_F(x-y,\lambda)$$
$$+ \int_{|q|<k_{min}} \frac{d^2q}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{dq_0}{2\pi} e^{-iq\cdot(x-y)} \frac{1}{q^2-\lambda^2+i\epsilon} \qquad (8.69)$$

where

$$q_0 = \sqrt{q^2+\lambda^2} \qquad k_{min} \gg \lambda \qquad (8.70)$$

The regions of momentum space in which the two propagators are modified are shown in Fig. 8.11.

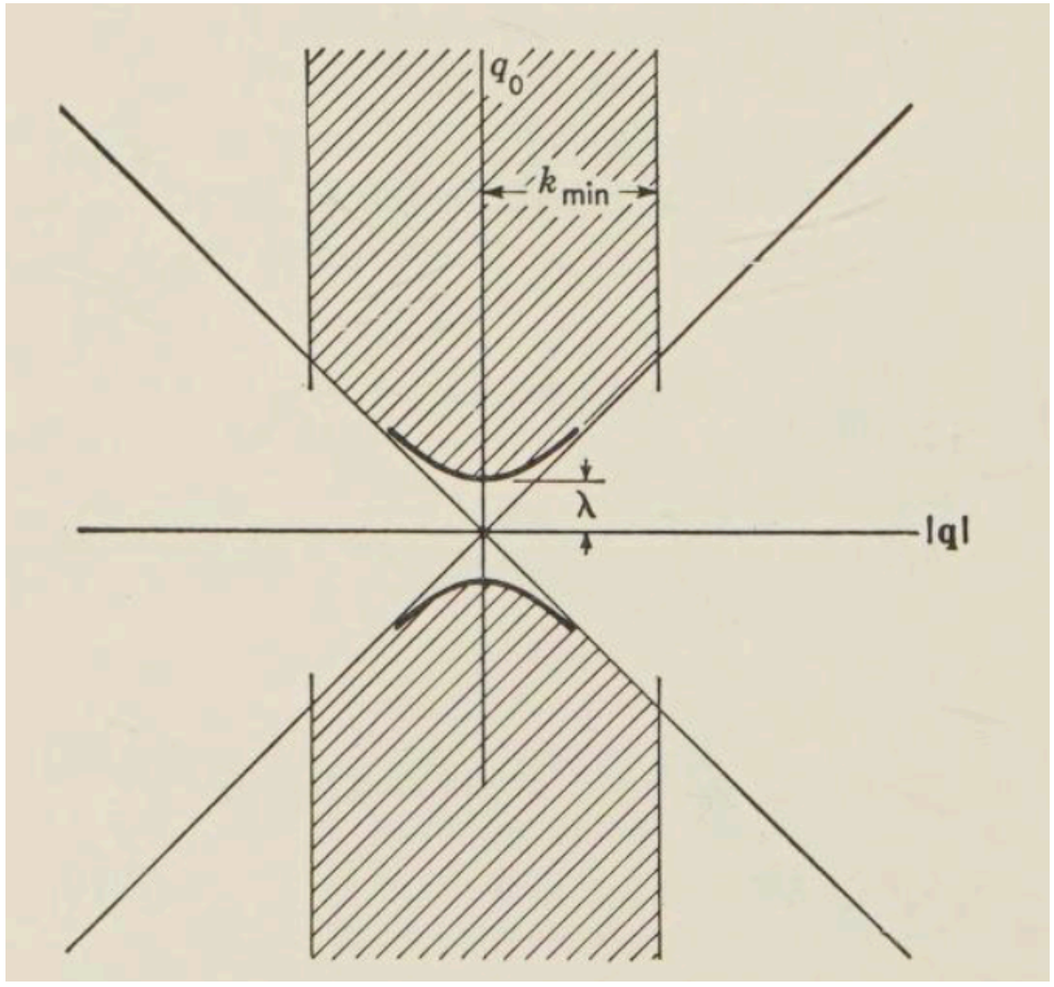

Fig. 8-11 Regions of momentum space modified by infrared cutoffs.

The change $\delta\Lambda_\mu(p',p)$ in the vertex (8.49) is then

$$\begin{aligned}\delta\Lambda_\mu(p',p) &= \Lambda_\mu(p',p,\lambda) - \Lambda_\mu(p',p,k_{\min}) \\ &= -ie^2 \int_{|\mathbf{k}|<k_{\min}} \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \\ &\times \frac{\gamma_\nu(\not{p}' - \not{k} + m)\gamma_\mu(\not{p} - \not{k} + m)\gamma^\nu}{(k^2 - \lambda^2 + i\epsilon)[(p' - k)^2 - m^2 + i\epsilon][(p - k)^2 - m^2 + i\epsilon]}\end{aligned} \tag{8.71}$$

This expression is free from ultraviolet divergence, since the range of integration is limited.

To compute (8.71), we do the k_0 integration first by performing a contour integral in the k_0 plane and using Cauchy's theorem.

Three simple poles are enclosed, as indicated in Fig. 8.12.

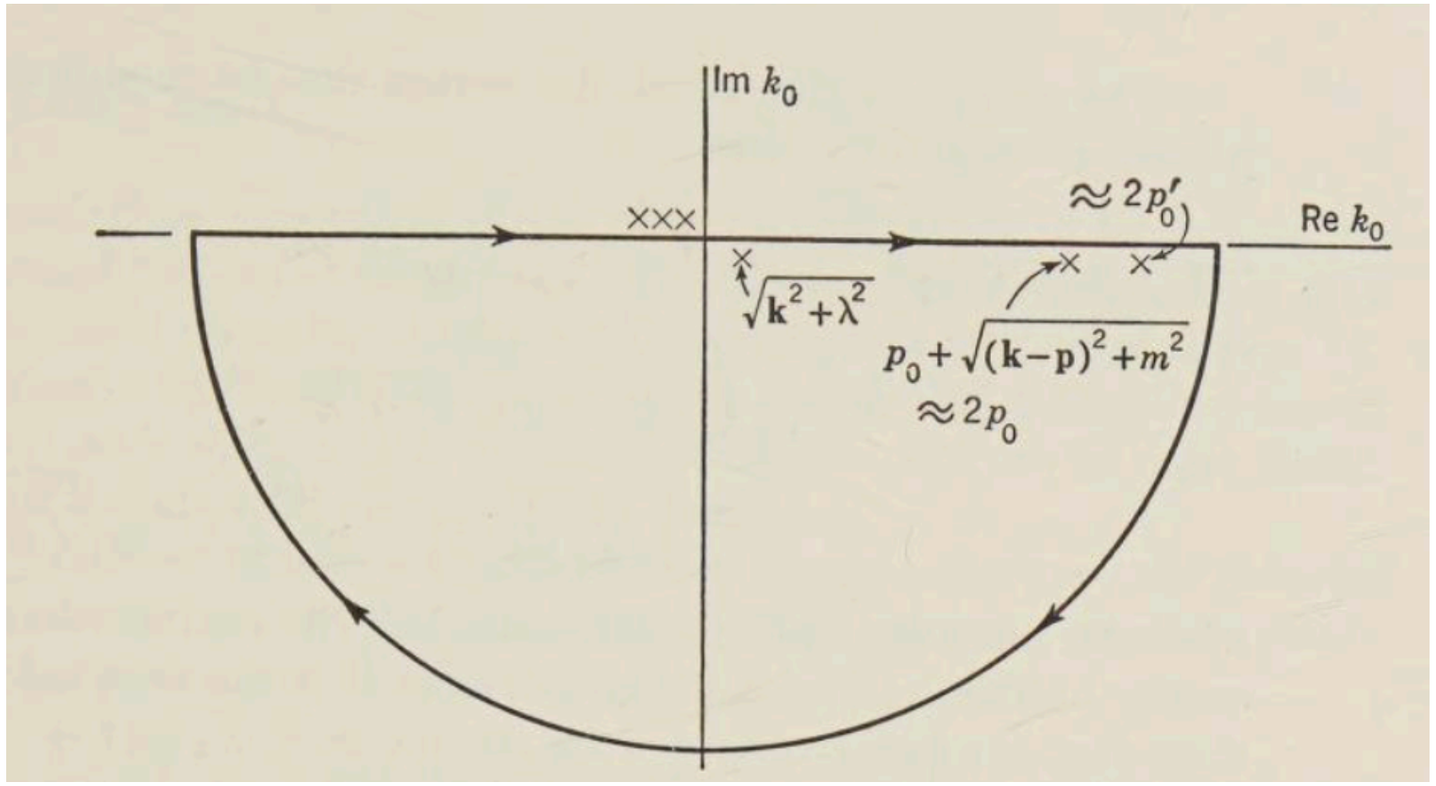

Fig. 8-12 Singularities in the k_0 plane encountered in computing $\delta\Lambda_\mu(p',p)$.

In the limit $k_{min} \ll m$, only the residue from the pole at $k_0 = \sqrt{\mathbf{k}^2+\lambda^2}$ survives in $\delta\Lambda_\mu(p',p)$, which simplifies to

$$\delta\Lambda_\mu(p',p) = -e^2 \int_{|\mathbf{k}|<k_{\min}} \frac{d^3k}{(2\pi)^3 2\sqrt{\mathbf{k}^2+\lambda^2}} \frac{\gamma_\nu(\not{p}'+m)\gamma_\mu(\not{p}+m)\gamma^\nu}{(2k\cdot p)(2k\cdot p')}$$

$$= -e^2\gamma_\mu \int_{|\mathbf{k}|<k_{\min}} \frac{d^3k}{(2\pi)^3 2\sqrt{\mathbf{k}^2+\lambda^2}} \frac{p\cdot p'}{(k\cdot p)(k\cdot p')} \qquad (8.72)$$

where we anticipate sandwiching $\delta\Lambda_\mu$, between free electron spinors.

The renormalization is now delicate because the introduction of the photon cutoff is a noncovariant operation.

Since (8.51) is still valid, we may use our previous conclusion ($Z_1 = Z_2$) that no renormalization of Λ_μ is necessary, provided the self-energy parts are included properly.

However, $\sum$ changes because of the changes in the photon propagator; indeed

$$\begin{aligned}\delta\Sigma(p) &= \Sigma(p,\lambda) - \Sigma(p,k_{\min}) \\ &= -ie^2 \int_{|\mathbf{k}|<k_{\min}} \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \gamma_\mu \frac{1}{\not p - \not k - m + i\epsilon} \gamma^\mu \frac{1}{k^2 - \lambda^2 + i\epsilon}\end{aligned} \tag{8.73}$$

This must be computed through order $p^2 - m^2$, since the modification in the renormalization constant Z_2 is what is needed here (δm is free of infrared divergence).

We integrate over k_0 first, as for the vertex, and obtain, through first order in $p^2 - m^2$,

$$\begin{aligned}\delta\Sigma(p) &= -e^2 \int_{|\mathbf{k}|<k_{\min}} \frac{d^3k}{(2\pi)^3\, 2\sqrt{\mathbf{k}^2+\lambda^2}} \frac{\gamma_\mu(\not p + m)\gamma^\mu}{\lambda^2 - 2k\cdot p + (p^2 - m^2)} \\ &= +e^2 \int_{|\mathbf{k}|<k_{\min}} \frac{d^3k\ (p^2 - m^2)\gamma_\mu(\not p + m)\gamma^\mu}{(2\pi)^3\, 2\sqrt{\mathbf{k}^2+\lambda^2}\ (\lambda^2 - 2k\cdot p)^2} + 0((p^2 - m^2)^2) \\ &\qquad + 0(k_{\min}) \\ &\cong +e^2 \int_{|\mathbf{k}|<k_{\min}} \frac{d^3k}{(2\pi)^3\, 2\sqrt{\mathbf{k}^2+\lambda^2}} \frac{m^2}{(k\cdot p)^2} (\not p - m)\end{aligned} \tag{8.74}$$

The complete change of the vertex due to modification of the photon propagator is then

$$\delta\Lambda_\mu + \frac{1}{2}\,\delta\Sigma(p')\,\frac{1}{\not p' - m}\,\gamma_\mu + \frac{1}{2}\,\gamma_\mu\,\frac{1}{\not p - m}\,\delta\Sigma(p)$$
$$= -\gamma_\mu e^2 \int_{|\mathbf{k}|<k_{\min}} \frac{d^3k}{(2\pi)^3\, 2\sqrt{\mathbf{k}^2+\lambda^2}}$$
$$\times\left[\frac{p\cdot p'}{(p\cdot k)(p'\cdot k)} - \frac{m^2}{2(p\cdot k)^2} - \frac{m^2}{2(p'\cdot k)^2}\right] \tag{8.75}$$

where we must remember to take only half the contribution of the self-energy bubbles of Fig. 8.10c and d, since it is the external wave functions that are being corrected by the factor $\sqrt{Z_2} \cong 1 + \frac{1}{2}(Z_2 - 1)$.

Evaluating (8.75) in the nonrelativistic limit $|q^2/m^2| \ll 1$ gives

$$\delta\Lambda_\mu^c(p',p) = \gamma_\mu \frac{\alpha}{3\pi}\frac{q^2}{m^2}\left(\log\frac{2k_{\min}}{\lambda} - \frac{5}{6}\right) \tag{8.76}$$

and thus, from (8.62) and (8.71),

$$\Lambda_\mu^c(p',p,k_{\min}) = \frac{\alpha}{2\pi}\left[\frac{i\sigma_{\mu\nu}q^\nu}{2m} + \frac{2}{3}\frac{q^2}{m^2}\gamma_\mu\left(\log\frac{m}{2k_{\min}} + \frac{5}{6} - \frac{3}{8}\right)\right] \tag{8.77}$$

For $|q^2/m^2| \gg 1$ one finds for the infrared divergent terms

$$\delta\Lambda_\mu^c(p',p) = -\gamma_\mu\frac{\alpha}{\pi}\left[\log\left(\frac{-q^2}{m^2}\right) - 1\right]\left(\log\frac{m}{\lambda} - \log\frac{E}{k_{\min}}\right) \tag{8.78}$$

and consequently, from (8.63) and (8.71),

$$\Lambda_\mu^c(p',p,k_{\min}) = -\gamma_\mu \frac{\alpha}{\pi} \log \frac{E}{k_{\min}} \left[\log\left(\frac{-q^2}{m^2}\right) - 1 \right] \tag{8.79}$$

We see that in terms of the k_{min} cutoff the infrared part of the elastic scattering is given, instead of (8.65), by

$$\left(\frac{d\sigma}{d\Omega}\right)_{k_{\min}} = \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1 - \frac{2\alpha}{\pi} \log \frac{E}{k_{\min}} \chi(q^2) \right] \tag{8.80}$$

Adding on the bremsstrahlung cross section (8.67) gives the infrared part of the scattering cross section, including emission of photons of energy less than k_{max}:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{infrared}} = \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1 - \frac{2\alpha}{\pi} \log \frac{E}{k_{\max}} \chi(q^2) \right] \tag{8.81}$$

It is completely independent of k_{min} and λ.

8.7 The Lamb Shift

The scattering correction (8.77) may be considered to be due to an additional "effective potential" which acts between the electron and the source of photons, which we take here to be a nucleus of charge Ze.

The change in the atomic energy levels due to this added interaction is the Lamb shift, which we are now able to discuss in more detail than in the physical argument of Chap. 4.

The effective interaction in momentum space between the electron and a current source $eA^\nu(q)$ is given by (8.77) plus the vacuum polarization contribution (8.26),

$$\bar{u}(p')\left\{\gamma_\mu\left[1+\frac{\alpha}{3\pi}\frac{q^2}{m^2}\left(\log\frac{m}{2k_{\min}}+\frac{5}{6}-\frac{3}{8}-\frac{1}{5}\right)\right]+\frac{i\alpha}{4\pi m}\sigma_{\mu\nu}q^\nu\right\}u(p)\times eA^\mu(q) \tag{8.82}$$

Equation (8.82) contains the corrections of order a to the electron's current operator, $\bar{u}(p')\gamma_\mu u(p)$, due to photons of momentum greater than k_{min}, and it is valid for small momentum transfers $q_\mu = p'_\mu - p_\mu$, that is, for $|q^2/m^2| << 1$.

The current source is $eA^\mu(q) = -(Ze^2/|q^2|, 0)$ for an electron in the Coulomb field of a nucleus of charge Ze, and (8.82) becomes

$$-u^\dagger(p')\left\{\frac{Ze^2}{|\mathbf{q}|^2}\left[1-\frac{\alpha}{3\pi}\frac{|\mathbf{q}|^2}{m^2}\left(\log\frac{m}{2k_{\min}}+\frac{5}{6}-\frac{3}{8}-\frac{1}{5}\right)\right]+\frac{\alpha}{4\pi m}\boldsymbol{\gamma}\cdot\mathbf{q}\right\}u(p) \tag{8.83}$$

The first term is spin-independent and is the Fourier transform of an effective interaction potential of the form

$$-\frac{Z\alpha}{r}+\frac{4\alpha}{3}\frac{Z\alpha}{m^2}\left(\log\frac{m}{2k_{\min}}+\frac{11}{24}-\frac{1}{5}\right)\delta^3(r)$$

In hydrogen-like atoms this leads to an energy shift due to photons of momentum > k_{min} which is found from a first-order perturbation calculation to be

$$\Delta E_n^{>}=\frac{4\alpha}{3}\frac{Z\alpha}{m^2}|\psi_{nlm}(0)|^2\left(\log\frac{m}{2k_{\min}}+\frac{11}{24}-\frac{1}{5}\right) \tag{8.84}$$

To this must be joined the contribution from soft photons of momentum less than k_{min}.

One expects a natural cutoff of order $k_{min} \leq (Z\alpha)m$, that is, for photon wavelengths large compared to the size of the atom.

Indeed, k_{min} cannot be chosen arbitrarily small, because the propagator of a bound electron is modified from the free-particle form for values of $p^2 - m^2 \sim (Z\alpha)^2 m^2$.

In the atom

$$p^\mu \sim (m + V, \mathbf{p})$$

with

$$V \sim (Z\alpha)^2 m \qquad \text{and} \qquad |\mathbf{p}| \sim Z\alpha m$$

and in our previous calculation of the self-energy part $\Sigma(p)$ as well as the vertex Λ_μ [see (8.74)] we assumed

$$k \cdot p \approx k_{\min} m \gg p^2 - m^2 \sim (Z\alpha)^2 m^2$$

For photons of momenta less than k_{min}, relativistic corrections should be small, that is, involve higher powers of $Z\alpha$, and one uses a completely nonrelativistic calculation, first carried out by Bethe.

From old-fashioned second-order perturbation theory, the energy shift due to emission and reabsorption of a photon by an electron in state n is

$$\Delta E_n^< = e^2 \int_0^{k_{\min}} \frac{d^3k}{2k(2\pi)^3} \sum_{m,\epsilon} \frac{\langle n|\boldsymbol{\alpha} \cdot \boldsymbol{\varepsilon}\, e^{i\mathbf{k}\cdot\mathbf{r}}|m\rangle \langle m|\boldsymbol{\alpha} \cdot \boldsymbol{\varepsilon}\, e^{-i\mathbf{k}\cdot\mathbf{r}}|n\rangle}{E_n - k - E_m} \tag{8.85}$$

where the sum is over transverse photon polarizations and all electron states.

We now choose k_{min} such that

$$(Z\alpha)^2 m \ll k_{\min} \ll (Z\alpha) m$$

[for instance let $k_{min} \sim (Z\alpha)^{1/2} m$] and make the dipole approximation, admittedly a somewhat questionable procedure.

Because the electron states are nonrelativistic, a may be replaced by $\mathbf{v} = \mathbf{p}/m$.

The integrations over k may then be done, yielding

$$\Delta E_n^{<} = \frac{2\alpha}{3\pi}\left[-k_{\min}\langle n|\mathbf{v}^2|n\rangle + \sum_m \frac{E_m - E_n}{m^2} \log \frac{|E_m - E_n + k_{\min}|}{|E_m - E_n|} |\langle n|\mathbf{p}|m\rangle|^2 \right] \tag{8.86}$$

We now must carry out the mass renormalization for this part of the calculation.

Since the electromagnetic mass δm of the electron is already contained in its experimental mass m, there will be a mass counter term of the form

$$\frac{\mathbf{p}^2}{2(m - \delta m)} - \frac{\mathbf{p}^2}{2m} \approx \frac{1}{2}\left(\frac{\mathbf{p}}{m}\right)^2 \delta m$$

in the hamiltonian.

This leads to an energy shift.

$$\delta E_n = \tfrac{1}{2}\, \delta m\, \langle n|\mathbf{v}^2|n\rangle$$

which is just the structure of the first term of (8.86).

It therefore is absorbed into the mass renormalization.

Since $k_{min} > E_n - E_m \sim (Z\alpha)^2 m$, the nonrelativistic part of the Lamb shift is given by

$$\begin{aligned}\Delta E_n^{<} &= \frac{2\alpha}{3\pi m^2}\sum_m (E_m - E_n)\log\frac{k_{\min}}{|E_n - E_m|}\,|\langle n|\mathbf{p}|m\rangle|^2 \\ &= \frac{2\alpha}{3\pi m^2}\sum_m (E_m - E_n)\log\frac{k_{\min}}{\bar{E}}\,|\langle n|\mathbf{p}|m\rangle|^2\end{aligned} \tag{8.87}$$

which serves to define $\bar{E}$, expected to be $\sim (Z\alpha)^2 m$.

The sum over states can now be performed with some commutator algebra

$$\sum_m (E_m - E_n)\langle n|\mathbf{p}|m\rangle^2 = \tfrac{1}{2}\langle n|[[\mathbf{p},H],\mathbf{p}]|n\rangle$$

yielding

$$\Delta E_n^{<} = \frac{\alpha}{3\pi m^2}\log\frac{k_{\min}}{\bar{E}}\,\langle n|\nabla^2 V|n\rangle = \frac{4\alpha(Z\alpha)}{3m^2}\log\frac{k_{\min}}{\bar{E}}\,|\psi_{nlm}(0)|^2 \tag{8.88}$$

$\bar{E}$ has been evaluated by Bethel et al. and found to be $8.9\alpha^2 m$ in hydrogen in agreement with our anticipations.

To (8.89) must still be added the contribution of the anomalous magnetic moment term in (8.83) in order to complete the Lamb shift to order $\alpha(Z\alpha)^4$.

In this chapter we have shown how to extend the rules for writing S-matrix elements by one power of a beyond the lowest order amplitudes.

The divergence difficulties which were encountered in this development were surmounted by showing that the infinite expressions could all be isolated in a well-defined way and included in constants which renormalize the charge and mass of the electron and the wave functions describing the propagation of the electron and photon.

The need for such renormalizations is clear on physical grounds.

To the mass parameter in the Dirac equation, the electromagnetic mass must be added, since this is already contained in the experimental mass.

Also, the charge must be renormalized to include the effect of the static polarizability of the vacuum.

Finally, the wave function must be renormalized, as in ordinary nonrelativistic perturbation theory illustrated by (8.47) and (8.48), to correct for the amplitude of observing an electron in the presence of the fluctuations induced by the interaction.

Delicate care in carrying out the renormalization program has been demanded by the unfortunate fact that the Z_1, Z_2, Z_3, and m diverge.

However, we have seen that the remaining physical effects are finite and independent of our cutoff.

Moreover, they agree with experiment, as, for example, in the Lamb shift and anomalous magnetic moment observations.

It is natural to ask at this point what new problems we face as we push on further to higher orders in α.

The answer is none beyond the demand of added computational labor.

We have already introduced all the renormalizations required.

The ideas and procedures introduced in this chapter suffice to carry us in a well-defined way to unique, finite, and cutoff-independent answers to all physical amplitudes in a calculation of the S matrix to any (finite) order in α.

9 The Klein-Gordon Equation

9.1 Introduction

The propagator formalism may be used and techniques of calculation may be developed for processes involving particles of zero spin.

We attempt to describe these particles by a scalar wave function φ(x) with only one single component, and consequently we are led back to the Klein-Gordon equation

$$(\Box + m^2)\varphi(x) = 0 \tag{9.1}$$

for the free particle.

Such an equation was abandoned in Chap. 1 because of the impossibility of defining a conserved positive definite probability.

However, this original motivation for rejecting (9.1) has by now disappeared and we reexamine it in the light of the Feynman interpretation of the negative-energy states propagating backward in time.

The spin of the particle does not enter crucially into such an interpretation which we shall find to be applicable to spin-zero particles as well as to electrons.

As in the case of the electron, we shall again be led to this picture: along with a at meson, for instance, which is described by a positive-energy solution of the Klein-Gordon equation, there emerges its antiparticle, the a meson, which is interpreted as a π^+ meson of negative energy propagating backward in time.

Let us consider for a moment for which particles in nature one would hope to use the Klein-Gordon equation.

There are no known stable elementary particles of spin zero; however, the π mesons and K mesons are nearly stable candidates.

They are experimentally found to be copiously created and destroyed one at a time, for example, in reactions (p = proton, n = neutron, Λ^0 = neutral lambda particle, π^+ = positively charged meson, etc.)

$$
\begin{aligned}
p + p &\to p + n + \pi^+ \\
&\to p + p + \pi^0 \\
&\to p + \Lambda^0 + K^+ \\
\pi^- + p &\to \Lambda^0 + K^0 \\
K^- + p &\to \Lambda^0 + \pi^0 \\
&\to \Sigma^- + \pi^+
\end{aligned}
\tag{9.2}
$$

Therefore the wave equation for these spin-zero mesons must take into account their possible production and annihilation.

One cannot follow the world lines of these particles throughout a scattering process as was possible in our discussion of the electron world lines interacting with photons.

This is also true if we consider just the interactions of charged π and K mesons with photons, since graphs such as Fig. 9.1 contribute.

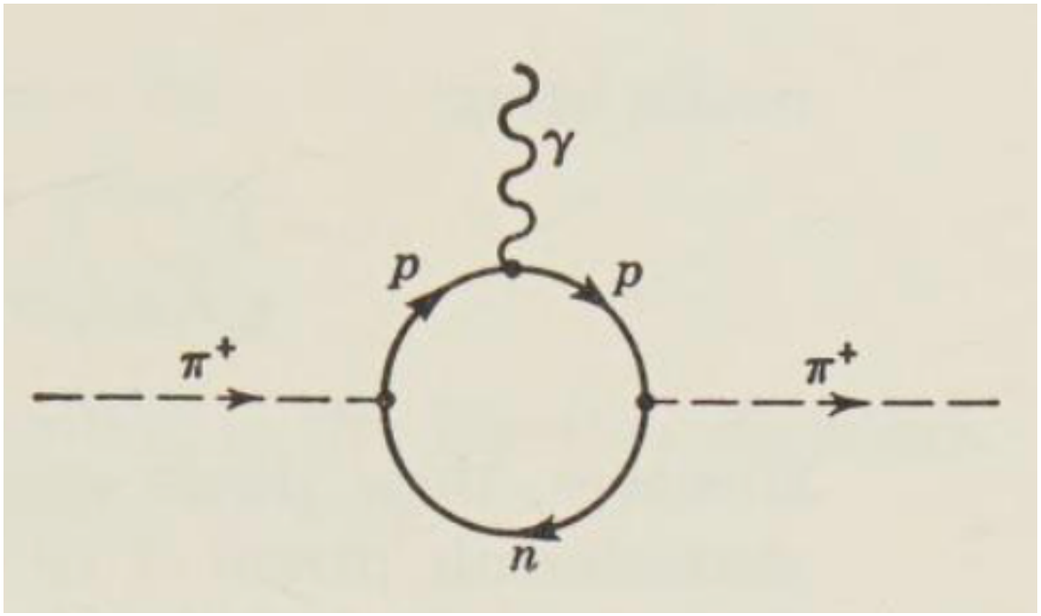

Fig. 9-1 Contribution to the electromagnetic structure of a meson.

This possibility of creation and destruction of single spinless particles as confirmed by experimental observation requires that a theory of their interactions be a many-particle theory.

The quantum field theory formalism is best suited to a discussion of this problem, but again, as with the electrons and photons, we shall find it possible to understand and calculate a great deal by extending our propagator approach to a study of the mesons coupled to source terms added to the right-hand side of (9.1).

If we include the weak interactions, the spin-zero mesons are also destroyed by reactions of the type, for example (μ = mu meson and υ = neutrino),

$$\begin{aligned} \pi^+ &\to \mu^+ + \nu \\ K^+ &\to \pi^+ + \pi^+ + \pi^- \\ &\to \pi^0 + \mu^+ + \nu \end{aligned} \tag{9.3}$$

Because of the exceedingly small magnitude of these weak decay interactions (9.3), the charged π and K mesons have very long half-lives $\tau \sim 10^{-8}$ sec, which greatly exceed the natural unit of time formed from $\hbar$, c, and the π- or K-meson masses, $\hbar/mc^2 < 10^{-23}$ sec.

To first order in an expansion in a perturbation series in powers of the weak interaction constant we may therefore ignore the decays (9.3) and the finite lifetimes $\tau \sim 10^{-8}$ sec in discussing strong interaction amplitudes such as (9.2).

In this approximation the π and K mesons are treated as stable particles and are represented by initial or final free wave functions.

The neutral π^0 and K^0 mesons which we also wish to include in these discussions have shorter half-lives, their predominant decay modes being

$$\pi^0 \to \gamma + \gamma \qquad \tau_{\pi^0} \sim 10^{-16} \text{ sec}$$

$$K^0 \to \pi^+ + \pi^- \qquad \tau_{K^0} \sim 10^{-10} \text{ sec}$$

However, these decay rates are still very long compared with the characteristic period of 10^{-23} sec, and the interactions responsible for them need be included in lowest order only.

The π^0 and K^0 will therefore also be treated as stable in strong reactions as in (9.2).

In addition to the interactions of the spin-zero particles illustrated by (9.2) and (9.3), the charged π and K mesons interact with photons and with external electromagnetic fields.

In order to emphasize first the similarity to the electrodynamics of a Dirac electron, we shall limit the discussions of this chapter to the electrodynamic interactions of charged spin-zero particles.

The propagator development follows the physical lines given for the electron theory.

In order to discuss the low-energy properties of mesons in external fields, for example, the bound states of the π-mesic atoms, we also make a systematic nonrelativistic reduction and interpretation of the Klein-Gordon equation.

More general weak decay and strong nuclear couplings are discussed in the following chapter.

9.2 The Propagator for Klein-Gordon Particles

Solutions of the Klein-Gordon equation satisfy a continuity equation, as derived in (1.12):

$$\frac{\partial j^{\mu}(x)}{\partial x^{\mu}} = \frac{\partial}{\partial x^{\mu}}\left(i\varphi^* \frac{\partial \varphi}{\partial x_{\mu}} - i\varphi \frac{\partial \varphi^*}{\partial x_{\mu}}\right) = 0$$

By the divergence theorem, the integral

$$Q = \int d^3x\, j_0(x) = i\int d^3x\, \varphi^* \overleftrightarrow{\partial_0} \varphi \tag{9.4}$$

where

$$a\overleftrightarrow{\partial_0} b \equiv a\left(\frac{\partial b}{\partial t}\right) - \left(\frac{\partial a}{\partial t}\right) b$$

is a useful shorthand, is conserved for solutions of (9.1).

The plane-wave solutions of the Klein-Gordon equation with both positive and negative frequencies form a complete set.

Normalized in a box of volume V, the solutions are

$$f_p^{(\pm)}(x) = \frac{e^{\mp ip\cdot x}}{\sqrt{2\omega_p V}}$$

with $\omega_p = p_0 > 0$ and $p^2 = m^2$ according to the Einstein condition.

In the continuum normalization language we write

$$f_p{}^{(\pm)}(x) = e^{\mp ip\cdot x}\frac{1}{\sqrt{(2\pi)^3 2\omega_p}} \tag{9.5}$$

for the positive-and negative-frequency solutions, respectively.

They satisfy the orthogonality and normalization relations

$$\begin{aligned}\int d^3x\, f_{p'}^{(\pm)*}(x)\overleftrightarrow{i\partial_0} f_p^{(\pm)}(x) &= \pm\delta^3(\mathbf{p}-\mathbf{p}')\\ \int d^3x\, f_{p'}^{(\pm)*}(x)\overleftrightarrow{i\partial_0} f_p^{(\mp)}(x) &= 0\end{aligned} \tag{9.6}$$

Notice that Q is positive for a superposition of positive-frequency solutions, that is, for

$$\begin{aligned}\varphi^{(+)}(x) &= \int d^3p\, a_+(p) f_p^{(+)}(x)\\ Q = i\int d^3x\, \varphi^{(+)*}(x)\overleftrightarrow{\partial_0}\varphi^{(+)}(x) &= +\int d^3p\, |a_+(p)|^2\end{aligned} \tag{9.7}$$

and for negative-frequency solutions Q is negative, that is, for

$$\varphi^{(-)}(x) = \int d^3p\, a_-^*(p) f_p^{(-)}(x)$$

$$Q = i\int d^3x\, \varphi^{(-)*}(x) \overleftrightarrow{\partial_0} \varphi^{(-)}(x) = -\int d^3p\, |a_-(p)|^2 \tag{9.8}$$

Herein lies the difficulty for a probability interpretation for the solutions of the Klein-Gordon equation, since Q may take on negative as well as positive values for a general superposition of plane-wave solutions.

To construct the Feynman propagator for the Klein-Gordon equation, we want to find a solution of

$$(\Box_{x'} + m^2)\Delta_F(x' - x) = -\delta^4(x' - x) \tag{9.9}$$

which propagates positive-frequency parts of waves forward in time and negative-frequency ones backward in time.

Proceeding in analogy with the Dirac theory, (6.40) to (6.46), we Fourier-transform to momentum space in which Δ_F has the representation

$$\Delta_F(x' - x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x'-x)} \frac{1}{p^2 - m^2 + i\epsilon} \tag{9.10}$$

The small negative imaginary part added to the mass in (9.10) assures that (9.10) meets the desired boundary condition of propagating only the positive frequencies forward in time and the negative frequencies backward.

As discussed in Chap. 6, there is no other choice of integration contour in (9.10) which accomplishes this.

We may verify that (9.10) is the correct choice of contour by integrating over dp_0 to obtain, with the use of the Cauchy theorem,

$$\begin{aligned}\Delta_F(x' - x) &= -i \int \frac{d^3p}{(2\pi)^3 2\omega_p} e^{-i\omega_p|t'-t|} e^{i\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})} \\ &= -i \int d^3p\, f_p^{(+)}(x') f_p^{(+)*}(x)\theta(t' - t) \\ &\quad - i \int d^3p\, f_p^{(-)}(x') f_p^{(-)*}(x)\theta(t - t')\end{aligned} \tag{9.11}$$

By direct computation we find from (9.6) and (9.11) that $\Delta_F(x' - x)$ propagates only the positive-frequency part of a general wave

$$\varphi(x) = \varphi^{(+)}(x) + \varphi^{(-)}(x) \tag{9.12}$$

as formed in (9.7) and (9.8), forward in time,

$$-i\theta(t' - t)\varphi^{(+)}(\mathbf{x}',t') = \int d^3x\, \Delta_F(x' - x) i\overleftrightarrow{\partial}_0 \varphi^{(+)}(\mathbf{x},t) \tag{9.13}$$

and the negative-frequency part backward

$$-i\theta(t - t')\varphi^{(-)}(\mathbf{x}',t') = -\int d^3x\, \Delta_F(x' - x) i\overleftrightarrow{\partial}_0 \varphi^{(-)}(\mathbf{x},t) \tag{9.14}$$

Equations (9.13) and (9.14) are analogous to (6.49) and (6.50) for the Dirac equation.

9.3 Introduction of Electromagnetic Potentials

Interaction of a charged spin-zero meson with the electromagnetic field is introduced by the minimal substitution

$$p^\mu \longrightarrow p^\mu - eA^\mu(x) \tag{9.15}$$

as for the Dirac equation.

We first consider $A^\mu(x)$ as an applied external potential.

Introducing (9.15) into (9.1), we obtain

$$\left[\left(i\frac{\partial}{\partial x_\mu} - eA^\mu\right)^2 - m^2\right]\phi(x) = 0 \tag{9.16}$$

Equation (9.16) still has a conserved current which we find, as in (1.12), to be

$$\begin{aligned} j^\mu = \phi^*(x)\left[\left(i\frac{\partial}{\partial x_\mu} - eA^\mu(x)\right)\phi(x)\right] \\ - \phi(x)\left[\left(i\frac{\partial}{\partial x_\mu} + eA^\mu(x)\right)\phi^*(x)\right] \end{aligned} \tag{9.17}$$

The corresponding conserved charge is

$$Q = \int d^3x\, \phi^*(x)[i\overleftrightarrow{\partial}_0 - 2eA^0(x)]\phi(x) \tag{9.18}$$

A plane wave representing a free incident charged meson will scatter in this potential, with the amplitude given by the solution of (9.16).

Adopting the Feynman boundary condition that scattered waves of positive frequency only are to propagate forward in time and negative-frequency waves backward, we integrate (9.16) with the Feynman propagator (9.10) :

$$(\Box_x + m^2)\phi(\mathbf{x},t) = -ie\left(\frac{\partial}{\partial x^\mu}A^\mu + A^\mu\frac{\partial}{\partial x^\mu}\right)\phi + e^2 A_\mu A^\mu \phi$$

$$\phi(\mathbf{x},t) = \varphi(\mathbf{x},t) + \int d^4y\, \Delta_F(x-y)V(y)\phi(y) \qquad (9.19)$$

with

$$V(y) = ie\left(\frac{\partial}{\partial y^\mu}A^\mu(y) + A^\mu(y)\frac{\partial}{\partial y^\mu}\right) - e^2 A_\mu(y)A^\mu(y)$$

Equation (9.19) is the analogue of (6.53) for Dirac particles, and the physical interpretation of the solution is again very similar to that given for the electron.

In order to ensure that only positive-frequency waves, representing positive-energy particles, emerge into the future after a scattering, we have integrated with the Feynman propagator in (9.19).

This leads by (9.11) to

$$\phi(\mathbf{x},t) = \varphi(\mathbf{x},t) - i\int d^3p\, f_p^{(+)}(x)\int d^4y\, \theta(t-y_0)f_p^{(+)*}(y)V(y)\phi(y)$$
$$- i\int d^3p\, f_p^{(-)}(x)\int d^4y\, \theta(y_0-t)f_p^{(-)*}(y)V(y)\phi(y) \qquad (9.20)$$

which also contains negative-frequency waves propagating backward to earlier times.

However, from the point of view of an observer watching his instruments, the absorption in the past of a negative-energy particle of charge e is equivalent to the emission of a positive-energy one of charge -e.

In this way we are led to the fundamental and experimentally verified prediction that there exists an oppositely charged antiparticle for each particle in nature.

A particle may have no charge, in which case it may be identical with its antiparticle.

Such a particle is found in nature, the neutral spin-zero a meson, π^0.

Though it does not share in the electromagnetic couplings introduced in (9.15), the propagator for free π^0 mesons can be developed in complete analogy with Sec. 9.2.

Since the current and charge (9.4) vanish for it, the π^0 will be represented in absence of interactions by a real solution $\varphi = \varphi^*$ of the free Klein-Gordon equation.

The Feynman propagator (9.11) will then propagate the positive-frequency parts of φ forward and the negative-frequency parts backward in time as for the charged mesons.

9.4 Scattering Amplitudes

By moving the world lines around - as in Fig. 9.2 - so that they move both backward and forward in time, we include in our scattering formalism for mesons, as for electrons, the amplitudes for production and annihilation of particle-antiparticle pairs along with the direct scattering amplitudes.

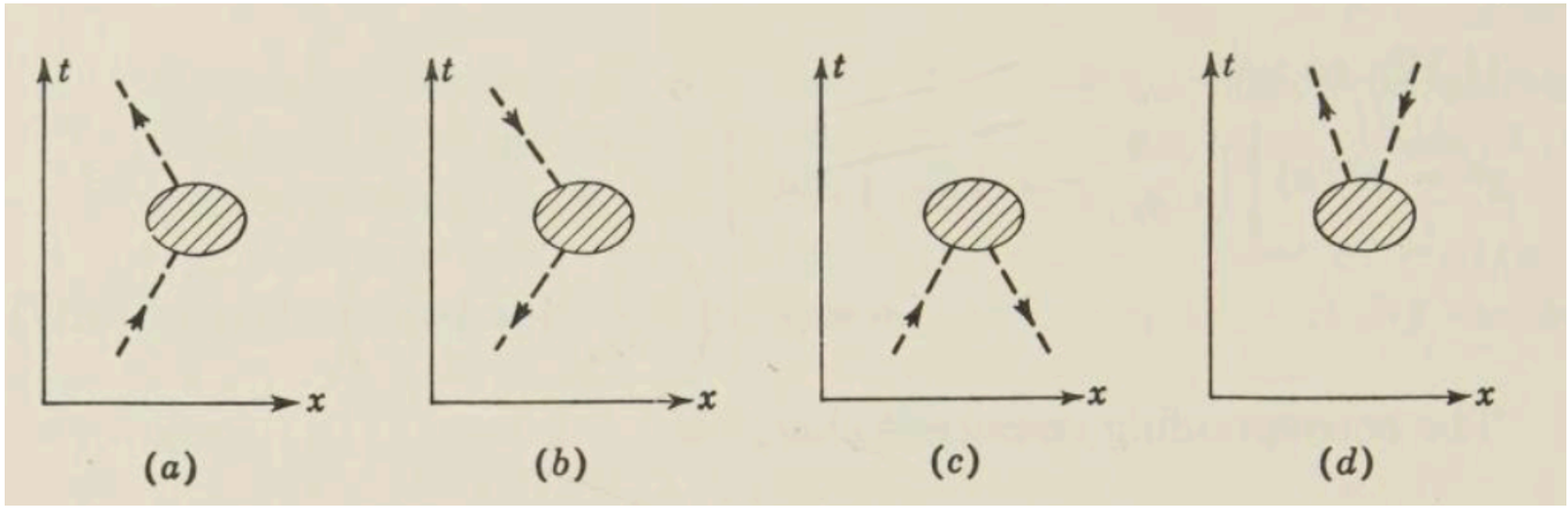

Fig. 9-2 Diagrams for particle and antiparticle scattering and for pair production and annihilation.

In order to compute a scattering or transition amplitude, (9.19) is iterated until ϕ is evaluated to the desired accuracy.

The free solution φ in (9.19) represents the normalized free-particle wave in the absence of scattering.

The transition amplitude to a particle state of given momentum, say, p'_+ is found by projecting the scattered wave emerging from the interaction onto a normalized free wave of momentum p'_+.

The transition probability is then given by the absolute square of this amplitude, or by the intensity of the projection of the scattered wave.

For ordinary scattering of mesons (Fig. 9.2a), positive-frequency waves emerge after the scattering as $t \to \infty$ with a scattering amplitude that is calculated by projecting out the positive-frequency part of the scattered wave (9.20):

$$\begin{aligned} S_{p_+',p_+} &= \lim_{t\to\infty} \int d^3x\, f^{(+)*}_{p_+'} i\overleftrightarrow{\partial}_0 \phi(x) \\ &= \delta^3(\mathbf{p}'_+ - \mathbf{p}_+) - i\int d^4y\, f^{(+)*}_{p_+'}(y)\, V(y)\, \phi(y) \end{aligned} \tag{9.21}$$

where ϕ(y) is given by (9.19), with $f_{\mathbf{p}+}{}^{(+)}$(y) representing the incident meson wave of positive frequency.

The transition probability is then $|S_{p_+',p_+}|^2$.

For pair production (Fig. 9.2d), we again project out the positive-frequency waves as in (9.21), but now ϕ(y) represents a scattered wave developing from an incident negative-frequency wave $f_{\mathbf{p}_-}^{(-)}$(y) in (9.19).

According to (9.14), the negative-frequency wave is "incident" at $y_0 \to +\infty$, since $\Delta_F(x - y)$ propagates it backward in time only.

In complete analogy with the ground rules developed in Chap. 6 for positron theory, we associate the backward propagation of the negative-frequency solution with quantum numbers p_- with the emergence of the antimeson — say, the π^- meson — of positive energy and four-momentum p_-.

For the pair annihilation amplitude, Fig. 9.2c, we project out the negative-frequency part of the scattered wave (9.20) as $t \to -\infty$:

$$\begin{aligned} S_{p_-,p_+} &= -\lim_{t\to-\infty} \int d^3x\, f_{p_-}^{(-)*}(x) i\overleftrightarrow{\partial}_0 \phi(y) \\ &= -i\int d^4y\, f_{p_-}^{(-)*}(y) V(y) \phi(y) \end{aligned} \tag{9.22}$$

Here ϕ(y) is given by (9.19) with $f_{\mathbf{p}_+}^{(+)}$(y) representing the incident π^+-meson wave of positive frequency with four-momentum p_+.

As usual, the incident π^- meson with positive energy and four-momentum p_- is represented by a negative frequency wave $f_{\mathbf{p}_-}^{(-)}$(y) propagating backwards into the past out of the interaction V(y).

Finally, the π^- (or anti-) meson scattering , Fig. 9.2b, is given by (9.22), where $\phi(y)$ is still determined by (9.19).

However, the "incident" π^+ meson wave of negative frequency is now given by $f_{\mathbf{p}^-}^{(-)}(y)$, representing the final π^- emerging with a positive energy and four-momentum p'_- *after* the scattering, that is,

$$S_{p_-,p_-'} = \delta^3(\mathrm{p}_- - \mathrm{p}'_-) - i\int d^4y\, f_{p^-}^{(-)*}(y)\, V(y)\phi(y) \tag{9.23}$$

Comparison with the propagator formulation of positron theory in Chap. 6 shows that the S-matrix rules discussed here have the same physical origin and interpretation as discussed there.

The practical rules for calculation of transition rates for spin-zero mesons under electromagnetic interactions can be developed by calculating several simple examples as was done for the electron in Chap. 7.

9.5 Low-order Scattering Processes

As a first example we consider the Coulomb scattering of a π^+ meson to lowest order in e.

The $e^2 A_\mu A^\nu$ term in the interaction (9.19) does not contribute to this order and may be neglected.

The transition amplitude corresponding to the graph, Fig. 9.3, is found from (9.21) with $\phi(y) \sim f_{\mathbf{p}^+}^{(+)}(y)$.

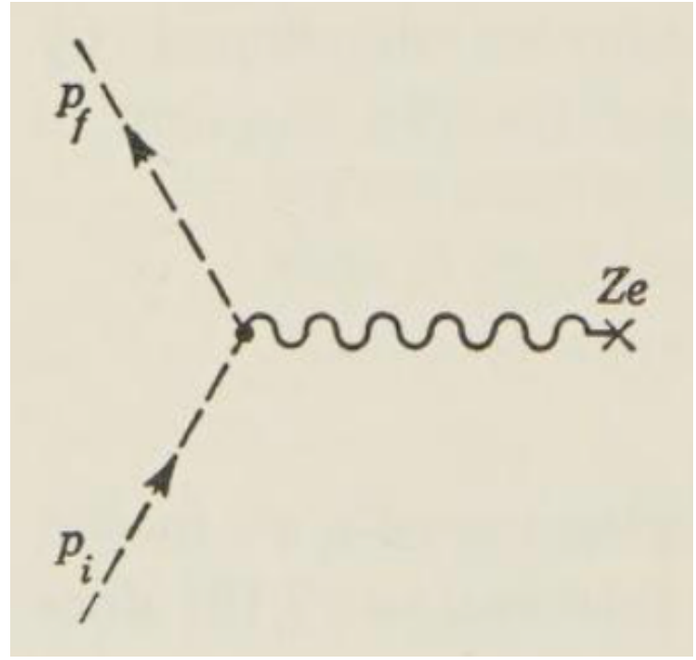

Fig. 9-8 Coulomb scattering of a π^+ meson.

For $q = p_f - p_i \neq 0$ the δ-function term vanishes, so that

$$S_{p_f,p_i} = \frac{-ie}{(2\pi)^3}\int d^4y \frac{1}{\sqrt{2\omega_f \cdot 2\omega_i}} e^{iq\cdot y}(p_f + p_i)_\mu A^\mu(y)$$
$$= -\frac{ie(p_f + p_i)_\mu}{(2\pi)^3\sqrt{2\omega_f \cdot 2\omega_i}} A^\mu(q) \qquad (9.24)$$

where $$A^\mu(q) \equiv \int d^4y\, e^{iq\cdot y} A^\mu(y)$$

The form of the current in (9.24) is reminiscent of the spin-independent term in the Gordon decomposition of the electron current.

Inserting $A^\nu(q)$ in (9.24) for a static Coulomb potential

$$A^\mu(q) = \frac{Ze}{|q|^2} 2\pi\delta(\omega_f - \omega_i) g^{\mu 0} \qquad (9.25)$$

we obtain the cross section by the usual procedure of squaring, summing over final states, and dividing by the incident flux.

In analogy with (7.10), we find

$$d\sigma = \frac{(2\pi)^3}{|\mathbf{v}|} d^3p_f \, 2\pi\delta(\omega_f - \omega_i) \cdot \left[\frac{Ze^2(\omega_f + \omega_i)}{(2\pi)^3\sqrt{2\omega_f \cdot 2\omega_i}} \frac{1}{|\mathbf{q}|^2}\right]^2 \tag{9.26}$$

and

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{4p^2\beta^2 \sin^4(\theta/2)}$$

which lacks the factor $1 - \beta^2 \sin^2(\theta/2)$ found for the electron in (7.22) and associated with the spin.

A similar result is obtained for the Coulomb scattering of π^+ mesons.

From (9.24), with f-* (y) representing the π^- with momentum p_- before the scattering and $\phi(y) \sim f_{\mathbf{p}_-}^{(-)}(y)$, the final π^- emerging after the scattering, as in Fig. 9.4,

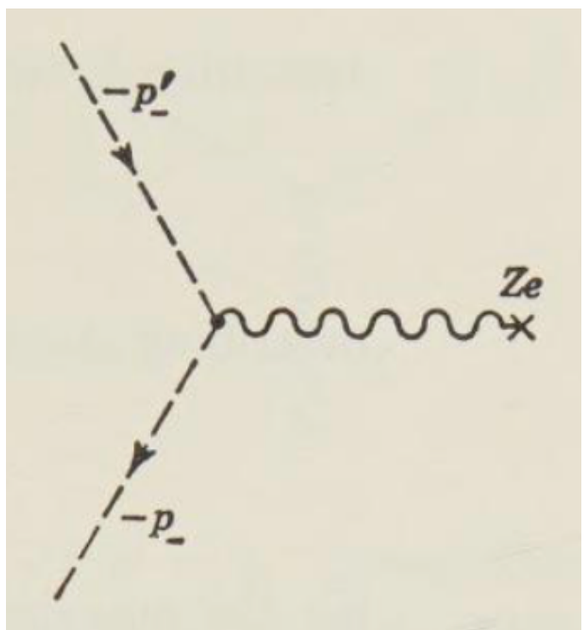

Fig. 9-4 Coulomb scattering of a π^- meson.

we find

$$S_{p_-,p_-'} = + \frac{ie(p_- + p'_-)_\mu A^\mu(q)}{(2\pi)^3 \sqrt{2\omega_- \cdot 2\omega'_-}} \tag{9.27}$$

with q = p′- - p- again representing the momentum transfer.

Equations (9.24) and (9.27) differ in sign only, corresponding to the change in sign of the π^+ and π^- charge, and lead to identical cross sections (9.26).

The lesson we learn from this calculation is that to the π-meson vertex we attach a factor $e(p_\mu + p'_\mu)$ instead of $e\gamma_\mu$ as for the electron.

The wave function normalization factor is $1/\sqrt{2\omega}$, , which replaces the $\sqrt{m/E}$ for the electron, and of course there are no spinors.

To obtain the rules for the $e^2 A_\mu A^\mu$ term in V of (9.19), we turn to Compton scattering of a charged meson.

The "external potential" in this example consists of the absorbed and emitted photons described "in continuum normalization" by the two terms, respectively see (7.53)]

$$A_\mu(x) = \frac{\epsilon_\mu(l,\lambda)e^{-il\cdot x}}{\sqrt{2l(2\pi)^3}} + \frac{\epsilon_\mu(l,\lambda)e^{il\cdot x}}{\sqrt{2l(2\pi)^3}} \tag{9.28}$$

where l, λ refer to the momentum and polarization.

Since the lowest order Compton amplitude is proportional to e^2, the terms in V linear in e must be iterated once.

The S matrix to order e^2, corresponding to the Feynman graphs of Fig. 9.5, is then

$$S_{fi} = (-ie)^2 \int d^4y\, d^4z\, f_{p'}^{(+)*}(y) i \left[\frac{\partial}{\partial y_\mu} A_\mu(y) + A_\mu(y) \frac{\partial}{\partial y_\mu} \right]$$
$$\times i\Delta_F(y-z) i \left[\frac{\partial}{\partial z_\nu} A_\nu(z) + A_\nu(z) \frac{\partial}{\partial z_\nu} \right] f_p^{(+)}(z)$$
$$+ ie^2 \int d^4y\, f_{p'}^{(+)*}(y) A_\mu(y) A^\mu(y) f_p^{(+)}(y) \qquad (9.29)$$

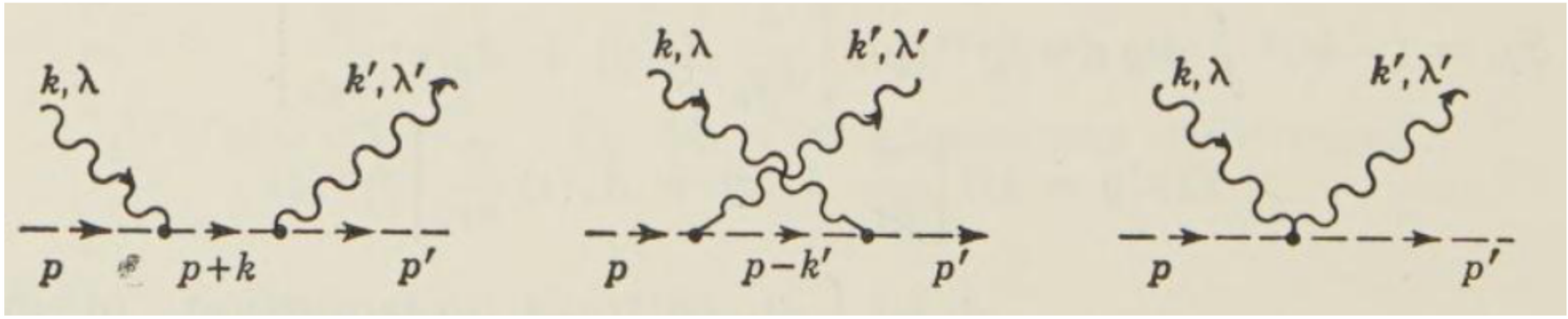

Fig. 9-5 Compton scattering of a t meson.

Inserting A_μ from (9.28) and keeping only the cross terms between

$$\frac{\epsilon_\mu e^{-ik\cdot x}}{\sqrt{(2\pi)^3 2k}}$$

describing absorption of a photon with (k,λ), and

$$\frac{\epsilon'_\mu e^{ik'\cdot x}}{\sqrt{(2\pi)^3 2k'}}$$

for the emission of one with (k′, λ'), we find after performing the coordinate integrations

$$\begin{aligned} S_{fi} = {} & \frac{(-ie)^2}{(2\pi)^6\sqrt{2\omega'\cdot 2\omega\cdot 2k'\cdot 2k}}\,(2\pi)^4\delta^4(p+k-p'-k') \\ & \times\left[\epsilon\cdot(2p+k)\,\frac{i}{(p+k)^2-m^2}\,\epsilon'\cdot(2p'+k')\right. \\ & \left. +\,\epsilon\cdot(2p'-k)\,\frac{i}{(p-k')^2-m^2}\,\epsilon'\cdot(2p-k')-2i\epsilon\cdot\epsilon'\right] \end{aligned} \tag{9.30}$$

Notice in particular the factor of 2 appearing with the $A_\mu A^\mu$ term.

This occurs because there are two ways of associating absorbed and emitted photons with the two factors of $A^\mu(x)$ and is easy to forget.

As a useful check on (9.30), we may apply the test of gauge invariance as applied to the electron amplitudes in Chap. 7.

Thus S_{fi} is invariant, as is readily checked, under a transformation of the gauge

$$\epsilon^\mu \longrightarrow \epsilon^\mu + \lambda k^\mu \tag{9.31}$$

employed for the initial photon, as well as under a gauge change

$$\epsilon^\mu \longrightarrow \epsilon^\mu + \lambda k^\mu \tag{9.32}$$

for the final one.

A convenient gauge choice to make in (9.30) is

$$\epsilon \cdot p = \epsilon' \cdot p = 0$$

corresponding to transversely polarized photons in the laboratory frame, in which the meson is initially at rest, p = (m,0).

Then only the $A_\mu A^\mu$ term survives in (9.30), since $e \cdot k = e' \cdot k' = 0$.

Proceeding to a cross section in the by-now-familiar manner, we find, upon squaring (9.30), dividing out one power of $(2\pi)^4(p + k - p' - k')$, multiplying by the final-state phase-space factor

$$d^3p'\, d^3k'$$

and by the reciprocal of the incident flux in the laboratory, $(2\pi)^3$, as well as the reciprocal of the density of target particles, $(2\pi)^3$,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{lab}} = \frac{\alpha^2}{m^2} \frac{(\epsilon \cdot \epsilon')^2}{[1 + (k/m)(1 - \cos\theta)]^2}$$

This reduces to the classical Thomson limit in the limit of low photon energy k $\rightarrow$ 0.

Summing over final photon polarizations ϵ' and averaging over the incident ϵ for unpolarized light, we obtain

$$\left(\frac{\overline{d\sigma}}{d\Omega}\right)_{\text{iab}} = \frac{\alpha^2(1+\cos^2\theta)}{2m^2[1+(k/m)(1-\cos\theta)]^2} \tag{9.33}$$

9.6 Higher-order Processes

We may continue to imitate the development of propagator theory for the electron and infer the rules of computation for higher-order graphs from the preceding examples.

The major changes from the electron rules are:

1. At a vertex scattering a meson from p_μ to p'_μ, as illustrated in Fig. 9.6, for any directions of the lines forward or backward in time we replace

$$e\gamma^\mu \rightarrow e(p^\mu + p^{\mu\prime}) \tag{9.34}$$

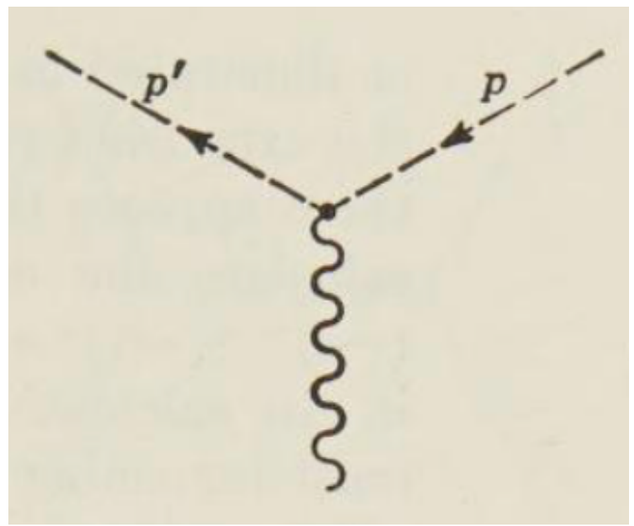

Fig. 9-6 Vertex graph representing the interaction $e\varphi^*(p' + p)_\mu \varphi A^\mu$.

2. The additional $A_\mu A^\mu$ interaction term in (9.19) contributes with a factor[1]

$$2ie^2 g_{\mu\nu} \tag{9.35}$$

as illustrated in the amplitude (9.30). The factor i appears because the expansion parameter for this term is e^2. Therefore, to order e^n there appears the factor $(-i)^n$ from n iterations of (9.19) when we calculate the contributions of (9.34) to this order. A factor of $(-i)^{n-m}(-1)^m = (-i)^{n-2m}(i)^m$ arises if the $A_\mu A^\mu$ term appears m times in the calculation; hence the factor of i in (9.35). Throughout we must remember that the expansion parameter is e and not the order of interaction. The factor 2 in (9.35) appears because there are always two ways of associating the factors $A_\mu A^\mu$ with the quanta to be destroyed, created, or scattered in the vertex (Fig. 9.7). The test of gauge invariance may be applied to the interaction amplitude representing the sum of all graphs which contribute to any given order of e. As in (9.30) to (9.32), this provides a very simple and useful check in the relative factors coming from the $\mathbf{p}\cdot A + A\cdot\mathbf{p}$ and the $A\cdot A$ terms in (9.19).

3. For the propagator of an internal line of momentum p, we replace

$$\frac{i}{\not{p} - m + i\epsilon} = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} \to \frac{i}{p^2 - m^2 + i\epsilon} \tag{9.36}$$

that is, the factor $\not{p} + m \to 1$.

4. For the normalization of the external lines we replace the electron spinors

$$\sqrt{\frac{m}{E}}\, u(p) \to \frac{1}{\sqrt{2\omega}} \tag{9.37}$$

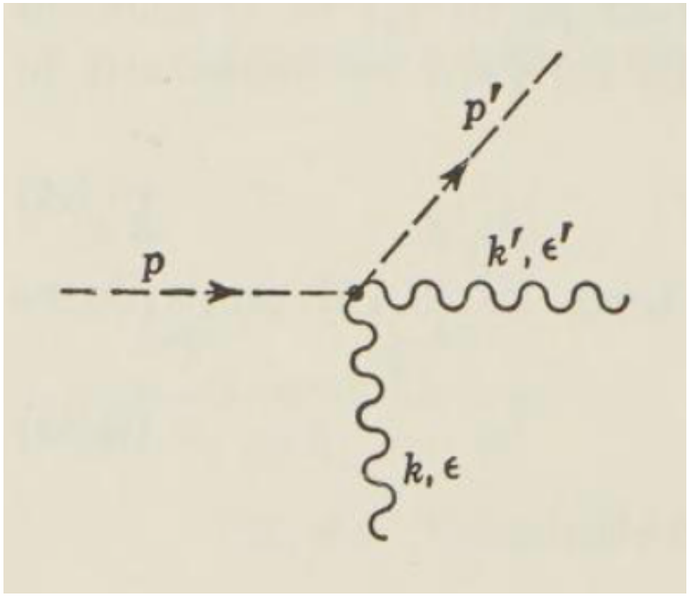

Fig. 9-7 Graph representing the interaction e?4,,44p*ф.

All other factors of i and 2π are precisely the same as for the electron.

Only a question of relative minus signs remains.

For electrons we were led by the Pauli principle to antisymmetrize the interchange of two identical particles.

On the other hand, experimental evidence indicates that mesons are bosons, that is, they satisfy the symmetric statistics of Bose-Einstein.

In particular, in the reaction

$$K^+ \rightarrow \pi^+ + \pi^+ + \pi^-$$

the two π^+ mesons are emitted in a relative s state.

In addition, there are strong theoretical reasons, given originally by Pauli, for a connection between spin and statistics, with particles of half-integer spin obeying the exclusion principle (fermions) and integer-spin particles symmetrized as bosons.

These arguments are best discussed within the framework of field theory.

Here we shall simply assume that the spin-zero particles being described are bosons obeying symmetric statistics.

This means that there must be a relative plus, instead of a minus, sign between graphs differing only by the interchange of bosons.

There are no longer factors of (-1) appearing with closed loops or between scattering and annihilation graphs.

These were introduced in the electron graphs, as in the amplitudes (8.2) and (8.5) for the processes in Figs. 8.1b and 8.1e, by applying the Pauli principle to the states of hole theory.

For bosons we have no filled negative-energy sea and must argue for relative signs in a different way.

In the Coulomb scattering of like bosons, the relative sign between the amplitudes for the two graphs in Fig. 9.8 is plus

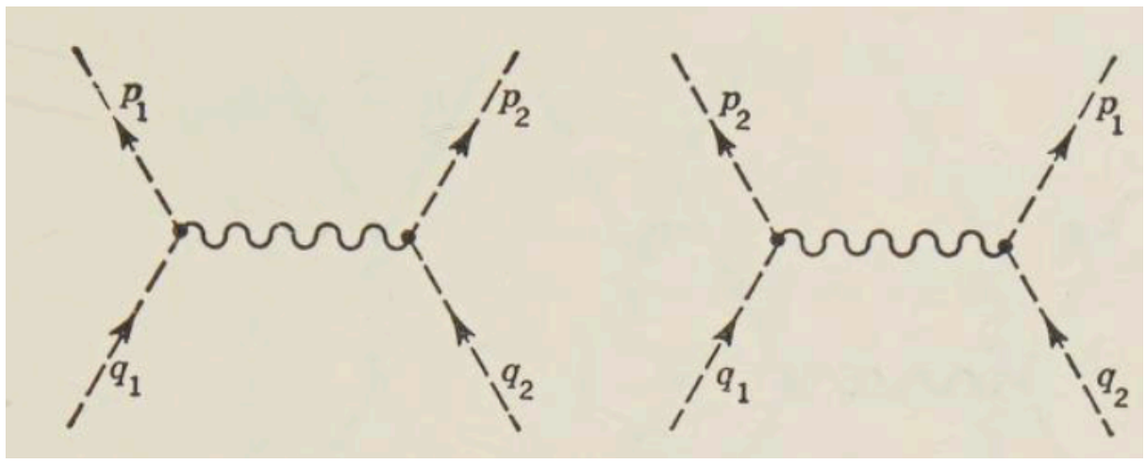

Fig. 9-8 π^+ - π^+ Coulomb scattering.

We obtain from this the amplitude for boson-antiboson scattering by changing the sign of energy of two of the lines; for example, by the substitution

$$q_2 \leftrightarrow -p_2 \tag{9.38}$$

we obtain the amplitudes for the graphs illustrated in Fig. 9.9. The relative sign between the two amplitudes corresponding to the graphs of Fig. 9.9 remains positive if the substitution (9.38) is the only change in going from the scattering graphs 9.8 to the graphs 9.9.

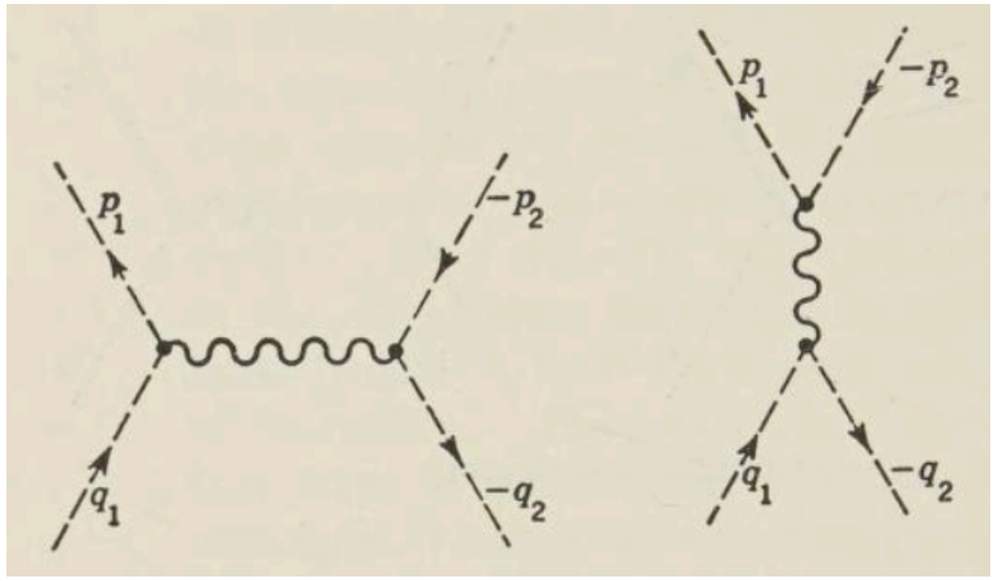

Fig. 9-9 π^+ - π^- Coulomb scattering.

Equation (9.38) is an example of the substitution rule already encountered for electron processes in (7.85) and extended now to boson amplitudes.

It leads to relative plus signs for all three amplitudes shown in Fig. 9.10.

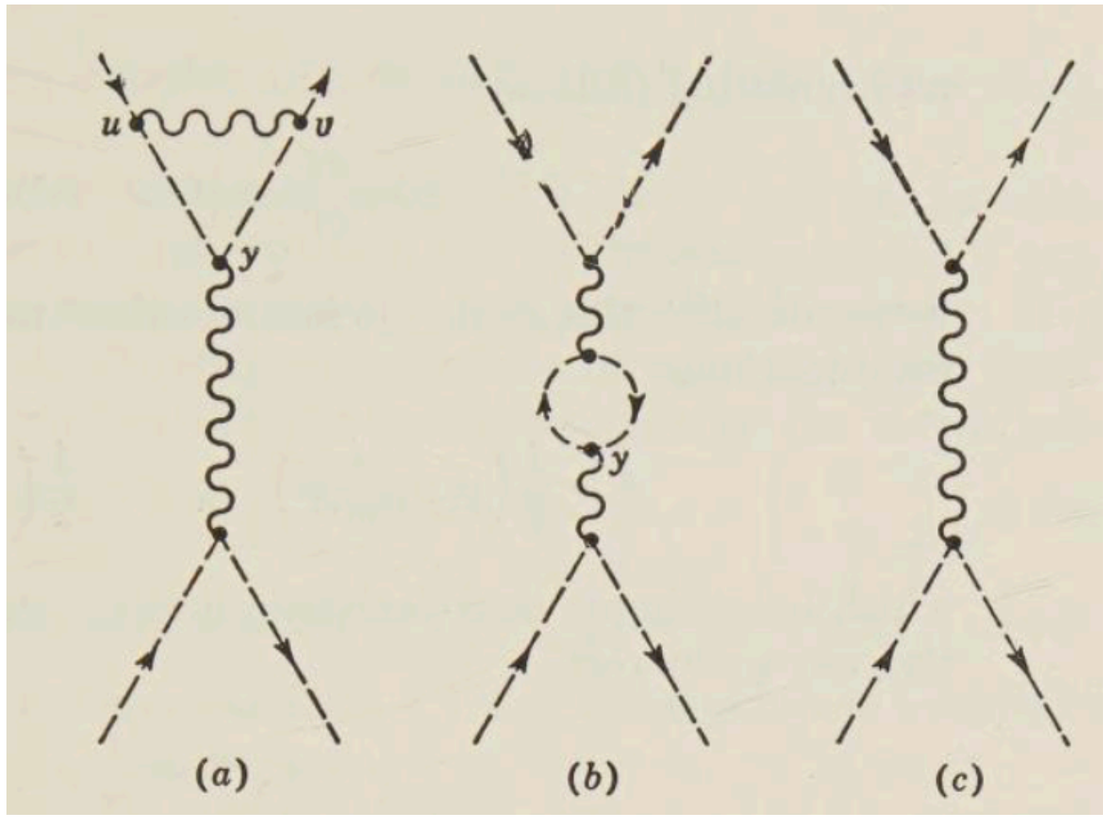

Fig.9-10 Fourth-order contributions to Coulomb scattering.

Graphs 9.10a and 9.10b are identical below vertex y and therefore have a relative plus sign between them.

Since no minus sign accompanies the introduction of the additional scattering interaction between u and v in graph 9.10a relative to 9.10c, we conclude that no factor of (-1) appears along with the closed loop in Fig. 9.10b, as stated at the beginning of the paragraph.

Higher-order calculations of the electromagnetic interactions of the spin-zero bosons such as π and K mesons showing also the renormalization effects can be pursued in complete analogy with the considerations of the preceding chapter.

We do not go into these in detail here because the much stronger interactions of the a and K mesons with themselves and with nucleons must also be included before comparison with physical observations is possible.

A discussion of these nonelectromagnetic couplings is introduced in the following chapter.

9.7 Nonrelativistic Reduction and Interpretation of the Klein-Gordon Equation

There exist physical situations in which an approximate description of mesons in terms of ordinary one-particle quantum mechanics with a probability interpretation is very desirable.

For instance, the interactions of charged π mesons with atomic electric and magnetic fields in matter or with applied external fields, as well as the properties of π-mesic atoms, may be studied from this point of view.

These are similar to the situations in which the one-particle Dirac electron theory is successfully applied and interpreted.

In these cases we would like to exhibit a nonrelativistic reduction to the Schrödinger equation as well as a classical correspondence limit.

In the face of the impossibility of constructing an exact one-particle quantum mechanics with a probability interpretation we were led to abandon the second-order Klein-Gordon equation at the very beginning in Chap. 1.

We did so in favor of the Dirac equation which was first order in the time derivative, as in the nonrelativistic Schrödinger theory.

By now, however, we have amply seen that the one-particle picture of the Dirac equation survives only in limited circumstances such as weak, slowly varying fields in which there still remains a broad gap $\sim 2mc^2$ between the positive- and negative-energy spectra.

It is to such physical situations that we now turn in search of an approximate one-particle quantum mechanics of the Klein-Gordon equation.

Our first step to bring the Klein-Gordon equation to Schrödinger form containing only first-order time derivatives is to rewrite (9.1) as a pair of first-order equations.

Defining

$$\xi \equiv \frac{\partial \varphi}{\partial t} \equiv \dot{\varphi} \tag{9.39}$$

and rewriting (9.1) as

$$\dot{\xi} \equiv \frac{\partial \xi}{\partial t} = (\nabla^2 - m^2)\varphi \tag{9.40}$$

we accomplish this goal.

It is convenient to introduce the two linear combinations

$$\theta = \frac{1}{2}\left(\varphi + \frac{i}{m}\dot{\varphi}\right) \qquad \chi = \frac{1}{2}\left(\varphi - \frac{i}{m}\dot{\varphi}\right) \tag{9.41}$$

which have simple nonrelativistic limits.

For a free particle of positive energy at rest

$$\varphi \propto e^{-imt} = \frac{i}{m}\dot{\varphi} \tag{9.42}$$

and, in this limit,

$$\theta = \varphi \propto e^{-imt} \qquad \chi = 0 \tag{9.43}$$

For the negative-energy, or antiparticle, solution

$$\theta = 0 \qquad \text{and} \qquad \chi = \varphi \propto e^{+imt} \tag{9.44}$$

in this limit.

Thus θ plays the role analogous to the large components, and χ to the small components, of the Dirac spinor.

In terms of θ and χ the Klein-Gordon equation now reads

$$i\frac{\partial\theta}{\partial t} = -\frac{\nabla^2}{2m}(\theta+\chi) + m\theta$$
$$i\frac{\partial\chi}{\partial t} = +\frac{\nabla^2}{2m}(\theta+\chi) - m\chi \tag{9.45}$$

We introduce the more compact two-component notation

$$\Phi = \begin{bmatrix}\theta\\ \chi\end{bmatrix} \tag{9.46}$$

and write

$$i\frac{\partial\Phi}{\partial t} = H_0\Phi \tag{9.47}$$

with the free-particle hamiltonian operator H_0 given by

$$H_0 = -\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\frac{\nabla^2}{2m} + \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}m \tag{9.48}$$

Although it is in Schrödinger form, (9.47), in analogy to (9.1), does not lead to a conserved positive definite probability, because H_0 is not a hermitian operator.

The non-hermitian matrix $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ in the kinetic-energy term couples the "large" and "small" components together.

Neglecting the ∇^2 to lowest order for a slowly moving particle, we reduce to a Schrödinger equation and to the solutions (9.43) and (9.44) in the two cases of positive and negative frequency.

By borrowing the Foldy-Wouthysen technique directly from the Dirac theory, Chap. 4, we can systematically incorporate the corrections from the kinetic-energy term.

Here the matrix $\rho = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is the non-hermitian analogue of α and $\eta \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is the analogue of β in the reduction of the Dirac equation,

Using the arguments leading to (4.1), we let

$$\Phi' = e^{iS}\Phi \tag{9.49}$$

with

$$S = \eta\rho\Theta(\mathbf{p}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\Theta(\mathbf{p}) \tag{9.50}$$

and find that the odd operators ρ are removed from the hamiltonian when

$$\Theta(\mathbf{p}) = -\frac{i}{2}\tanh^{-1}\frac{\mathbf{p}^2/2m}{m + \mathbf{p}^2/2m} \tag{9.51}$$

with $\mathbf{p} \equiv \frac{1}{i}\nabla.$

The transformation (9.49) to (9.51) is not unitary and leads from a hamiltonian (9.48) which is not hermitian to a new one which is:

$$H'_0 \equiv e^{iS} H_0 e^{-iS} = \eta \sqrt{m^2 + \mathbf{p}^2} \tag{9.52}$$

In this form the positive- and negative-energy solutions are completely decoupled and the energy-momentum relation is the same as for free electrons.

The only difference of (9.52) from (4.1) is that there is now no doubling of the solutions for the spin degree of freedom.

Since H'_0 is hermitian, we are free to give a probability interpretation to the solutions Φ' in this representation.

For positive-frequency solutions

$$\Phi'^{(+)}(x) = e^{-i\omega_p t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} a^{(+)}(\mathbf{x}) \tag{9.53}$$

we have

$$\sqrt{m^2 + \mathbf{p}^2}\, a^{(+)}(\mathbf{x}) = \omega_p a^{(+)}(\mathbf{x}) \tag{9.54}$$

with

$$P(\mathbf{x}) = |a^{(+)}(\mathbf{x})|^2 \tag{9.55}$$

representing the probability density and

$$\omega_p = \int \Phi'^{(+)*}(x) H'_0(\mathbf{x}) \Phi'^{(+)}(x)\, d^3x \tag{9.56}$$

the energy.

For negative-frequency solutions we write

$$\Phi'^{(-)}(x) = e^{i\omega_p t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} a^{(-)}(\mathbf{x}) \tag{9.57}$$

The energy eigenvalue equation is similar to (9.54)

$$\sqrt{m^2 + \mathbf{p}^2}\, a^{(-)}(\mathbf{x}) = \omega_p a^{(-)}(\mathbf{x}) \tag{9.58}$$

and the probability may be defined as by (9.55):

$$P(\mathbf{x}) = |a^{(-)}(\mathbf{x})|^2 \tag{9.59}$$

Now, however, the expectation value of the hamiltonian is the negative of the energy eigenvalue due to the η in (9.52):

$$\omega_p = -\int \Phi'^{(-)*}(x) H'_0(\mathbf{x}) \Phi'^{(-)}(x)\, d^3x \tag{9.60}$$

We associate Φ'^{*} with the antiparticle wave function because $H'_0 = H'_0{}^*$ by (9.52) for a free particle and, according to the propagator picture [(9.11), (9.10), and (9.5)], it is the complex conjugate of a negative energy solution that propagates forward in time.

In the presence of external electromagnetic fields we can no longer, in general, diagonalize the Klein-Gordon equation into separate positive- and negative-frequency parts.

Proceeding in analogy with (4.2) to (4.4), however, we can achieve an approximate diagonalization in the presence of weak, slowly varying fields.

It we again introduce the field interaction by the minimal prescription

$$p_\mu \rightarrow p_\mu - eA_\mu$$

in (9.1), the Klein-Gordon equation in the two-component language of (9.48) becomes

$$i\frac{\partial \Phi(x)}{\partial t} = \left\{ \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \frac{\boldsymbol{\pi}^2}{2m} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} m + e\Phi(x) \right\} \Phi(x)$$

where $\pi = \mathbf{p} - e\mathbf{A}$.

Φ is defined as in (9.46) and (9.41), with

$$\xi = \left[\frac{\partial}{\partial t} + ie\Phi(x) \right] \phi$$

replacing (9.39).

Identifying in (4.2)

$$\beta = \eta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\mathcal{O} = \rho \frac{\boldsymbol{\pi}^2}{2m} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\boldsymbol{\pi}^2}{2m} \tag{9.61}$$

$$\mathcal{E} = e\Phi + \eta \frac{\boldsymbol{\pi}^2}{2m}$$

we arrive at (4.4), with β replaced by η.

In the special case of static external fields we find the approximate Schrödinger equation, through terms of order $1/m^4$:

$$i\frac{\partial \Phi'}{\partial t} = H'\Phi' \qquad \Phi' = e^{iS}\Phi$$

with

$$H' = \eta\left(m + \frac{\boldsymbol{\pi}^2}{2m} - \frac{\boldsymbol{\pi}^4}{8m^3} + \cdots\right) + e\Phi + \frac{1}{32m^4}[\boldsymbol{\pi}^2,[\boldsymbol{\pi}^2,e\Phi]] + \cdots \tag{9.62}$$

The first term is the binomial expansion of $\sqrt{m^2 + \boldsymbol{\pi}^2}$ showing the correct relativistic mass increase as in the Dirac theory.

In applied magnetic field it reduces the orbital g factor by m/E, as we saw for a Dirac particle.

The last term of (9.62) is the Darwin term, correcting the classical electrostatic interaction of a point charge, $e\Phi(\mathbf{x})$, in analogy to the zitterbewegung correction of the Dirac theory.

Here, however, it first appears in order $1/m^4$ in contrast with (4.5) and (4.7).

As long as we limit ourselves to physical problems for which the Foldy-Wouthysen procedure converges and, with a few terms of the series (4.4) or (9.62), leads to a good approximation to the exact description, we can discuss meson interactions as in nonrelativistic quantum mechanics.

To the accuracy of the terms retained in H', the positive- and negative-frequency solutions are decoupled in this representation and the hamiltonian is hermitian, and we can make the conventional nonrelativistic quantum-mechanical probability interpretation according to the postulates given in Chap. 1.

Taking (9.62) as an example and writing for positive-frequency solutions in this representation, in analogy with (9.53),

$$\Phi_n'^{(+)}(x) = e^{-iE_n t}\begin{bmatrix}1\\0\end{bmatrix}\psi_n^{(+)}(\mathbf{x}) \tag{9.63}$$

we have

$$\left[\left(m + \frac{\boldsymbol{\pi}^2}{2m} - \frac{\boldsymbol{\pi}^4}{8m^3} + \cdots\right) + e\Phi + \frac{1}{32m^4}[\boldsymbol{\pi}^2,[\boldsymbol{\pi}^2,e\Phi]] \cdots\right]\psi_n^{(+)}(\mathbf{x}) \tag{9.64}$$

$$= E_n\psi_n^{(+)}(\mathbf{x})$$

The probability density is given by

$$P(x) = |\Phi'^{(+)}_n(x)|^2 = |\psi^{(+)}_n(\mathbf{x})|^2 \tag{9.65}$$

and the energy eigenvalue E_n coincides with the expectation value of H as in (9.56):

$$E_n = \int \psi^{(+)*}_n(\mathbf{x}) H'(\mathbf{x},e) \psi^{(+)}_n(\mathbf{x})\, d^3x \tag{9.66}$$

where H′(**x**, e) is the operator on the left-hand side of (9.64):

$$H'(\mathbf{x},e) \equiv \left(m + \frac{\boldsymbol{\pi}^2}{2m} - \frac{\boldsymbol{\pi}^4}{8m^3} + \cdots\right) + e\Phi + \frac{1}{32m^4}[\boldsymbol{\pi}^2,[\boldsymbol{\pi}^2,e\Phi]] + \cdots \tag{9.67}$$

For negative-frequency solutions we write as in (9.57)

$$\Phi'^{(-)}_n(x) = e^{+iE_n t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \psi^{(-)}_n(\mathbf{x}) \tag{9.68}$$

and find from (9.62),

$$H'(\mathbf{x},-e)^* \psi^{(-)}_n(\mathbf{x}) = E_n \psi^{(-)}_n(\mathbf{x}) \tag{9.69}$$

Again it is the complex conjugate of the negative-energy solution which we associate with the antiparticle, since $\psi'_n{}^*(\mathbf{x})$ satisfies

$$H'(\mathbf{x},-e) \psi^{(-)*}_n(\mathbf{x}) = E_n \psi^{(-)*}_n(\mathbf{x})$$

which differs from (9.64) only by the sign of e.

The probability density is

$$P(x) = |\Phi'^{(-)}_n(x)|^2 = |\psi^{(-)}_n(\mathbf{x})|^2 \tag{9.70}$$

and the expectation value of the hamiltonian is again the negative of the energy eigenvalue as found in (9.60):

$$E_n = -\int \Phi'^{(-)*}_n(x) H' \Phi'^{(-)'}_n(x)\, d^3x \tag{9.71}$$

theSince the positive- and negative-frequency solutions differ only by the sign of charge in (9.64) and (9.69), it is attractive for us to redefine the probability and energy expectation values in this Foldy-Wouthysen transformed representation by inserting the diagonal matrix $\eta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ introduced earlier:

$$Q_n(x) \equiv \Phi'^{*}_n(x) \eta \Phi'_n(x) \tag{9.72}$$

$$E_n = \int \Phi'^{*}_n(x) \eta H' \Phi'_n(x)\, d^3x \tag{9.73}$$

This redefinition does not alter (9.65) and (9.66) for the positive-frequency solutions, but it changes the signs in (9.70) and (9.71) for the negative-frequency ones.

The energy eigenvalue coincides with the expectation value of H which is now positive for both positive- and negative-frequency solutions as for standard quantum mechanics.

However, Q(x) is now ≥0 for the positive-frequency and ≤ 0 for the negative-frequency solutions and is interpreted as the *charge density* for the particle and antiparticle, respectively.

We can proceed further in the Foldy-Wouthysen representation which, to the approximation that the terms are retained in H' and the series in powers of 1/m converges, is the same formalism as standard quantum mechanics.

Energy levels and transition rates for π-mesic atoms can be computed from (9.62) with the relativistic mass and Darwin corrections to the Schrödinger theory, for example.

Also, a classical correspondence can be established and Ehrenfest relations derived from

$$\frac{d}{dt}\langle O\rangle = i\langle [H,O]\rangle + \left\langle \frac{\partial O}{\partial t}\right\rangle \tag{9.74}$$

The one-particle probability interpretation is limited only to those cases where the positive- and negative-frequency solutions can be decoupled by the Foldy-Wouthysen procedure.

It will not apply to physical problems with strong or rapidly varying fields for which one must take into account the existence of $\pi^+\pi^-$ pairs.

With the ansatz that the definitions (9.72) and (9.73) shall apply in the general case as well, we can return to the original representation and study the structure of these inner products by undoing the transformation according to

$$\Phi = e^{-iS}\Phi' \tag{9.7}$$

We must proceed with some care here because the transformation connecting the two representations is not unitary.

Equations (9.50) and (9.51) show that for the free-particle case

$$S = -S^\dagger \tag{9.76}$$

For the energy we find, since $\eta S = -S\eta$,

$$\begin{aligned}
\omega_p &= \int \Phi'^*_p(x)\eta H'_0 \Phi'_p(x)\, d^3x \\
&= \int \Phi^*_p(x) e^{-iS^\dagger} \eta H'_0 e^{iS} \Phi_p(x)\, d^3x \\
&= \int \Phi^*_p(x)\eta e^{-iS} H'_0 e^{iS} \Phi_p(x)\, d^3x \\
&= \int \Phi^*_p(x)\eta H_0 \Phi_p(x)\, d^3x
\end{aligned} \tag{9.77}$$

which has the same form.

Similarly, for the charge we find, using (9.46) and (9.41),

$$\begin{aligned}
\int Q(x)\, d^3x &= \int \Phi'^*_p(x)\eta \Phi'_p(x)\, d^3x \\
&= \int \Phi^*_p(x)\eta \Phi_p(x)\, d^3x \\
&= \int d^3x\, [\theta^*(x)\theta(x) - \chi^*(x)\chi(x)] \\
&= \frac{i}{2m} \int d^3x\, \varphi^*(x) \overleftrightarrow{\partial_0} \varphi(x)
\end{aligned} \tag{9.78}$$

which (times 2m) was already identified as a conserved charge in (9.4).

Similar results are obtained when interactions are present.

The Foldy-Wouthysen transformation then has the form

$$\Phi' = \cdots e^{iS''} e^{iS'} e^{iS} \Phi$$

as in Chap. 4, with each $S^{(\prime)}$ satisfying

$$S^{(\prime)\dagger} = -S^{(\prime)} \qquad S^{(\prime)}\eta = -\eta S^{(\prime)}$$

as in (9.76).

Therefore, the charge density defined with the n present as in (9.78) again takes a simple form in the original representation and coincides with (9.17) for the charge density.

Similarly, for the energy levels of a π^+ or π^- meson in static external fields we have

$$E = \int \Phi'^*(x)\eta H'(\mathbf{x})\Phi'(x)\, d^3x = \int \Phi^*(x)\eta H(\mathbf{x})\Phi(x)\, d^3x \tag{9.79}$$

This simple correspondence of expectation values between the two representations suggests that we insert the matrix $\eta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ in the general definition of expectation values so that

$$\begin{aligned} \langle O' \rangle &\equiv \int \Phi'^*(x)\eta O'(x)\Phi'(x)\, d^3x \\ &= \int \Phi^*(x)\eta O(x)\Phi(x)\, d^3x \\ &= \langle O \rangle \end{aligned} \tag{9.80}$$

with $O'(x) = e^{iS}O(x)e^{-iS}$.

Without the matrix η we obtain no such simplicity of correspondence of forms between the two representations.

The physical effect of introducing η into the definition (9.80) is to multiply the expectation values of the physical observables for the system in a state of negative frequency by -1.

This is connected with the requirement that negative-frequency solutions propagate backward in time, thus reversing the role of emission and absorption and associating physical observables with minus the parameters of these negative-energy solutions.

In positron theory, the boundary condition of backward propagation for negative frequencies was ensured by hole theory; for bosons, there is no hole theory and one must be content with the arguments of propagator theory or turn to the formalism of quantum-field theory.

In concluding, we recall that we have now given a probability interpretation to solutions of physical problems for bosons for which the Foldy-Wouthysen procedure converges.

In particular, for free particles we constructed the exact transformation decoupling the positive- and negative-frequency, or particle and antiparticle, solutions.

This justifies our interpretation in (9.21), (9.22), and (9.24) of the S matrix as a probability amplitude.

The charge that distinguishes the boson from the antiboson need not be electric but may be of an entirely different character. In nature there occurs, for example, the K_0 and , which are electrically neutral and are each other's antiparticle differing by the sign of their "strangeness charge.

Also, the boson may bear no charge whatever, in which case it coincides with its own antiparticle; the π^0 is such an example.

In this case the wave function is real and $Q(x) \equiv 0$.

10 Nonelectromagnetic Interactions

Remember this is the 1960s in this book, so no Standard Model ideas exists yet.

10.1 Introduction

Armed with a propagator theory developed for their electromagnetic interactions, we attack the other known interactions of "elementary" particles of spin 0 and ½.

These interactions are divided into three distinct classes.

The first is the gravitational interaction, which at customary laboratory energies is characterized by an extremely small dimensionless coupling constant of $(M_1M_2G/\hbar c) \sim 10^{-40}$ and which we shall neglect.

The second class is the weak interactions, which lead to transmutations between particles such as in β decay and the decay of π, K, and μ mesons [see (9.3)].

These are characterized in the region of low and moderate energies ($\leq$ 1 GeV) by a dimensionless coupling constant of $\sim 10^{-5}$ to 10^{-6}.

Finally, there exist the strong interactions, characterized by coupling constants > 1, which are responsible for the forces which bind nuclei and which provide the mechanism for producing π, K, Λ, Σ, and Ξ particles in reactions such as (9.2).

The understanding of the weak and the strong interactions has not progressed to such a degree that their effects can be derived from a general principle such as the equivalence principle and general covariance in gravitational theory or the principle of "minimal electromagnetic interaction" which instructs the introduction of electrodynamic couplings with the substitution $\mathbf{p}_\mu \rightarrow \mathbf{p}_\mu - eA_\mu$.

In the absence of such a lofty starting point it is necessary to appeal directly to the available experimental evidence along with symmetry principles, notably Lorentz invariance, to limit the possible forms of the interaction.

"What are the vertices?" is the central question in discussing the weak and the strong interactions, and to this we now turn.

We proceed within the framework of the propagator approach and limit ourselves to lowest order calculations in the coupling parameters.

From the point of view of detailed experimental comparison this is a very severe limitation.

For the strong interactions, the expansion parameter exceeds unity; for the weak interaction theory in its present primitive form, higher-order diagrams are dependent on the cutoff in the closed-loop momentum integrals in an unpleasant way which does not permit the divergences to be isolated into renormalization constants as illustrated in the electrodynamic calculations in Chap. 8.

10.2 Strong Interactions

In 1935, Yukawa' made an analogy between the strong, short-ranged nuclear force and the electromagnetic force between particles.

If the Coulomb force is due to the exchange of a virtual quantum, or photon, perhaps the nuclear force is likewise due to a virtual particle, necessarily of integral spin, exchanged between nucleons.

For a particle of spin zero and mass u we may use the Klein-Gordon propagator (9.36) in writing the first-order scattering amplitude corresponding to Fig. 10.1:

$$\mathfrak{M} \sim \frac{g_0^2}{q^2 - \mu^2 + i\epsilon} \tag{10.1}$$

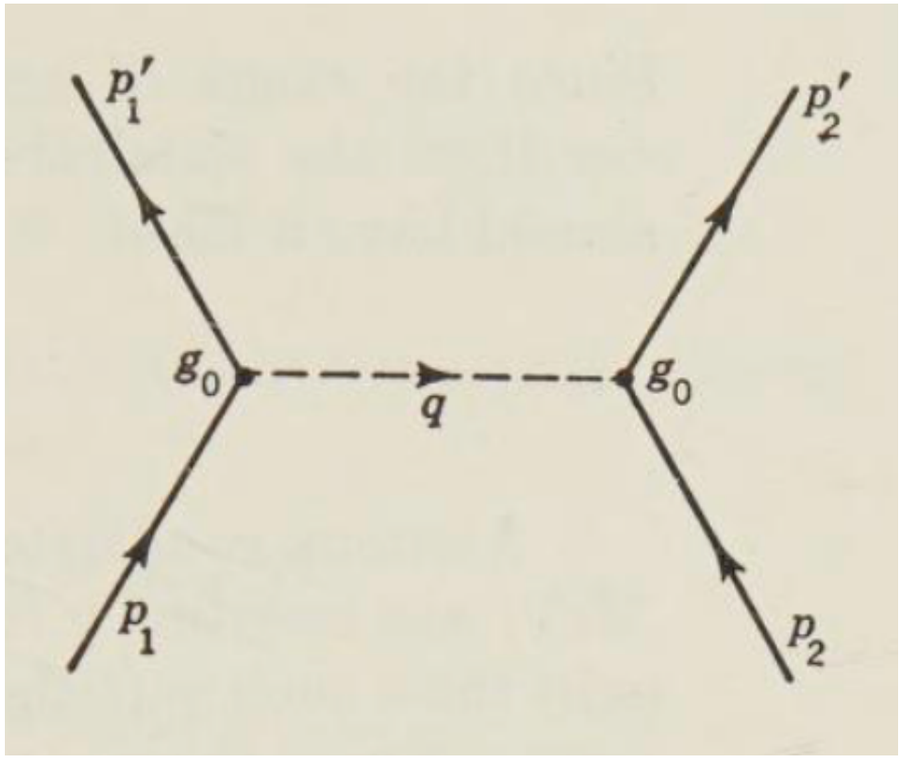

Fig. 10-1 Nucleon-nucleon interaction with single π^+ exchange.

In writing (10.1) we have suppressed all factors coming from the vertices at which the particle, represented by the dashed line, is absorbed or emitted by the two nucleons, drawn as solid lines with initial and final momenta p_1, p_2 and p'_1, p'_2, respectively.

The invariant momentum transfer $q^2 = (p_1 - p'_1)^2 = (p'_2 - p_2)^2$ is space-like ($q^2 < 0$).

In the nonrelativistic limit in which the recoil kinetic energies of the nucleons are neglected relative to their rest energies, $q^2 \sim -|\mathbf{q}|^2$ and we may approximate (10.1) to

$$\mathfrak{M} \sim \frac{g_0^2}{|\mathbf{q}|^2 + \mu^2} \tag{10.2}$$

Fourier-transforming to coordinate space, we see that $\mathfrak{M}$ corresponds to the Born approximation amplitude for scattering in a Yukawa potential

$$V(r) \sim g_0^2 \frac{e^{-\mu r}}{r}$$

Since the range of nuclear forces extends to ~10^{-13} cm, roughly one-third the classical electron radius, the particle being exchanged should have a mass

$$\mu \sim \frac{3m_e}{\alpha} \sim 200 \text{ MeV}$$

A strong candidate for this particle, the π meson of mass ~140 MeV, was eventually found in 1947; indeed there are now known to exist three such particles, the π^+, π^-, and π^0 of approximately the same mass.

These three π-mesons of charge +e, -e, and 0, respectively, are believed to be the major contributors to the nuclear force at large distances, although heavier particles such as the K meson may also play a role for small impact parameter collisions with large q^2.

It has been determined experimentally that the spins of the π mesons are zero but that their "intrinsic parity" is odd.

For the charged mesons application of detailed balance to the reactions

$$\pi^+ + d \leftrightarrow p + p$$

gives the spin as zero, since the ratio of these processes proceeding in the two directions is determined by the statistical weights.

The "intrinsic parity" was determined by observation of capture of a π^- from the K shell in a deuterium atom, leading to two neutrons:

$$\pi^- + d \rightarrow n + n$$

The only state of J = 1 that can be formed by two neutrons is the $^3P_1$ according to the exclusion principle and has parity -1.

If parity conservation is to apply in this strong reaction, the a must also have odd parity.

In this assignment of parity we follow the usual convention of choosing proton and neutron to have the same intrinsic parity +1; that is, the same phase $\varphi = 0$ is assigned to their wave functions under the spatial reflection (2.33) so that $y'(\mathbf{x}',t) = + \gamma_0 \psi(\mathbf{x},t)$ for $\mathbf{x}' = -\mathbf{x}$.

Since the π^- is captured from a spherically symmetric s orbit, the parity - 1 assigned to its wave function is referred to as its "intrinsic parity."

The properties of zero spin and negative intrinsic parity are shared by both π^+ and π^- mesons which - as interpreted in Chap. 9 - are each other's antiparticle.

For the π^0-meson observation of the two-photon decay

$$\pi^0 \rightarrow \gamma + \gamma$$

coupled with the observation of the correlation in the planes of polarization' of the "Dalitz pairs"

$$\pi^0 \rightarrow (e^+ + e^-) + (e^+ + e^-)$$

determines the spin to be zero and parity to be -1.

With this information we continue to model the nuclear force discussion along the lines of electrodynamics, starting with a more detailed discussion of proton-proton scattering due to the exchange of a single π^0, as in Fig. 10.1.

Consider, for instance, that proton 1 is scattered in a "-meson field" which is produced by proton 2, in analogy with the electromagnetic potential $A_\mu(x)$ in the discussion of electron-proton scattering, Eq. (7.31).

To describe this process, we write a tentative Dirac equation, analogous to (6.52), which will be of the form

$$(i\nabla - M_p)\psi_p(x) = g_0\Gamma\psi_p(x)\varphi_0(x) \tag{10.3}$$

where g_0 is the analogue of e and Γ is a Dirac matrix to be determined.

The proposed equation for the meson field φ_0, analogous to (7.27) and (7.33), will then be, within a sign $\eta_0 = \pm 1$ to be determined,

$$(\Box + \mu_0^2)\varphi_0(x) = -g_0\bar{\psi}_p(x)\Gamma\psi_p(x)\eta_0 \tag{10.4}$$

Parity is observed to be conserved to high accuracy for nuclear forces, and we consequently demand that (10.3) and (10.4) conserve parity as well as be Lorentz covariant.

It is then necessary to choose $\Gamma = i\gamma_5$ to make the right-hand side of (10.4) transform as a real pseudoscalar, as does the left-hand side.

We may also verify that there is a charge conjugation transformation which leaves (10.3) invariant so that we can carry over directly the hole-theory discussions of Chap. 5 and reinterpret the negative-energy solutions as antiprotons.

The antiproton wave function is formed as in (5.5) by

$$\psi_{\bar{p}} = C\bar{\psi}_p^T$$

Fig. 10.1 —> Fig. 10.2

and satisfies the same Dirac equation as (10.3) if the π^0 field is identical with the charge conjugate field, that is,

$$(\varphi_0)_c = +\varphi_0$$

This simple form of the interaction is in all likelihood incorrect or at best incomplete.

The analogy with electrodynamic interactions, together with simplicity, was our sole motivation in writing (10.3) and (10.4).

We have, for example, arbitrarily excluded the possibility of interaction terms containing derivatives of the fields, although we have already encountered such terms in the preceding chapter in discussing the electrodynamic couplings of charged π mesons.

We should therefore regard (10.3) and (10.4) as no more than a rough, simple model, since nature undoubtedly has more imagination than is exhibited in those equations.

The virtue of the model is that it permits discussion of general features of the nuclear interaction which survive in a more general treatment.

Now that there is a candidate for the π^0-p vertex, we may compute the amplitude for the graph of Fig. 10.1.

From (10.4), the first-order π^0 field produced by the "transition current" of particle 2,

$$-g_0\eta_0\bar{\psi}_{p_2'}(x)i\gamma_5\psi_{p_2}(x)$$

is

$$\varphi_0(x) = -ig_0\int d^4x'\, i\Delta_F(x-x')[\bar{\psi}_{p_2'}(x')i\gamma_5\psi_{p_2}(x')]\,\eta_0 \tag{10.5}$$

This produces a change in the wave function of proton 1, according to (10.3),

$$\Delta\psi_{p_1}(x) = \int d^4x''\, S_F(x-x'')[g_0 i\gamma_5\psi_{p_1}(x'')\varphi_0(x'')]$$

and thus the scattering amplitude, by (6.53) and (6.56), is (10.6)

$$S_{fi} = (-ig_0)^2\int d^4x'\, d^4x''\, [\bar{\psi}_{p_1'}(x'')i\gamma_5\psi_{p_1}(x'')]i\Delta_F(x''-x')\eta_0 \times [\bar{\psi}_{p_2'}(x')i\gamma_5\psi_{p_2}(x')] \tag{10.7}$$

Comparing (10.7) with the analogous (7.32) and (7.33), we conclude that the rules for graphs are modified by replacement of the vertex $e\gamma_\mu$ in electrodynamies by $ig_0\gamma_5$ and of the photon propagator $ig_{\mu\upsilon}D_F(x - x')$ by $+i\Delta_F(x - x')$no for the π^0.

To (10.7) we must add the exchange contribution corresponding to Fig. 10.2 that is due to the identity of the two protons.

The amplitude for this is

$$S_{fi}^{ex} = -(-ig_0)^2 \int d^4x'\, d^4x'' \,[\bar{\psi}_{p_2'}(x'')i\gamma_5\psi_{p_1}(x'')]i\Delta_F(x'' - x')\eta_0 \times [\bar{\psi}_{p_1'}(x')i\gamma_5\psi_{p_2}(x')] \tag{10.8}$$

and differs from (10.7) by the interchange of final proton wave functions $\bar{\psi}_{p_2'}(x) \leftrightarrow \bar{\psi}_{p_1'}(x)$ as well as by the important minus sign which, as in electron-electron scattering (7.82), assures antisymmetry of the initial and final proton wave functions under interchange of the two protons.

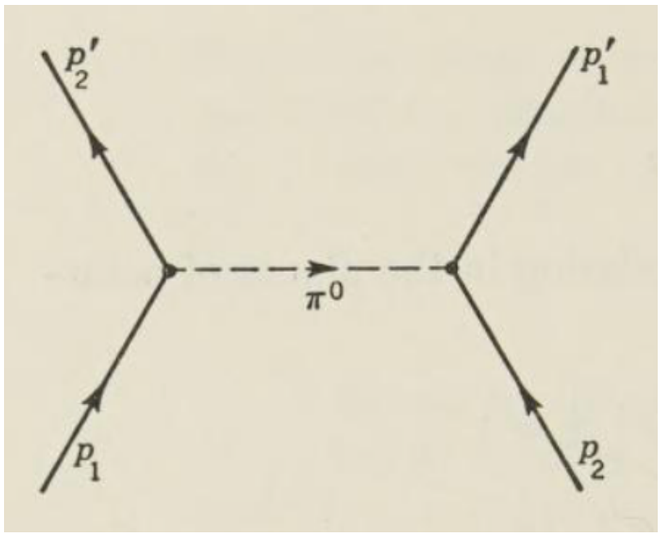

Fig. 10-2 Pauli exchange graph for two protons.

Neutron-neutron scattering is described in a similar fashion.

We must write a wave equation for a neutron which includes an interaction term with the π^0.

In addition, we add to (10.4) a neutron source term for the π^0.

Important experimental guidance here comes from the observed equality of p-p and n-n forces within corrections due to electromagnetic interactions, such as the Coulomb force between protons.

This suggests that, within a sign $\epsilon_0 = \pm 1$ still to be determined, the coupling between neutrons and the π^0 be the same as for protons.

We therefore write for the neutron wave functions

$$(i\nabla\!\!\!/ - M_n)\psi_n(x) = -g_0\epsilon_0 i\gamma_5\psi_n(x)\varphi_0(x) \tag{10.9}$$

and we replace (10.4) by

$$(\Box + \mu_0^2)\varphi_0(x) = -g_0[\bar{\psi}_p(x)i\gamma_5\psi_p(x) - \epsilon_0\bar{\psi}_n(x)i\gamma_5\psi_n(x)]\eta_0 \tag{10.10}$$

The small mass difference between neutrons and protons, $M_n - M_p \sim 0.002M_p$, is attributed to electromagnetic effects due to the proton charge and is neglected in this approximation along with all the other electromagnetic interactions.

Equations (10.9) and (10.10) lead to an n-n scattering amplitude which is identical to the p-p amplitude (10.7) and (10.8), since $\epsilon_0^2 = +1$.

We must also take into account the coupling to charged π^+ and π^- mesons when we come to the p-n scattering.

In the lowest order approximation of including only one meson exchange in the scattering amplitude these give rise to diagrams (Fig. 10.3b) of the charge exchange type in addition to the noncharge exchange scattering (Fig. 10.3a).

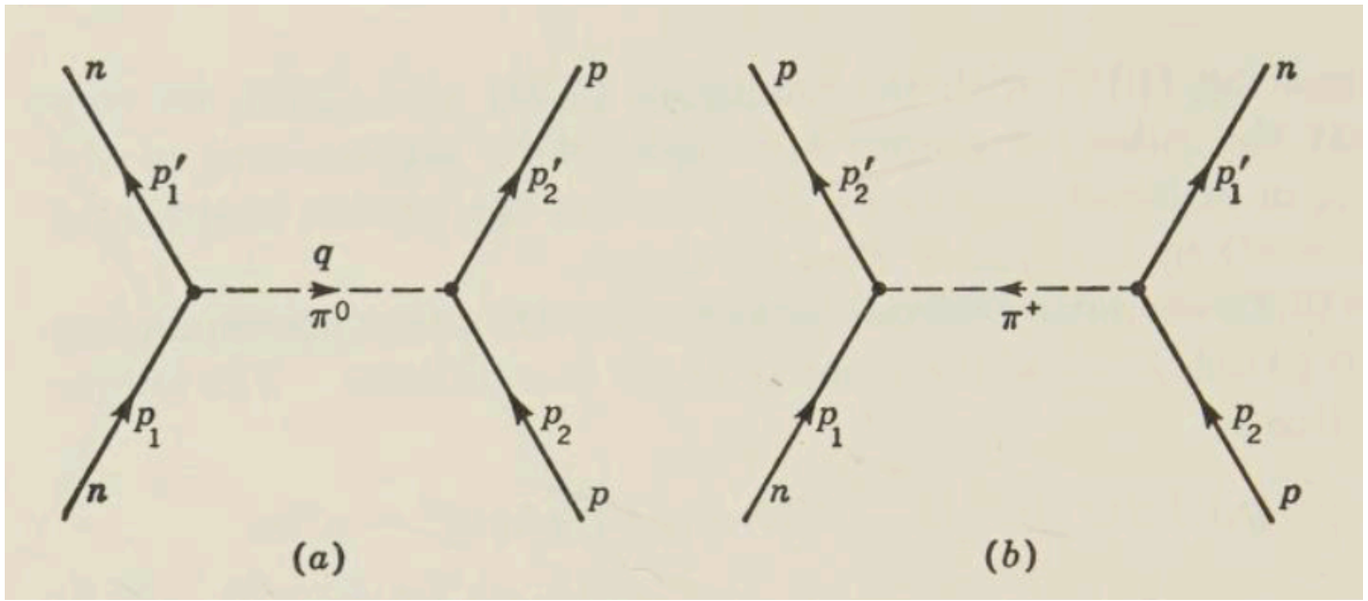

Fig. 10-3 Noncharge-exchange and charge-exchange diagrams for n-p scattering.

In writing a wave equation for the π^+ we are again guided by the observed equality of the n-p force to the p-p force, within electromagnetic corrections, in the states available to the p-p system.

In analogy with (10.10) for the π^0, we write for the at π^+

$$(\Box + \mu_+^2)\varphi_+(x) = -g_+\eta_+\bar{\psi}_n(x)i\gamma_5\psi_p(x) \qquad (10.11)$$

where the sign $\eta_+ = \pm 1$ and the coupling constant g_+ will be determined relative to g_0 and η_0 later.

The right-hand side of (10.11) gives rise to vertices (Fig. 10.4) from which a at emerges in the transmutation of a proton to a neutron.

the the

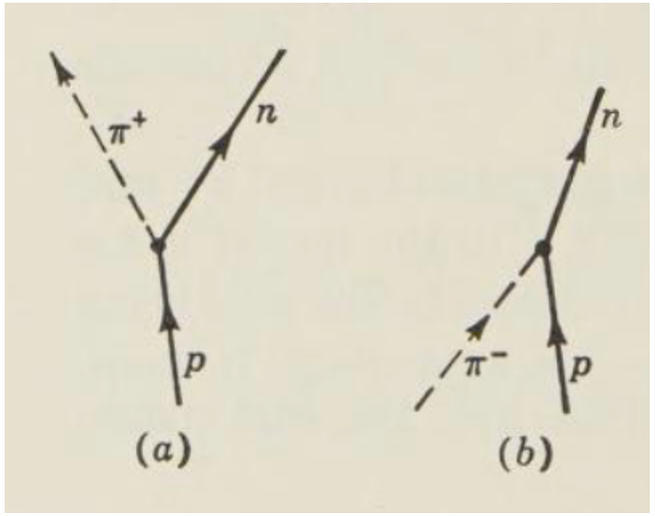

Fig. 10-4 Vertices for p → n.

The π^+ may then propagate forward or backward in time.

If backward, as discussed in the propagator theory for $\pi^\pm$ mesons in the preceding chapter, it is interpreted as a π^- coming forward in time with positive energy to be absorbed at the vertex as in Fig. 10.4b.

The complex conjugate of (10.11) gives the equation for the charge-conjugate particle, the π^-:

$$(\Box + \mu_+^2)\varphi_+^*(x) = (\Box + \mu_+^2)\varphi_-(x) = -g_+^*\eta_+\bar{\psi}_p(x)i\gamma_5\psi_n(x) \tag{10.12}$$

The right-hand side of (10.12) provides the vertices of Fig. 10.5.

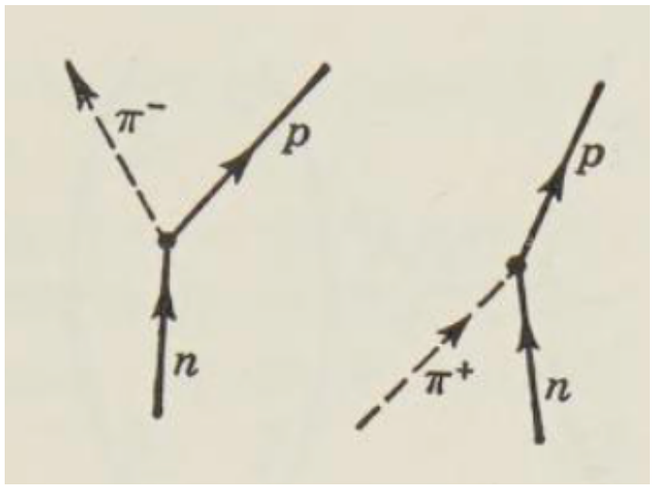

Fig. 10-5 Vertices for n → p.

We must now modify the wave equations for the neutron and proton with additional terms allowing for the transitions in Figs. 10.4 and 10.5.

Comparing (10.11) and (10.12) with (10.3), (10.9), and (10.10), we are led to write for the complete wave equations for protons and neutrons,

$$
\begin{aligned}
(i\not\partial - M_p)\psi_p(x) &= g_0 i\gamma_5\psi_p(x)\varphi_0(x) + g_+^* i\gamma_5\psi_n(x)\varphi_+(x) \\
(i\not\partial - M_n)\psi_n(x) &= -g_0\epsilon_0 i\gamma_5\psi_n(x)\varphi_0(x) + g_+\epsilon_+ i\gamma_5\psi_p(x)\varphi_-(x)
\end{aligned}
\tag{10.13}
$$

where an additional sign $\epsilon_+ = \pm 1$ remains to be determined.

In order to limit the constants remaining in the wave equations, we consider n-p scattering and write the amplitudes associated with the two lowest order graphs, Fig. 10.3.

Considering the incident neutron (p_1) to scatter in the meson field produced by the incident proton (p_2), we find as before

$$
\begin{aligned}
S_{fi} = (-ig_0)^2(-\epsilon_0)\eta_0 \int d^4x'\, d^4x''\, [\bar\psi_{p_1'}(x')i\gamma_5\psi_{p_1}(x')]i\Delta_F(x'-x'') \\
\times [\bar\psi_{p_2'}(x'')i\gamma_5\psi_{p_2}(x'')] \\
+ (-ig_+)(-ig_+^*)\eta_+ \int d^4x'\, d^4x''[\bar\psi_{p_2'}(x')i\gamma_5\psi_{p_1}(x')]i\Delta_F(x'-x'') \\
\times [\bar\psi_{p_1'}(x'')i\gamma_5\psi_{p_2}(x'')]
\end{aligned}
\tag{10.14}
$$

Had we instead considered the incident proton to scatter in the field produced by the incident neutron, (10.14) would be modified by the replacement

$$\eta_+ \longrightarrow \eta_+\epsilon_+$$

Hence we set

$$\epsilon_+ = 1 \tag{10.15}$$

since the only change has been our point of view.

In order to determine the relative magnitudes of g_+ and g_0, we again appeal to the observed approximate equality of the n-p force to the p-p force in the states allowed for the p-p system by the exclusion principle.

Let us then compare p-p and p-n scattering for the two particles in antisymmetric states.

To do this, we temporarily *imagine* the neutron to be identical to the proton but still coupled to charged as well as neutral mesons, and we compute its scattering by adding exchange graphs, with the appropriate minus sign, to assure anti-symmetric states.

Figure 10.6 shows the two diagrams of Fig. 10.3 plus their exchange parts, with the momenta and coupling as indicated; diagram (c) is the exchange of (a) and (d) of (b).

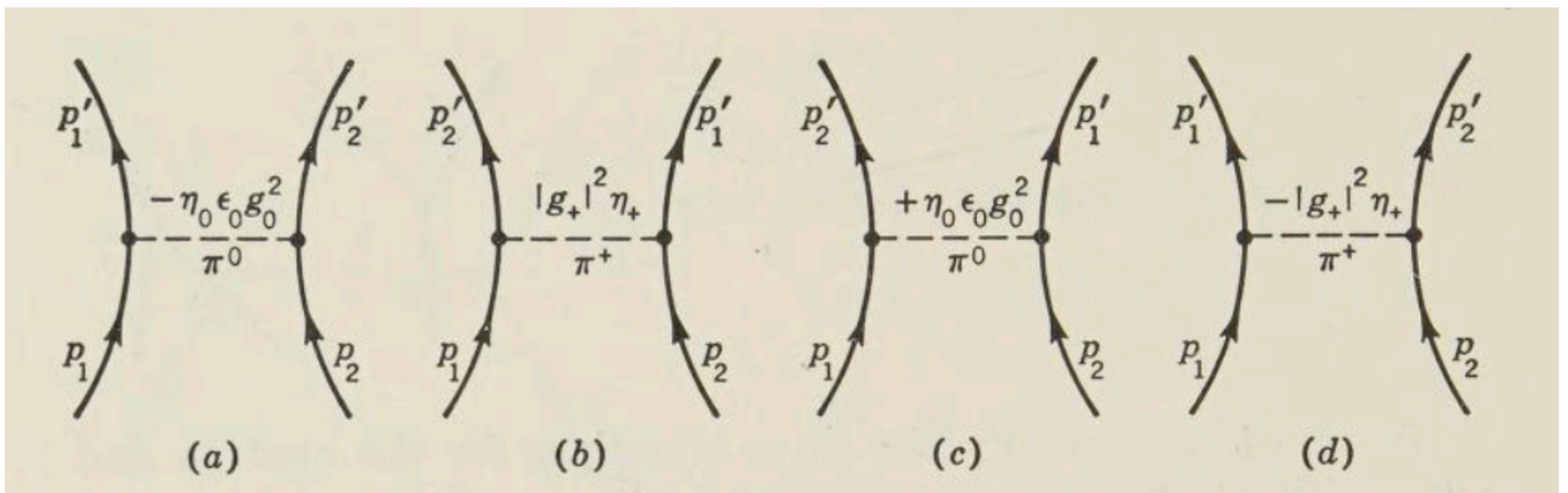

Fig. 10-6 Sum of graphs for p-n scattering with one pion exchange including Pauli- and charge-exchange contributions.

The sum of these four contributions to the scattering amplitude, S_{ji}, constructed as if n and p were identical and obeyed the exclusion principle is

$$\begin{aligned} S_{fi} = & -(-i)^2[\eta_0\epsilon_0 g_0^2 + \eta_+|g_+|^2] \\ & \times \int d^4x'\, d^4x'' \{[\bar{\psi}_{p_1'}(x')i\gamma_5\psi_{p_1}(x')]i\Delta_F(x'-x'')[\bar{\psi}_{p_2'}(x'')i\gamma_5\psi_{p_2}(x'')] \\ & - [\bar{\psi}_{p_2'}(x')i\gamma_5\psi_{p_1}(x')]i\Delta_F(x'-x'')[\bar{\psi}_{p_1'}(x'')i\gamma_5\psi_{p_2}(x'')]\} \end{aligned} \tag{10.16}$$

Comparing (10.16) with the sum of (10.7) and (10.8) shows that the condition on the coupling constants for equal n-p and p-p interactions in the antisymmetrical states available to the p-p system is

$$g_0^2\eta_0 = -|g_+|^2\eta_+ - g_0^2\eta_0\epsilon_0 \tag{10.17}$$

This has two solutions

$$|g_+|^2 = 0 \qquad \epsilon_0 = -1 \tag{10.18}$$

and

$$|g_+|^2 = 2g_0^2 \qquad \epsilon_0 = +1 \qquad \eta_0 = -\eta_+ \tag{10.19}$$

The first, (10.18), corresponds to the excahnge of only neutral mesons which clearly leads to equal n-p and p-p forces in corresponding states.

Since the at and a mesons exist and reactions are observed with vertices of the types in Figs. 10.4 and 10.5 at which single at or t mesons are produced, for example,

$$\gamma + p \rightarrow n + \pi^+$$

we are led to choose the second solution (10.19).

There might, of course, be additional single neutral mesons contributing to the nuclear force; but since no clear candidate exists, we ignore this possibility here.

According to (10.19) the coupling of charged mesons to a (p-n) vertex is greater by $\sqrt{2}$ than the π^0 coupling to a (p-p) vertex.

Since only the absolute square $|g_+|^2$ appears in the scattering amplitude, we may choose g_+ real for convenience, so that

$$g_+ = \sqrt{2}\, g_0 \tag{10.20}$$

We find it convenient to replace the condition $\eta_+ = -\eta_0$ in (10.19) by an additional rule in writing Feynman amplitudes.

We write

$$\eta_+ = +\eta_0 \tag{10.21}$$

and introduce the following additional rule for writing the amplitudes for each graph: Multiply by -1 if an odd number of charged mesons are exchanged by the nucleons in the graph.

This rule may be shown to be equivalent to the rule to antisymmetrize not only with respect to interchange of n-n and p-p lines in a graph but also to n-p lines.

This is because any graph with a charged pion exchange can be related to a graph with a neutral pion exchange by interchange of n and p lines (and the relative minus sign comes from the opposite signs η_0 and η_+ associated with the pion propagator.

This evidently is the case in lowest order; compare Figs. 10.6a and b.

In general, there are two cases to consider in making the anti-symmetrization prescription well-defined.

The first, illustrated in Fig. 10.7a, occurs when the π^+ is exchanged between two different nucleon lines.

This case is similar to that encountered above in lowest order.

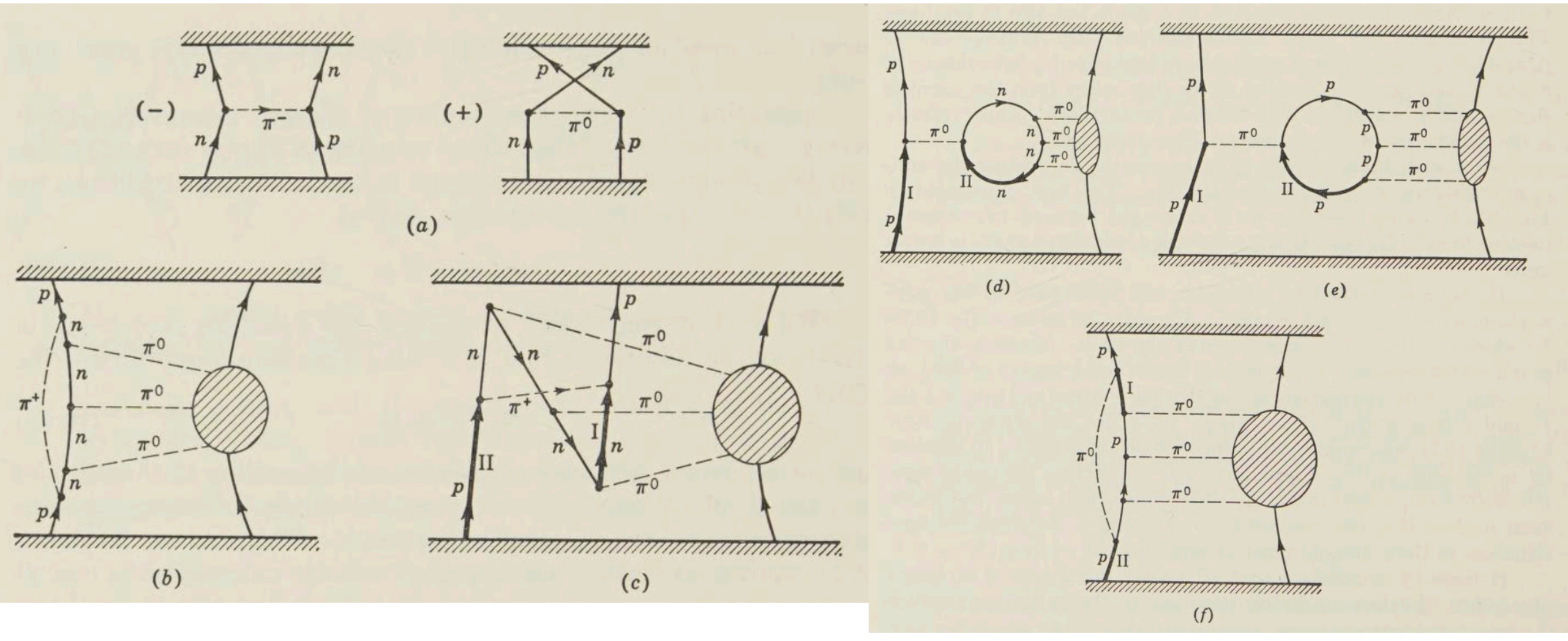

Fig. 10-7 Rule antisymmetrizing n-p exchanges.

If the is emitted by a nucleon and reabsorbed by the same nucleon, a little care must be taken.

Consider, for instance Fig. 10.7b, for which one time ordering is shown in Fig. 10.7c.

To relate this to a graph with a signature we know (that is, not involving the π^+ line), we interchange with appropriate minus sign the neutron and proton I and II and obtain graph 10.7d. Graph 10.7d has the same signature as graph 10.7e, whose signature is opposite that of graph 10.7f, obtained by interchange of lines I and II in (e).

Collecting the minus signs (there are two, we see that if the it is emitted and reabsorbed on the same nucleon line, one may set $\eta_+ = \eta_0$ and give the graph the same signature as the graph obtained by replacing the π^+ by a π^0.

It must be emphasized that all we are doing here is making a purely formal extension, by construction, of the antisymmetrization rule from the p-p scattering case, where it has a physical basis in the exclusion principle.

As applied to the n-p scattering case for which the particles are different, distinguishable, and can scatter in symmetric $^3S_1$, $^3P_1$, $^3D_{2,1,0}$, etc., states, it is no more than a convenient rule for bookkeeping on the signs, in the approximation of "charge independence," that is, of equal p-p, n-n, and n-p forces in corresponding states.

Our reason for introducing this formal and seemingly complicated generalization of the exclusion principle will be seen in the next section, where we develop a simple unified description of the proton and neutron as two aspects of the same particle, the nucleon.

According to this rule we introduce a relative minus sign into a p-n scattering amplitude when we interchange the final p and n, as in Figs. 10.3a and 10.3b, in the same way as we introduce a -1 between the amplitudes for Figs. 10.1 and 10.2 in p-p scattering as required by the exclusion principle for two identical fermions.

At this stage we have determined all parameters except η_0; $\epsilon_0 = \epsilon_+ = +1$ and $\eta_+ = \eta_0$ with the additional convention that graphs differing by exchange of an n and p line be antisymmetrized.

To obtain η_0, we observe that, according to the interactions in (10.13), as illustrated by Figs. 10.4 and 10.5, a proton is not always a proton, but is sometimes a neutron and a π^+; hence the electric charge is carried mutually by the π^+ and the proton.

This has the consequence that neither the electromagnetic current of the proton nor that of the it is separately conserved, as is seen by computing these currents (normalized to unit charge) from the above wave equations:

$$i\frac{\partial}{\partial x_\mu} j_\mu{}^p(x) = i\frac{\partial}{\partial x_\mu}(\bar{\psi}_p \gamma_\mu \psi_p) = g_+^* \bar{\psi}_p i\gamma_5 \psi_n \varphi_+ - g_+ \bar{\psi}_n i\gamma_5 \psi_p \varphi_-$$

$$i\frac{\partial}{\partial x_\mu} j_\mu{}^{\pi^+}(x) = i\frac{\partial}{\partial x_\mu}\left[i\varphi_+^*\left(\frac{\overrightarrow{\partial}}{\partial x^\mu} - \frac{\overleftarrow{\partial}}{\partial x^\mu}\right)\varphi_+\right] \tag{10.22}$$

$$= -\eta_+ g_+^* \bar{\psi}_p i\gamma_5 \psi_n \varphi_+ + \eta_+ g_+ \bar{\psi}_n i\gamma_5 \psi_p \varphi_-$$

There exists, however; a differential current conservation law for the sum of the proton and π^+ currents:

$$\frac{\partial}{\partial x_\mu}\left[j_\mu{}^p(x) + j_\mu{}^{\pi^+}(x)\right] = 0 \tag{10.23}$$

provided we choose

$$\eta_+ = +1 \tag{10.24}$$

in the π^+ equation (10.11).

We make this choice in order to rule out the unobserved possibility of local sources and sinks of electric charge.

It follows from (10.21) that

$$\eta_0 = \eta_+ = +1 \tag{10.25}$$

10.3 Isotopic Spin Formalism

Gathering together the rules (10.15), (10.19) to (10.21), and (10.25), as determined by the requirement of equal n-n, n-p, and p-p scattering amplitudes in corresponding states, we can write the wave equations (10.10) to (10.13) in terms of one real unknown coupling constant, as summarized below:

$$\begin{aligned}
(i\not\nabla - M_p)\psi_p &= g_0 i\gamma_5(\psi_p\varphi_0 + \sqrt{2}\,\psi_n\varphi_+) \\
(i\not\nabla - M_n)\psi_n &= g_0 i\gamma_5(-\psi_n\varphi_0 + \sqrt{2}\,\psi_p\varphi_-)
\end{aligned} \tag{10.26}$$

$$\begin{aligned}
(\Box + \mu_0^2)\varphi_0 &= -g_0(\bar\psi_p i\gamma_5\psi_p - \bar\psi_n i\gamma_5\psi_n) \\
(\Box + \mu_+^2)\varphi_+ &= -g_0\sqrt{2}\,\bar\psi_n i\gamma_5\psi_p \\
(\Box + \mu_+^2)\varphi_- &= (\Box + \mu_+^2)\varphi_+^* = -g_0\sqrt{2}\,\bar\psi_p i\gamma_5\psi_n
\end{aligned} \tag{10.27}$$

The similarity in the forms of the proton and neutron equations suggests that we introduce one nucleon wave function

$$\Psi = \begin{bmatrix} \psi_p \\ \psi_n \end{bmatrix} \tag{10.28}$$

to describe them.

The nucleon wave function is represented by an eight-component spinor, the top four components for the proton spinor and the lower four for the neutron.

The free Dirac equation is diagonal with no coupling between the p and n components; and in the "charge independent" approximation, with $M_p \approx M_n \approx M$, it is simply

$$(i\nabla\!\!\!/ - M)\Psi = 0$$

For the interaction terms in (10.26) we must introduce nondiagonal matrices mixing the n and p wave functions.

As a notation for this mixing it is convenient to introduce the three Pauli matrices

$$\tau_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \tau_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{10.29}$$

with each entry understood to operate on all four components of ψ_p or ψ_n in (10.28), for example,

$$\tau_1\Psi = \begin{bmatrix} \psi_n \\ \psi_p \end{bmatrix}$$

We label the matrices here by $\boldsymbol{\tau}$ in order to distinguish them from the Pauli matrices σ, which operate on the spin components of the neutron and proton.

Equations (10.26) may now be combined and rewritten as

$$(i\not\nabla - M)\Psi = g_0 i\gamma_5(\tau_3\Psi\varphi_0 + \sqrt{2}\,\tau_+\Psi\varphi_+ + \sqrt{2}\,\tau_-\Psi\varphi_-) \tag{10.30}$$

where

$$\tau_+ \equiv \tfrac{1}{2}(\tau_1 + i\tau_2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$\tau_- \equiv \tfrac{1}{2}(\tau_1 - i\tau_2) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{10.31}$$

are the charge "raising" and "lowering" operators, respectively.

We further reduce this to a very compact form if we introduce a "vector" $\boldsymbol{\varphi}$ with the three components

$$\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3) \tag{10.32}$$

where

$$\varphi_1 \equiv \frac{1}{\sqrt{2}}(\varphi_+ + \varphi_-) \qquad \varphi_2 \equiv \frac{i}{\sqrt{2}}(\varphi_+ - \varphi_-) \qquad \varphi_3 \equiv \varphi_0$$

This gives

$$(i\not\nabla - M)\Psi = g_0 i\gamma_5(\boldsymbol{\tau}\cdot\boldsymbol{\varphi})\Psi \tag{10.33}$$

In the same way, with neglect of the small $\pi^{\pm}$ - π^0 mass difference, $\mu_0 \approx \mu_+ \approx \mu$, Eqs. (10.27) may be combined into an approximate equation for the π mesons:

$$(\Box + \mu^2)\boldsymbol{\varphi} = -g_0\bar{\Psi}i\gamma_5\boldsymbol{\tau}\Psi \tag{10.34}$$

This compact "isotopic spin" notation for m mesons and nucleons represents purely formal progress and has been accompanied by no new physical input.

In terms of a fictitious "isotopic space" we may suppose that Ψ transforms as a spinor and ϕ as a vector.

Then the wave equations (10.33) and (10.34) are both covariant under rotations in isotopic space.

This covariance is a consequence of limiting the forms of the coupling terms so that protons and neutrons and charged and neutral π mesons share identical interactions and are therefore equivalent in the absence of electromagnetic effects.

Conversely, we may turn the whole procedure around and show that for any set of such wave equations which are invariant under rotations in isotopic space the n-n, n-p, p-p forces in corresponding states are equal.

It was with this aim of constructing a compact language in which the proton and neutron appear as two components of the nucleon wave function Ψ that we tailored our rules for Feynman amplitudes in the preceding section.

The isotopic spin formalism is mathematically identical to the three-dimensional angular-momentum formalism.

In the same way that the law of angular-momentum conservation follows from the covariance of the wave equation under rotations in ordinary three-dimensional space, a law of isotopic spin conservation emerges from the covariance of (10.33) and (10.34) in isotopic space.

However, the isotopic spin conservation is only an approximate law, since the symmetry of the equations is valid only with the neglect of electromagnetic couplings and of mass differences between p and n and $\pi^{\pm}$ and π^0.

In this approximation the states of systems of mesons and nucleons may be diagonalized with respect to the square of the total isotopic spin I^2 and, say, the third component of isotopic spin I_3, which is related to the total charge of the system.

The nucleon wave function (10.28) rotates in isotopic space as a two-component spinor; the nucleon is thus assigned one-half unit of isotopic spin I, with the component along the three-axis being ½ for the proton and -½ for the neutron.

The meson wave function (10.32) rotates as a vector, with a three-axis projection of 0 for the π^0.

The meson thus carries one unit I = 1 of isotopic spin.

Meson-nucleon scattering may be discussed in terms of the two isotopic channels of I = 3/2 and I = ½ through which a meson and nucleon may couple according to the familiar rules of angular-momentum combination.

In nucleon-nucleon scattering only one isotopic channel I = 1 is available for p-p and n-n scattering in the charge-independent approximation.

For the p-n system $I_S = 0$, and the scattering may be via both the I = 0 and I = 1 channels.

We shall discuss these examples in the isotopic spin language shortly.

To summarize, we list the rules for writing amplitudes corresponding to graphs in this model with the charge-independent interactions of (10.33) and (10.34) (compare Sec. 8.1):

1. Draw all connected graphs.
2. Associate with each graph an amplitude with a factor

$$-ig_0(i\gamma_5\tau_\alpha)\int d^4x$$

at each vertex.

3. Include a propagator $iS_F(x-y)\mathbf{1}$, with $\mathbf{1}$ the unit matrix in the 2×2 dimensional space of the nucleon isotopic spinors, for each internal nucleon line terminating at x and y. For each internal meson line include a propagator $i\Delta_F(x-y)\delta_{\alpha\beta}$, with $\delta_{\alpha\beta}$ a 3×3 unit matrix tying together the τ_α and τ_β operators at the vertices connected by the meson.

4. Introduce a wave function for each external line. For the nucleon line the projection operators $\frac{1}{2}(1+\tau_3)$ and $\frac{1}{2}(1-\tau_3)$ for proton and neutron, with isotopic wave functions $\chi_p=\begin{bmatrix}1\\0\end{bmatrix}$ and $\chi_n=\begin{bmatrix}0\\1\end{bmatrix}$, respectively, are useful. The wave function for an incident neutron with quantum numbers (p,s), for example, is

$$\frac{1}{(2\pi)^{3/2}}\sqrt{\frac{M}{E_p}}\,e^{-ip\cdot x}u(p,s)\chi_n=\frac{1}{(2\pi)^{3/2}}\sqrt{\frac{M}{E_p}}\,e^{-ip\cdot x}u(p,s)\left(\frac{1-\tau_3}{2}\right)\begin{bmatrix}0\\1\end{bmatrix} \tag{10.35}$$

For the meson line the wave function has the isotopic factor $\hat{\phi}$, where $\hat{\phi}$ is a unit vector in the three-dimensional isotopic spin space of the π mesons. In terms of the states of mesons with charge $+$, $-$, and 0, $\hat{\phi}$ has the components, by (10.32):

$$\hat{\phi}_+ = \frac{1}{\sqrt{2}}(1,i,0)$$

$$\hat{\phi}_- = \frac{1}{\sqrt{2}}(1,-i,0) \qquad (10.36)$$

$$\hat{\phi}_0 = (0,0,1)$$

Thus at the vertex on which an incident π^+ meson is absorbed, or a final π^- is emitted, corresponding to an incident π^+ of negative energy propagating backward in time, there appears the isotopic factor $\tau \cdot \hat{\phi}_+ = \frac{1}{\sqrt{2}}(\tau_1 + i\tau_2) = \sqrt{2}\,\tau_+$ as in (10.30). For emission of a final π^+ (or absorption of an incident π^-) the factor is $\tau \cdot \hat{\phi}_+^* = \sqrt{2}\,\tau_-$.

5. Because of the convention made at the end of Sec. 10.2, there must be a relative minus sign between two terms corresponding to graphs which differ topologically only by the interchange of two nucleon lines. There is also a factor -1 for each closed nucleon loop and a factor $(-1)^{\bar{n}}$, where $\bar{n}$ is the number of antiparticles in the initial state [see (6.56)].

We have slavishly imitated electrodynamics in formulating this model.

General features such as the isotopic spin conservation for the π-nucleon interactions are generally valid to all orders in their coupling, in the approximation of neglecting small mass differences and electromagnetic effects.

However, perturbation expansions of the interaction into a power series in go are not generally useful, since $g_0^2/4\pi$ appears to be ≈ 14 and is not small as is its electromagnetic analogue $\alpha \approx 137$.

Instead of converging, a power series expansion leads to a diverging perturbation series.

10.4 Conserved Currents

In the isotopic notation (10.28) and (10.32) the differential law of current conservation (10.23) takes the form

$$\frac{\partial}{\partial x_\mu}\left[\bar{\Psi}\left(\frac{1+\tau_3}{2}\right)\gamma_\mu\Psi + \left(\hat{\phi}\times\frac{\partial\hat{\phi}}{\partial x^\mu}\right)_3\right] = 0 \tag{10.37}$$

where vector notation applies to the isotopic space.

The conserved total electric charge, found by integrating the time component, $\mu = 0$, over all space is

$$Q = \int d^3x\left[\Psi^\dagger\left(\frac{1+\tau_3}{2}\right)\Psi + (\hat{\phi}\times\dot{\hat{\phi}})_3\right] \tag{10.38}$$

The conservation law emerging directly from (10.33) is

$$\frac{\partial}{\partial x_\mu}J^N_\mu = \frac{\partial}{\partial x_\mu}\bar{\Psi}\gamma_\mu\Psi = \frac{\partial}{\partial x_\mu}(\bar{\psi}_p\gamma_\mu\psi_p + \bar{\psi}_n\gamma_\mu\psi_n) = 0 \tag{10.39}$$

This is identified as the conservation of nucleonic charge.

The total nucleonic charge is given by

$$N = \int\Psi^\dagger\Psi\, d^3x = \int(\psi_p^\dagger\psi_p + \psi_n^\dagger\psi_n)\, d^3x \tag{10.40}$$

and is a constant in this model, since the total number of nucleons (protons + neutrons) minus the total number of antinucleons is conserved.

This is seen to follow from the graphs in Figs. 10.4 and 10.5, since a continuous nucleon line propagates through each vertex with all possible time orderings.

Introducing (10.40) into (10.38) gives

$$Q = \frac{N}{2} + I_3 \tag{10.41}$$

where

$$I_3 = \int d^3x \, [\tfrac{1}{2}\Psi^\dagger \tau_3 \Psi + (\boldsymbol{\phi} \times \dot{\boldsymbol{\phi}})_3] \tag{10.42}$$

is identified as the third component of the "isotopic spin."

From the invariance of Eqs. (10.33) and (10.34) under rotations in isotopic space we suspect that not only I_3, but all three components of the isotopic spin should be conserved.

By construction we indeed find this to be true.

The conserved isotopic current is

$$\mathbf{J}_\mu = \tfrac{1}{2}\bar{\Psi}\gamma_\mu \boldsymbol{\tau}\Psi + \boldsymbol{\phi} \times \frac{\partial \boldsymbol{\phi}}{\partial x^\mu} \tag{10.43}$$

and the constant total isotopic spin is

$$\mathbf{I} = \int d^3x \, [\tfrac{1}{2}\Psi^\dagger \boldsymbol{\tau}\Psi + (\boldsymbol{\phi} \times \dot{\boldsymbol{\phi}})] \tag{10.44}$$

Conservation of electric and nucleonic charge and of isotopic spin are general features of a theory based on Eqs. (10.33) and (10.34).

Both Q and N are observed' to be rigorously conserved in nature; isotopic spin I, on the other hand, is conserved only to the approximation of neglecting electromagnetic and weak interactions.

These couplings destroy the charge independence of the strong interactions which led to the symmetric form of (10.33) and (10.34), and thence to (10.43) and (10.44), leaving only the third component I_3 as rigorously conserved according to (10.41).

The usefulness of the isotopic spin formalism has been evident in this discussion of conservation laws in permitting us to "see" better what the equations say.

10.5 Approximate Calculations; Nucleon-Nucleon Scattering

To illustrate both applications of the isotopic formalism and some of the general physical features of the a-meson and nucleon interactions, we consider briefly two examples: the one -meson exchange contribution to the nucleon-nucleon interaction, and -nucleon scattering.

The nucleon-nucleon scattering graphs which we considered, Figs. 10.1 to 10.3, may now all be combined into a compact form.

Following our rules and going as usual into momentum space gives for the scattering amplitude

$$
\begin{aligned}
S_{fi} = {} & \frac{(-ig_0)^2 M^2}{(2\pi)^6 \sqrt{E_1 E_2 E_1' E_2'}} (2\pi)^4 \delta^4(p_1 + p_2 - p_1' - p_2') \\
& \times \left\{ [\chi_1^\dagger, \bar{u}(p_1') i\gamma_5 \tau u(p_1) \chi_1] \frac{i}{(p_1' - p_1)^2 - \mu^2} \cdot [\chi_2^\dagger, \bar{u}(p_2') i\gamma_5 \tau u(p_2) \chi_2] \right. \\
& \left. - [\chi_2^\dagger, \bar{u}(p_2') i\gamma_5 \tau u(p_1) \chi_1] \frac{i}{(p_2' - p_1)^2 - \mu^2} \cdot [\chi_1^\dagger, \bar{u}(p_1') i\gamma_5 \tau u(p_2) \chi_2] \right\}
\end{aligned}
\tag{10.45}
$$

Comparison of (10.45) with (7.82) for electron-electron scattering shows the correspondence of factors

$$e\gamma_\mu \rightarrow ig_0\gamma_5\tau$$
$$g_{\mu\nu}D_F(k^2) \rightarrow \Delta_F(k^2)$$

according to our rules.

The substitution of $(2\pi)^6$ for V^2 come from the transition to continuum normalization of the external lines.

For p-p scattering the isotopic factors become

$$(\chi_p^\dagger\tau\chi_p)\cdot(\chi_p^\dagger\tau\chi_p) = (\chi_p^\dagger\tau_3\chi_p)(\chi_p^\dagger\tau_3\chi_p) = 1 \qquad (10.46)$$

and the two terms in (10.45) correspond to the direct and exchange scattering of the two identical fermions, as in (10.7) and (10.8); similarly for n-n scattering.

For p-n scattering the isotopic factors are

$$(\chi_p^\dagger\tau\chi_p)\cdot(\chi_n^\dagger\tau\chi_n) = -1$$

for the first term, and

$$(\chi_n^\dagger\tau\chi_p)\cdot(\chi_p^\dagger\tau\chi_n) = +2$$

for the exchange term, and the scattering amplitude differs from that for p-p and n-n scattering.

If, however, we ask for scattering into the symmetric isotopic state with $I = 1$ and $I_3 = 0$,

$$\frac{1}{\sqrt{2}}[\chi_p(1)\chi_n(2) + \chi_n(2)\chi_p(1)] \qquad (10.47)$$

formed by the proton and neutron the isotopic factors in (10.45) become, both for the first term and the second term,

$$\frac{1}{\sqrt{2}}(\chi_p^\dagger \boldsymbol{\tau}\chi_p \cdot \chi_n^\dagger \boldsymbol{\tau}\chi_n + \chi_n^\dagger \boldsymbol{\tau}\chi_p \cdot \chi_p^\dagger \boldsymbol{\tau}\chi_n) = \frac{1}{\sqrt{2}}(-1+2) = \frac{1}{\sqrt{2}} \tag{10.48}$$

The scattering amplitude equals $1/\sqrt{2}$ times that for p-p and n-n scattering and is antisymmetric under interchange of their space variables.

For scattering into an antisymmetric isotopic state with $I = 0, I_3 = 0$

$$\frac{1}{\sqrt{2}}[\chi_p(1)\chi_n(2) - \chi_n(2)\chi_p(1)] \tag{10.49}$$

the isotopic factors in (10.45) are

$$\frac{1}{\sqrt{2}}(\chi_p^\dagger \boldsymbol{\tau}\chi_p \cdot \chi_n^\dagger \boldsymbol{\tau}\chi_n - \chi_n^\dagger \boldsymbol{\tau}\chi_p \cdot \chi_p^\dagger \boldsymbol{\tau}\chi_n) = -\frac{3}{\sqrt{2}}$$

for the first term and +3/ $\sqrt{2}$ for the second term.

Hence the scattering amplitude is symmetric under the interchange of the p and n space (coordinate and spin) variables when antisymmetric under interchange of their isotopic variables.

These examples show how the generalized exclusion principle applies in N-N scattering.

They interact only in states antisymmetric under interchange of both their isotopic and space variables.

The charge-independent approximation equates the p-p and n-n scattering amplitudes to the amplitude for a p-n system in a symmetrie isotopic state (10.47) and hence interacting in an antisymmetric space state.

This equality follows from (10.45), (10.46), and (10.48) when we recall that the total p-p and n-n cross sections are integrated only over one-half the phase space as in (7.81) in order not to count the protons or neutrons twice; thus ½ compensates the $(1/\sqrt{2}\,)^2$ from (10.48).

In the nonrelativistic limit the spinor matrix elements in (10.45) simplify to

$$\bar{u}(p_1',s_1')\gamma_5 u(p_1,s_1) \cong u^\dagger(s_1')\,\frac{\boldsymbol{\sigma}\cdot(\mathbf{p}_1-\mathbf{p}_1')}{2M}\,u(\mathbf{s}_1) \tag{10.50}$$

where u($\mathbf{s}_1$) stands for a two-component Pauli spinor; (10.50) is readily verified by writing out the spinors in the nonrelativistic limit.

In this limit the meson propagators reduce to the Fourier transform of a Yukawa potential

$$\frac{1}{(p_1'-p_1)^2-\mu^2} \approx \frac{-1}{(\mathbf{p}_1'-\mathbf{p}_1)^2+\mu^2} = -\frac{1}{4\pi}\int d^3r\, e^{i(\mathbf{p}_1-\mathbf{p}_1')\cdot\mathbf{r}}\,\frac{e^{-\mu r}}{r}$$

and (10.45) is seen to be the scattering which to order g_0^2 results from a potential

$$V(\mathbf{r}_1,\mathbf{r}_2) = \frac{f^2}{\mu^2}(1-P_{ex})(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)(\boldsymbol{\sigma}_1\cdot\nabla_1)(\boldsymbol{\sigma}_2\cdot\nabla_1)\,\frac{e^{-\mu|\mathbf{r}_1-\mathbf{r}_2|}}{|\mathbf{r}_1-\mathbf{r}_2|} \tag{10.51}$$

where

$$f^2 \equiv \frac{g_0^2}{4\pi}\left(\frac{\mu}{2M}\right)^2$$

and P_{ex} is the operator exchanging neutron and proton wave functions and providing the second term of (10.45).

The two nucleons must be in a state which is antisymmetric under their interchange as required by the generalized exclusion principle; otherwise, $(1 - P_{ex})$ destroys the state.

For the two nucleons in an s state the potential is attractive for finite separations $r \equiv |\mathbf{r}_1 - \mathbf{r}_2| > 0$ as seen from the angular average of (10.51).

$$V_s(r) \equiv \int \frac{d\Omega_{12}}{4\pi} V(\mathbf{r}_1,\mathbf{r}_2)$$
$$= \frac{f^2}{\mu^2}(1 - P_{ex})\frac{1}{3}\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2\, \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \left[\mu^2 \frac{e^{-\mu r}}{r} - 4\pi\delta^3(\mathbf{r})\right]$$

and from the observation that this state, symmetric in space coordinates, must be antisymmetric under either spin or isotopic spin interchanges.

Therefore, in an s state

$$\langle \tfrac{1}{3}\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2\, \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \rangle_s = \langle \tfrac{1}{12}[(\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2)^2 - 6][(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)^2 - 6] \rangle_s$$
$$= \tfrac{4}{3}[T(T+1) - \tfrac{3}{2}][S(S+1) - \tfrac{3}{2}] = -1 \qquad (10.52)$$

and

$$V_s(r) = -2f^2 \left[\frac{e^{-\mu r}}{r} - \frac{4\pi}{\mu^2}\delta^3(\mathbf{r})\right] \qquad (10.53)$$

The δ-function repulsion is spread out into a short-range repulsive core interaction when nonstatic corrections are made to this simple calculation.

By itself, (10.53) is inadequate to explain deuteron and low-energy scattering parameters.

This is not surprising, since there is little reason to expect the static one-meson exchange approximation to the scattering to be reliable.

Indeed, higher-order contributions from other additional diagrams involving more mesons are important because of the large coupling constant $g_0^2/4\pi \sim 14$.

These, however, have been shown to contribute predominantly for smaller separations r and fall off as $e^{-n\mu r}$ for $\mu r > 1$, where n is the number of exchanged mesons.

It is therefore very encouraging that in the analysis of the high partial waves in an angular-momentum expansion of the nucleon-nucleon scattering amplitude, (10.45) or (10.51) reproduces the observed phase shifts accurately when $g_0^2/4\pi$ is set ≈ 14 ($f^2 = 0.08$) in agreement with its observed value in the meson-nucleon p-wave scattering analysis.

10.6 Meson-Nucleon Scattering

The Feynman graphs of Fig. 10.8 describe meson scattering from a nucleon to lowest order in $g_0^2/4\pi$.

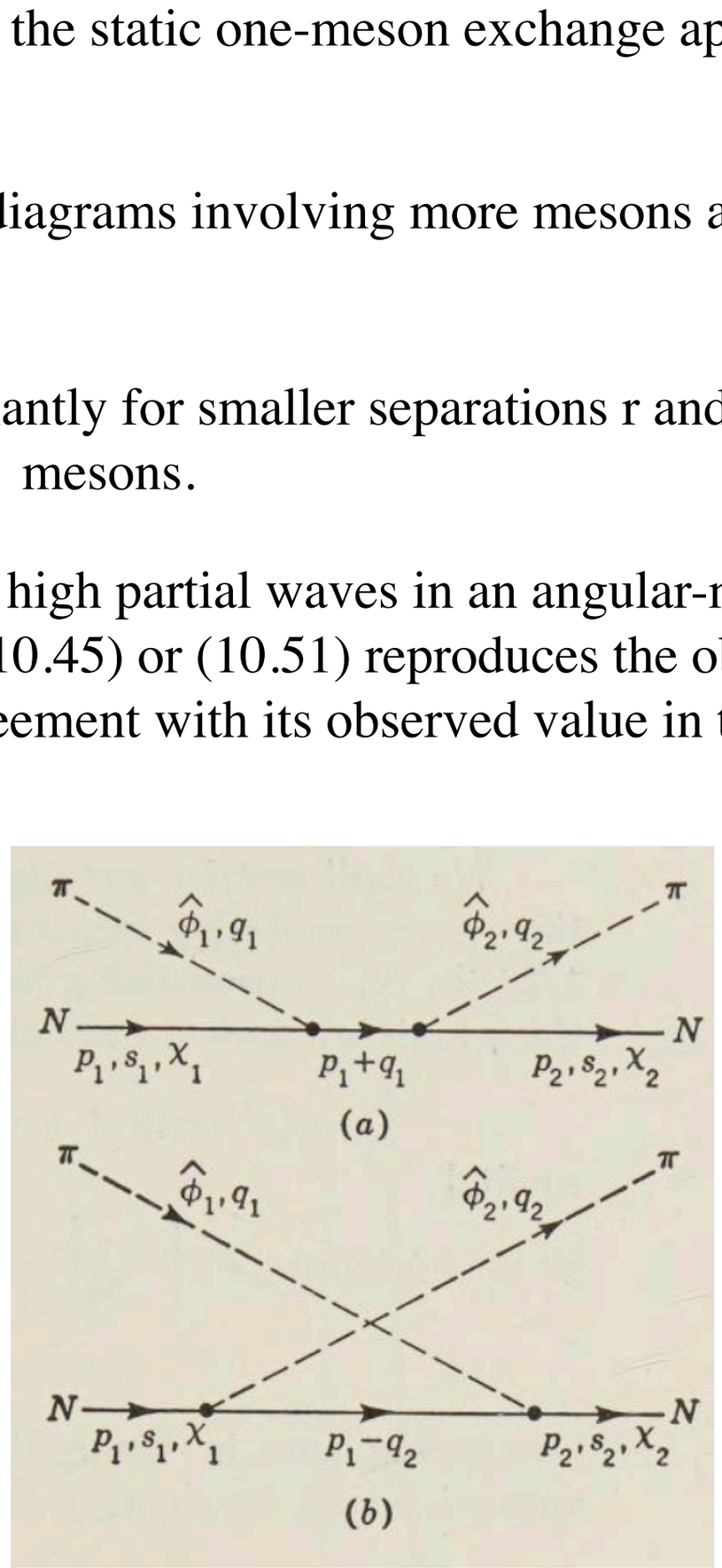

Fig. 10-8 Lowest order meson-nucleon scattering corresponding to Eq. (10.54)

The scattering amplitude according to our rules is given by

$$S_{fi} = \frac{1}{(2\pi)^6}\sqrt{\frac{M^2}{E_{p_1}E_{p_2}2\omega_{q_1}2\omega_{q_2}}}\,(2\pi)^4\delta^4(q_1+p_1-q_2-p_2)\mathfrak{M}$$

with

$$\mathfrak{M} = (-ig_0)^2\chi_2^\dagger\bar{u}(p_2,s_2)\left[\boldsymbol{\tau}\cdot\hat{\boldsymbol{\phi}}_2^* i\gamma_5\frac{i}{\not p_1+\not q_1-M}i\gamma_5\boldsymbol{\tau}\cdot\hat{\boldsymbol{\phi}}_1 + \boldsymbol{\tau}\cdot\hat{\boldsymbol{\phi}}_1 i\gamma_5\frac{i}{\not p_1-\not q_2-M}i\gamma_5\boldsymbol{\tau}\cdot\hat{\boldsymbol{\phi}}_2^*\right]u(p_1,s_1)\chi_1 \tag{10.54}$$

Notice the crossing symmetry of (10.54); it is invariant under the interchange

$$\hat{\boldsymbol{\phi}}_1 \leftrightarrow \hat{\boldsymbol{\phi}}_2^* \qquad q_1 \leftrightarrow -q_2 \tag{10.55}$$

in analogy with the crossing symmetry found in (7.67) in Compton scattering.

This symmetry under (10.55) is preserved to all higher orders.

From the Feynman diagrams it follows clearly when we observe that for each graph, as in Fig. 10.8a, in which the incident π is absorbed *before* the final π is emitted, there is one, such as Fig. 10.8b, differing only in that the initial π is absorbed *after* the final π is emitted.

We shall restrict our discussion to low energies, retaining only the s-wave scattering terms of order 1/M and the p waves of order $1/M^2$ in $\mathfrak{M}$.

Rationalizing the Feynman denominators and reducing the y matrices with the aid of

$$\bar{u}(p')i\gamma_5(\not{p} + \not{q} + M)i\gamma_5 u(p) = \bar{u}(p')\not{q}u(p)$$

we find

$$\mathfrak{M} = -ig_0^2\chi_2^\dagger\bar{u}(p_2,s_2)\left[\frac{(\boldsymbol{\tau}\cdot\hat{\boldsymbol{\xi}}_2^*)(\boldsymbol{\tau}\cdot\hat{\boldsymbol{\xi}}_1)q_1}{2p_1\cdot q_1+\mu^2} + \frac{(\boldsymbol{\tau}\cdot\hat{\boldsymbol{\xi}}_1)(\boldsymbol{\tau}\cdot\hat{\boldsymbol{\xi}}_2^*)(-q_2)}{-2p_1\cdot q_2+\mu^2}\right] \times u(p_1,s_1)\chi_1 \tag{10.56}$$

Computing now in the center-of-mass frame to the above stated approximation, we simplify (10.56) to

$$\mathfrak{M} \approx \frac{-ig_0^2}{M}[u^\dagger(\mathbf{s}_2)u(\mathbf{s}_1)(\chi_2^\dagger\chi_1)(\hat{\boldsymbol{\xi}}_2^*\cdot\hat{\boldsymbol{\xi}}_1)] - \frac{ig_0^2}{4M^2\omega}u^\dagger(\mathbf{s}_2)\chi_2^\dagger(\boldsymbol{\tau}\cdot\hat{\boldsymbol{\xi}}_2^*\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{\xi}}_1\,\boldsymbol{\sigma}\cdot\mathbf{q}_2\,\boldsymbol{\sigma}\cdot\mathbf{q}_1 - \boldsymbol{\tau}\cdot\hat{\boldsymbol{\xi}}_1\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{\xi}}_2^*\,\boldsymbol{\sigma}\cdot\mathbf{q}_1\,\boldsymbol{\sigma}\cdot\mathbf{q}_2)u(\mathbf{s}_1)\chi_1 \tag{10.57}$$

The first term is a spin and isotopic spin independent interaction which would be described nonrelativistically by the Born approximation potential

$$V(r) = +\frac{g_0^2}{2\mu M}\delta^3(\mathbf{r}) = 6f^2M\left[\frac{4\pi}{3\mu^3}\delta^3(\mathbf{r})\right] \tag{10.58}$$

In perturbation theory this gives an enormous s-wave scattering length of $(4Mf^2/\mu)1/\mu \sim 2/\mu \sim 2.8 \times 10^{-13}$ cm, where we have set $f^2 = 0.08$ as in the discussion of the nucleon-nucleon interaction.

However, since the interaction is repulsive and of short range-in this non-relativistic approximation (10.58) has zero range it actually has very little effect.

A strong, short-range, repulsive potential, as drawn in Fig. 10.9, produces an s-phase shift of the order $\delta \sim qa$, where a, the range of the potential, is the low-energy scattering length.

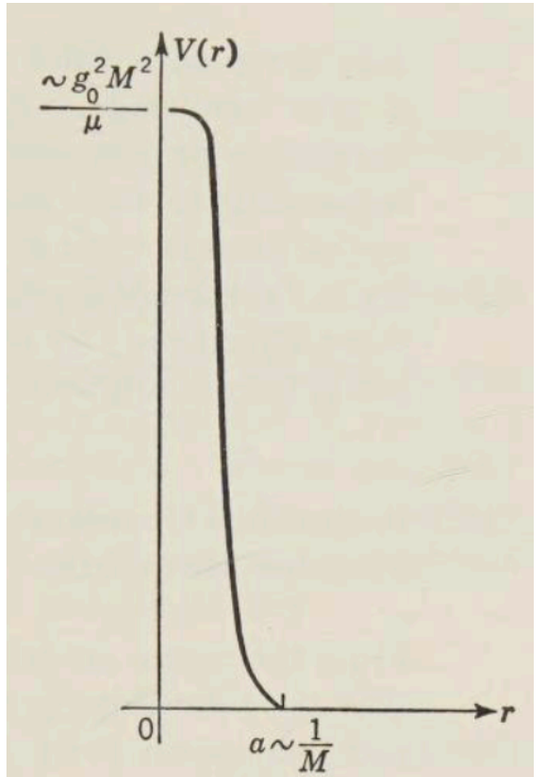

Fig. 10-9 Short-range repulsion for s-wave π-n scattering.

From recoil corrections one expects that a ~1/M and that the low-energv s-wave meson-nucleon scattering is small, in contrast with the large amplitude of ~1/μ obtained from the unjust application of Born approximation to (10.58).

This is indeed the case found experimentally.

The second term in (10.57) is the p-wave scattering.

Its form may be recognized as that obtained by applying second-order non-relativistic perturbation theory.

If we treat the nucleon as a non-relativistic spin one-half particle propagating in positive-energy states only in between the absorption of the initial meson and emission of the final one, we may use (10.50) to reduce the interaction vertex to the form $g_0(\sigma \cdot \nabla)$ $(\boldsymbol{\tau} \cdot \boldsymbol{\phi})1/2M$.

The factor $1/\omega$ in (10.57) comes from the energy denominator in this approximation, appearing with a plus sign for the amplitude, Fig. 10.8a, and with a minus sign for Fig. 10.8b.

In contrast, the first, or s-wave scattering, term in (10.57) comes from transitions of the nucleon in and out of the negative-energy sea in the intermediate state.

In this case $\bar{v}\gamma_5 u \sim -1$ for low nucleon momenta, and the energy denominator gives a factor -1/2M.

The uncertainty relation in the form $\Delta E\, \Delta t \sim 1$ suggests that the p-wave interaction via these graphs takes place over a longer time scale, $\sim 1/\omega$, than does the s wave, $\sim 1/M$.

Therefore, it is natural to expect a stronger energy dependence for the low-energy p-wave scattering amplitude than found for the s wave.

In fact, if there is a strong attractive p-wave "potential," a resonance may develop.

The crucial question to be asked, as first emphasized by Chew, is what the sign of the p-wave potential is.

Independently of the quantitative inadequacy of our perturbation approximation in writing (10.57), this sign will have the greatest influence on the scattering and may serve as a valid qualitative guide to an understanding of the p-wave amplitude.

To answer this question, it is especially convenient to project out the amplitudes for the various channels corresponding to each individual value of total angular momentum **J** and of total isotopic spin **I**, since transitions between channels differing in **J** are forbidden by angular-momentum conservation and between those differing in **I** by isotopic spin conservation, (10.44) (in the charge-independence approximation).

10.7 Projection Operators for Isotopic Spin and Angular Momentum

From the vector model for the addition of angular momenta we know that the total isotopic spin of a system of one nucleon (with I = ½) and one meson (with I = 1) is either I = ½ or I = 3/2.

The projection operators for these two states, $P_{1/2}$ and $P_{3/2}$, will be 3 x 3 matrices in the meson isotopic spin space spanned by the basis (10.36) and 2 x 2 matrices in the space of the nucleon isospinors (10.35).

$P_{1/2}$ and $P_{3/2}$ should have the basic properties of projection operators:

$$P_{1/2} + P_{3/2} = \mathbf{1} \tag{10.59a}$$

$$P_{1/2}^2 = P_{1/2} \tag{10.59b}$$

$$P_{3/2}^2 = P_{3/2}$$

where **1** is a unit matrix in the six-dimensional product space of the meson and nucleon isotopic spaces.

The search for these operators is made easy by observing that the uncrossed graph, Fig. 10.8a, must lead to a pure I = ½ amplitude, since there is only a single intermediate nucleon line with I = ½ and I is conserved at each vertex.

Thus the isotopic matrices in the uncrossed graph must be proportional to $P_{1/2}$:

$$\langle \hat{\boldsymbol{\phi}}_2 | P_{1/2} | \hat{\boldsymbol{\phi}}_1 \rangle = \alpha\, \boldsymbol{\tau} \cdot \hat{\boldsymbol{\phi}}_2^{*}\, \boldsymbol{\tau} \cdot \hat{\boldsymbol{\phi}}_1 \tag{10.60}$$

The coefficient α is determined to be ⅓ by squaring (10.60) and imposing (10.596):

$$\begin{aligned}
\langle \hat{\boldsymbol{\phi}}_2 | P_{1/2}^2 | \hat{\boldsymbol{\phi}}_1 \rangle &= \sum_{r=1}^{3} \langle \hat{\boldsymbol{\phi}}_2 | P_{1/2} | \hat{\boldsymbol{\phi}}_r \rangle \langle \hat{\boldsymbol{\phi}}_r | P_{1/2} | \hat{\boldsymbol{\phi}}_1 \rangle \\
&= \alpha^2 \sum_{r=1}^{3} (\boldsymbol{\tau} \cdot \hat{\boldsymbol{\phi}}_2^{*})(\boldsymbol{\tau} \cdot \hat{\boldsymbol{\phi}}_r)(\boldsymbol{\tau} \cdot \hat{\boldsymbol{\phi}}_r^{*})(\boldsymbol{\tau} \cdot \hat{\boldsymbol{\phi}}_1) \\
&= 3\alpha \langle \hat{\boldsymbol{\phi}}_2 | P_{1/2} | \hat{\boldsymbol{\phi}}_1 \rangle = \langle \hat{\boldsymbol{\phi}}_2 | P_{1/2} | \hat{\boldsymbol{\phi}}_1 \rangle
\end{aligned}$$

or $\alpha = ⅓$ and

$$\langle \hat{\boldsymbol{\phi}}_2 | P_{1/2} | \hat{\boldsymbol{\phi}}_1 \rangle = \tfrac{1}{3} \boldsymbol{\tau} \cdot \hat{\boldsymbol{\phi}}_2^{*}\, \boldsymbol{\tau} \cdot \hat{\boldsymbol{\phi}}_1 \tag{10.61}$$

Equation (10.59a) now gives $P_{3/2}$ directly:

$$\langle \hat{\boldsymbol{\phi}}_2 | P_{3/2} | \hat{\boldsymbol{\phi}}_1 \rangle = \hat{\boldsymbol{\phi}}_2^{*} \cdot \hat{\boldsymbol{\phi}}_1 - \tfrac{1}{3} (\boldsymbol{\tau} \cdot \hat{\boldsymbol{\phi}}_2^{*})(\boldsymbol{\tau} \cdot \hat{\boldsymbol{\phi}}_1) \tag{10.62}$$

Happily, the angular-momentum composition is identical to that for isotopic spin, since we are again coupling a spin S = ½ to L = 1 for the meson p-wave orbital angular momentum.

The orbital wave functions for the π meson now are the vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ in analogy with the $\hat{\boldsymbol{\phi}}_1$ and $\hat{\boldsymbol{\phi}}_2$ in isotopic space, and the angular-momentum projection operators analogous to (10.61) and (10.62) are

$$\langle \mathbf{q}_2|Q_{1/2}|\mathbf{q}_1\rangle = [\tfrac{1}{3}\boldsymbol{\sigma}\cdot\mathbf{q}_2\,\boldsymbol{\sigma}\cdot\mathbf{q}_1]\,\frac{3}{4\pi q^2}$$

$$\langle \mathbf{q}_2|Q_{3/2}|\mathbf{q}_1\rangle = [\mathbf{q}_2\cdot\mathbf{q}_1 - \tfrac{1}{3}(\boldsymbol{\sigma}\cdot\mathbf{q}_2)(\boldsymbol{\sigma}\cdot\mathbf{q}_1)]\,\frac{3}{4\pi q^2} \qquad (10.63)$$

with $q \equiv |\mathbf{q}_1| = |\mathbf{q}_2|$.

They are normalized according to

$$\int d\Omega_n \,\langle \mathbf{q}_2|Q_i|\mathbf{q}_n\rangle\,\langle \mathbf{q}_n|Q_j|\mathbf{q}_1\rangle = \delta_{ij}\langle \mathbf{q}_2|Q_i|\mathbf{q}_1\rangle \qquad (10.64)$$

where we have replaced the sum over the three orthogonal directions in the space of **q**, as used in the isotopic projection operators, by an integral over a sphere $\int d\Omega_n$.

This is an inessential difference in normalization conventions between the P_i and Q_i which is motivated by the fact that observed mesons are always oriented in isospace along one of the three directions (10.36) corresponding to charge $\pm 1, 0$, whereas their momentum vectors lie along a continuum of directions corresponding to different scattering angles.

The combined projection operators for isotopic spin and angular-momentum eigenstates are just the products of the P′s and O′s.

We define them by

$$\mathcal{P}_1 = \mathcal{P}_{11} = P_{1/2}Q_{1/2}$$
$$\mathcal{P}_2 = \mathcal{P}_{13} = P_{1/2}Q_{3/2}$$
$$\mathcal{P}_3 = \mathcal{P}_{31} = P_{3/2}Q_{1/2}$$
$$\mathcal{P}_4 = \mathcal{P}_{33} = P_{3/2}Q_{3/2} \qquad (10.65)$$

where the first index of $\mathcal{P}_{ij}$ is just twice the isotopic spin and the second index is twice the angular momentum.

The $\mathcal{P}_\alpha$, $\alpha = 1,\ldots, 4$ satisfy the properties (10.59) with the normalization condition

$$\sum_{r=1}^{3} \int d\Omega_n \, \mathcal{P}_\alpha |\hat{\boldsymbol{\xi}}_r \mathbf{q}_n\rangle \langle \hat{\boldsymbol{\xi}}_r \mathbf{q}_n | \mathcal{P}_{\alpha'} = \delta_{\alpha\alpha'} \mathcal{P}_\alpha \tag{10.66}$$

Introducing the projection operators into (10.57) we find for the π-nucleon scattering amplitude to second order in g_0^2 and in the non-relativistic approximation for our model

$$\begin{aligned} \mathfrak{M} \approx & -\frac{ig_0^2}{M} u^\dagger(\mathbf{s}_2) u(\mathbf{s}_1) \chi_2^\dagger (P_{1/2} + P_{3/2}) \chi_1 - \frac{ig_0^2}{4M^2}\left(\frac{4\pi q^2}{3}\right) \\ & \times u^\dagger(\mathbf{s}_2)\chi_2^\dagger \left\langle \hat{\mathbf{q}}_2 \hat{\boldsymbol{\xi}}_2 \left| \frac{9\mathcal{P}_{11}}{\omega} - \frac{4\mathcal{P}_{33} - 2\mathcal{P}_{13} - 2\mathcal{P}_{31} + \mathcal{P}_{11}}{\omega} \right| \hat{\mathbf{q}}_1 \hat{\boldsymbol{\xi}}_1 \right\rangle u(\mathbf{s}_1)\chi_1 \end{aligned} \tag{10.67}$$

Observe in (10.67) that the scattering amplitude is negative only in the (3, 3) channel, corresponding to an attractive potential for I = J = ½ only.

The experimental observations of a resonance in this state and of small phase shifts at low energies in the other three p-wave states are in qualitative agreement with what we would expect from a "potential" leading to (10.67).

10.8 Cross Sections for Pi-Nucleon Scattering

The scattering cross section is formed from (10.67) and (10.54) by squaring and multiplying by the customary phase space factors.

For fixed initial and final spins we have

$$d\sigma = \frac{|\mathfrak{M}|^2}{2\omega_{q_1}|\mathbf{v}_{q_1} - \mathbf{v}_{P_1}|} \int \left(\frac{M}{E_{P_1}}\right)\left(\frac{d^3q_2}{2\omega_{q_2}}\right)\left(\frac{M}{E_{P_2}}\, d^3P_2\right)\frac{1}{(2\pi)^2} \times \delta^4(q_1 + P_1 - q_2 - P_2)$$

which in the nonrelativistic limit $M \to \infty$ becomes, in the center-of-mass frame,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{c.m.}} \cong \frac{1}{16\pi^2}\,|\mathfrak{M}|^2 \tag{10.68}$$

To evaluate $d\sigma/d\Omega$ for a specific process, the appropriate π-meson isotopic wave functions $\hat{\boldsymbol{\phi}}_i$ and momenta $\mathbf{q}_i$ and the corresponding nucleon isotopic spinors χ_i are inserted into $\mathfrak{M}$.

For an unpolarized cross section the nucleon spins are summed over as usual.

As an example we consider π^+-p scattering, to which only the $I = 3/2$ channel contributes, since $I_3 = 3/2$.

Neglecting all but the $I = J = 3/2$ contribution, we have from (10.65)

$$\chi_p^\dagger \langle \mathbf{q}_2 \hat{\boldsymbol{\phi}}_+ | \mathcal{P}_{33} | \mathbf{q}_1 \hat{\boldsymbol{\phi}}_+ \rangle \chi_p = \frac{3}{4\pi q^2}\,(\mathbf{q}_2 \cdot \mathbf{q}_1 - \tfrac{1}{3}\boldsymbol{\sigma} \cdot \mathbf{q}_2\, \boldsymbol{\sigma} \cdot \mathbf{q}_1) \tag{10.69}$$

Summing over final and averaging over initial nucleon spins, we find with the aid of (10.69)

the

$$\frac{1}{2}\sum_{\text{spins}}|\mathfrak{M}_{33}|^2$$

$$= \frac{1}{2}\left(\frac{g_0^2}{4M^2}\right)^2\left(\frac{4}{\omega}\right)^2\sum_{\text{spins}}\left|u^\dagger(\mathbf{s}_2)\left(\mathbf{q}_2\cdot\mathbf{q}_1-\frac{1}{3}\boldsymbol{\sigma}\cdot\mathbf{q}_2\,\boldsymbol{\sigma}\cdot\mathbf{q}_1\right)u(\mathbf{s}_1)\right|^2$$

$$= \left(\frac{g_0^2}{M^2\omega}\right)^2\frac{1}{2}\,\mathrm{Tr}\left(\mathbf{q}_2\cdot\mathbf{q}_1-\frac{1}{3}\boldsymbol{\sigma}\cdot\mathbf{q}_2\,\boldsymbol{\sigma}\cdot\mathbf{q}_1\right)\left(\mathbf{q}_2\cdot\mathbf{q}_1-\frac{1}{3}\boldsymbol{\sigma}\cdot\mathbf{q}_1\,\boldsymbol{\sigma}\cdot\mathbf{q}_2\right)$$

$$= \left(\frac{g_0^2}{3M^2\omega}\right)^2[\mathbf{q}_2^2\mathbf{q}_1^2+3(\mathbf{q}_2\cdot\mathbf{q}_1)^2] \qquad (10.70)$$

Inserting into (10.68) we find for the I = J = 3/2 contribution to the differential π^+-p scattering cross section in the center-of-mass frame

$$\frac{d\sigma_{33}(\pi^+\text{-}p)}{d\Omega} = \left(\frac{4f^2}{3\omega\mu^2}\right)^2 q^4(1+3\cos^2\theta) \qquad (10.71)$$

where, as before, we have introduced

$$f^2 = \frac{g_0^2}{4\pi}\left(\frac{\mu}{2M}\right)^2$$

Equation (10.71) can hardly be considered reliable, since it is based upon Born approximation which, as already seen, fails badly for the s-wave scattering.

It has the important virtue, however, of predicting an angular distribution $1 + 3\cos^2\theta$ which is in approximate agreement with experiment for meson energies in the 150- to 200-MeV region.

Also in this energy region the ratios of cross sections are observed to be close to the computed values

$$\sigma(\pi^+\text{-}p \to \pi^+\text{-}p) : \sigma(\pi^-\text{-}p \to \pi^0\text{-}n) : \sigma(\pi^-\text{-}p \to \pi^-\text{-}p) = 9:2:1 \tag{10.72}$$

for scattering in the I = J = 3/2 channel only.

With these suggestions that the scattering is dominantly through the I = J = 3/2 channel in this energy region we try to extend the validity of (10.71) with the aid of two general observations.

We notice first that the energy dependence of (10.71) is unrealistic except near threshold since it predicts that $\sigma \to \infty$ as $\omega \to \infty$.

There is however, an upper bound on the magnitude of the total cross section coming from unitarity.

Purely within the framework of propagator theory it is difficult to discuss unitarity of the S matrix; here we simply use some general results of nonrelativistic scattering theory, namely:

1. For a given channel the scattering amplitude has the form

$$t \propto \frac{1}{q} e^{i\delta} \sin\delta = \frac{1}{q(\cot\delta - i)} \tag{10.73}$$

where q is the momentum of each particle in the center-of-mass system and δ the phase shift in this channel. δ is real if there are no competing inelastic channels with the same quantum numbers.

2. The contribution of a channel with orbital angular momentum l and total angular momentum $J = l \pm \frac{1}{2}$ to the total cross section is limited by

$$\sigma_{\text{tot}}^{J,l} \leq \frac{4\pi(2J+1)}{2} \frac{1}{q^2} \tag{10.74}$$

3. The effective range expansion

$$q^{2l+1} \cot \delta = a + b\omega + c\omega^2 + \cdots \tag{10.75}$$

provides a good approximation at low energies.

Secondly, we observe, as already noted in Sec. 10.6, that the small energy denominator ~ ω and relatively long time scale ~$1/\omega$ of the p-wave interaction make it natural to expect a strong energy dependence in the p-wave scattering phase shifts.

Therefore, in (10.75) for the (3,3) channel we may anticipate that higher corrections to the Born approximation will lead to a coefficient c that is nonnegligible and negative, enhancing the Born approximation attraction in this channel.

Using (10.73) and (10.74), we rewrite (10.71) in the form

$$\left(\frac{d\sigma_{33}}{d\Omega}\right)_{\pi^+-p} = \frac{1}{q^2} |e^{i\delta_{33}} \sin \delta_{33}|^2 (1 + 3\cos^2 \theta) \tag{10.76}$$

with

$$(e^{i\delta_{33}} \sin \delta_{33})_{\text{Born}} \cong + \frac{4f^2 q^3}{3\omega\mu^2} \tag{10.77}$$

To this order we may also write

$$(q^3 \cot \delta_{33})_{\text{Born}} = + \frac{3\omega\mu^2}{4f^2} \tag{10.78}$$

To determine the next coefficient c in the expansion (10.75) and to develop a formula including the effective range correction to the scattering length term, we must go beyond our low-energy Born approximation calculation.

We have already noted that (10.67) leads us to expect a negative coefficient c for the 33 channel, since the signs correspond to an attractive potential in it as opposed to the other channels.

Writing as a low-energy approximation

$$q^3 \cot \delta_{33} = + \frac{3\omega\mu^2}{4f^2}\left(1 - \frac{\omega}{\omega_r}\right) \tag{10.79}$$

we obtain a good fit to experiment on π^+-p scattering provided $f^2 = 0.08$, or equivalently, $g_0^2/4\pi \approx 14$ and $\omega_r \sim 2.2\mu$.

Equation (10.79) was first derived by Chew and Low from a meson theory with fixed nucleon sources ($\omega/M \to 0$) and without recourse to an expansion in powers of the coupling constant such as we made in (10.54).

The singularity in the scattering amplitude (10.77) at the point $\omega = 0$ in the "unphysical region" below the physical threshold at $\omega = \mu$ may be traced to the vanishing energy denominator in the Born amplitude (10.54).

Thus the rationalized Feynman propagators in (10.56) have simple poles for meson energies in the laboratory system $\omega_1 = -\mu^2/2M$ and $\omega_2 = +\mu^2/2M$; ω_1 and $\omega_2 \to 0$ in the nonrelativistic limit $\mu/M \to 0$.

All higher-order graphs with only one nucleon line propagating between the meson vertices as in Fig. 10.10 will also contribute terms with a pole at $\omega = 0$, and the residue at this pole in (10.78) and (10.79) includes the sum of their effects.

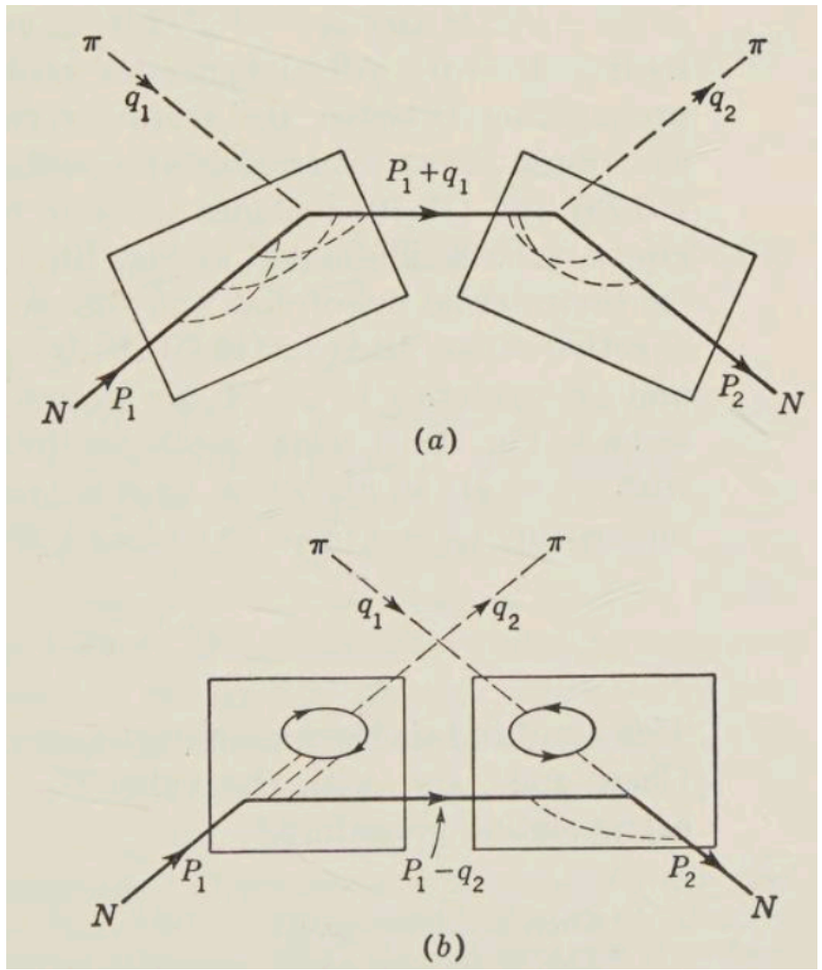

Fig. 10-10 Radiative corrections with a single intermediate nucleon line.

All other diagrams, such as illustrated in Fig. 10.11, are finite at an energy $\omega = 0$ for the external meson line and therefore contribute to the second, or effective range, term in (10.79).

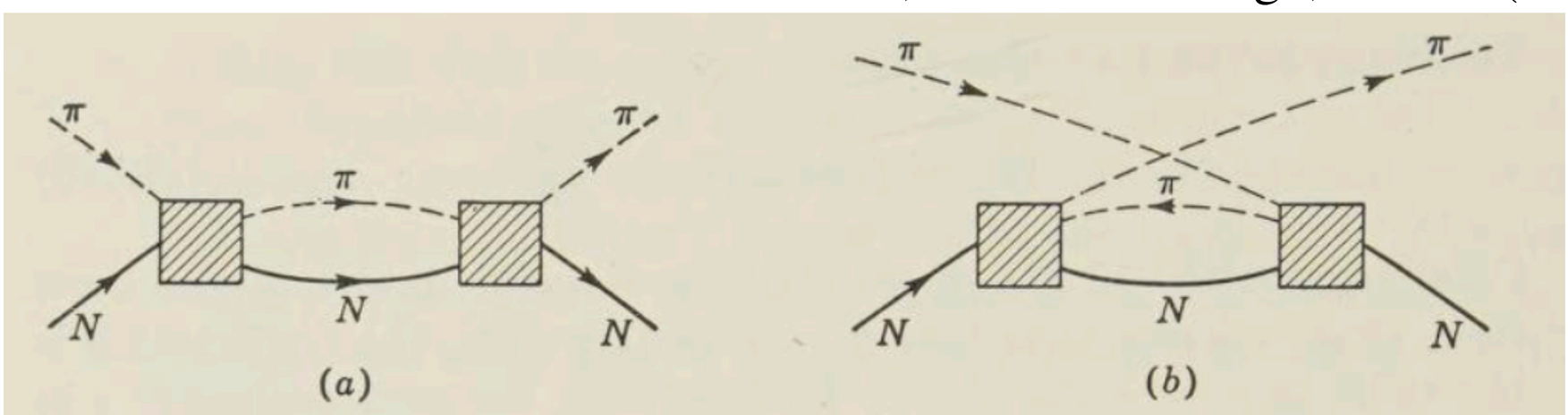

Fig. 10-11 A higher-order contribution.

By plotting $(q^2 \cot \delta_{33})/\omega$ versus ω and extrapolating to $\omega = 0$, we isolate the contribution of the amplitudes of Fig. 10.10, which measures the strength for a physical nucleon with $P^2 = M^2$ to absorb or emit zero-energy mesons with imaginary momentum $|q| = i\mu$ and to remain a physical nucleon with

$$(P + q)^2 = M^2$$

This amplitude is the meson-nucleon coupling constant as identified by Chew and Low, with the value $f^2 = 0.08$ as determined from the extrapolation procedure.

10.9 Electromagnetic Structure of Mesons and Nucleons

The electromagnetic interactions of mesons and nucleons are influenced by their strong interactions, as we have already remarked.

Indeed it has been known for a long time that the magnetic moment of a proton has the anomalously large value of 2.79 μ_B, where

$$\mu_B = \frac{e\hbar}{2M_p c}$$

is the nuclear Bohr magneton, instead of the value 1.0 μ_B predicted by the Dirac equation for a particle of charge e, such as the electron. (We neglect here the electromagnetic radiative corrections computed in Chaps. 7 and 8).

Similarly, the neutron has a magnetic moment of - 1.91 μ_B, whereas the free Dirac equation predicts a zero magnetic moment for a neutral particle.

It is possible to account for these anomalous moments by abandoning the principle of minimal electromagnetic coupling.

Instead of introducing electromagnetic interactions into the Dirac equation with

$$i\partial\!\!\!/ \rightarrow \gamma^\mu \left(i \frac{\partial}{\partial x^\mu} - eA_\mu \right) \tag{10.80}$$

we might also add a magnetic dipole term

$$i\partial\!\!\!/ \rightarrow \gamma^\mu \left(i \frac{\partial}{\partial x^\mu} - eA_\mu \right) - \frac{\kappa \mu_B}{2} \sigma_{\mu\nu} F^{\mu\nu} \tag{10.81}$$

with

$$\kappa_p = 1.79 \qquad \kappa_n = -1.91$$

A more fruitful approach eschews the temptation to introduce new parameters as in (10.81) and remains faithful to the minimal form (10.80).

This approach attributes all deviations from (10.80), including the anomalous magnetic moments κ, to the influence of the strong interactions.

In the same spirit we saw in Chap. 8 that the Lamb shift in the atomic energy levels and the anomalous electron magnetic moment could be explained, to the limit of present experimental precision, by including the effect of the interaction of the electron with photons.

Without entering into detailed calculations and by relying on invariance arguments alone, we can establish the general form of the modification of (10.80) produced by the strong interactions.

In the present instance the requirements of Lorentz covariance and of conservation of electromagnetic current severely limit the electromagnetic vertex of a particle.

Consider first the at meson and the graph in Fig. 10.12b, which is a "radiative correction" to the vertex in Fig. 10.12a.

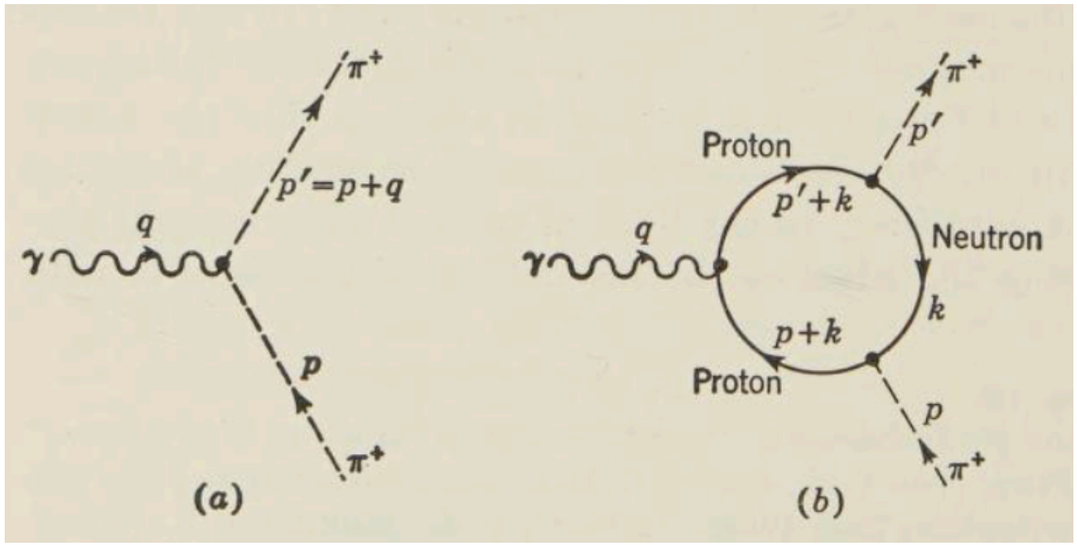

Fig. 10-12 Charged-pion electromagnetic vertex and radiative correction.

According to our rules the modification in the amplitude of Fig. 10.12a for the electromagnetic current of the π^+ due to the graph of Fig. 10.126 is

$$e(p_\mu + p'_\mu) \to e(p_\mu + p'_\mu) + (-ig_0\sqrt{2})^2(-)\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\,\frac{i}{k-M}\,i\gamma_5$$

$$\times \frac{i}{\not{p}' + k - M}\,e\gamma_\mu\,\frac{i}{\not{p} + k - M}\,i\gamma_5 = e(p_\mu + p'_\mu) + I_\mu(p',p) \qquad (10.82)$$

where the form factors f_1 and f_2 are scalar functions of the three independent scalars p^2, p'^2, and $(p - p')^2$ in the integral.

If we restrict our attention to the scattering of a real meson from an electromagnetic potential, the form factors become functions of the invariant momentum transfer $q^2 = (p' - p)^2$ only, since then $p^2 = p'^2 = \mu^2$.

A further restriction on (10.83) follows from the requirement of current conservation, for example; for the qth Fourier component of the current of a real physical meson

$$q^\mu I_\mu(p',p) = (p' - p)^\mu I_\mu(p',p) = 0 \tag{10.84}$$

In (10.83), with $p^2 = p'^2 = \mu^2$ this gives $f_1(q^2) = f_2(q^2)$, a result that may be verified directly from (10.82) in a similar manner-and with the same kind of ambiguity— to that discussed in connection with the vacuum polarization in Chap. 8.

We have now the general form of the electromagnetic current of a real π^+ meson scattering with momentum transfer q^2.

The point interaction current $e(p_\mu + p'_\mu)$ is modified to

$$e(p_\mu + p'_\mu) \rightarrow e(p_\mu + p'_\mu)F_\pi(q^2) \tag{10.85}$$

where $F_\pi(q^2)$ is the charged -meson form factor and depends only upon the invariant momentum transfer.

The form factor is normalized to 1 for zero momentum transfer, $F_\pi(0) = 1$, after the charge renormalization is carried out as in Chap. 8 and e is set equal to the observed physical π^+ charge.

Study of $F_\pi(q^2)$ requires more powerful techniques of calculation than perturbation theory.

Already (10.85) constitutes a powerful result in limiting the form of the differential cross section for scattering of a at meson by an electromagnetic field.

For example, to lowest order in α the ratios of cross sections at different energies and scattering angles, but with fixed q^2, are independent of $F_\pi(q^2)$ and equal to the ratios calculated as in Chap. 9 with neglect of the strong interactions.

A similar result is obtained in discussing the form of the electromagnetic current of the nucleon (and the strange hyperons as well).

Here the spin degree of freedom of the nucleon allows the possibility of two scalar form factors corresponding to the additional possibility of a spin one-half particle having a magnetic moment.

For the proton, for instance, one encounters graphs to order g_0^2 such as illustrated in Fig. 10.13.

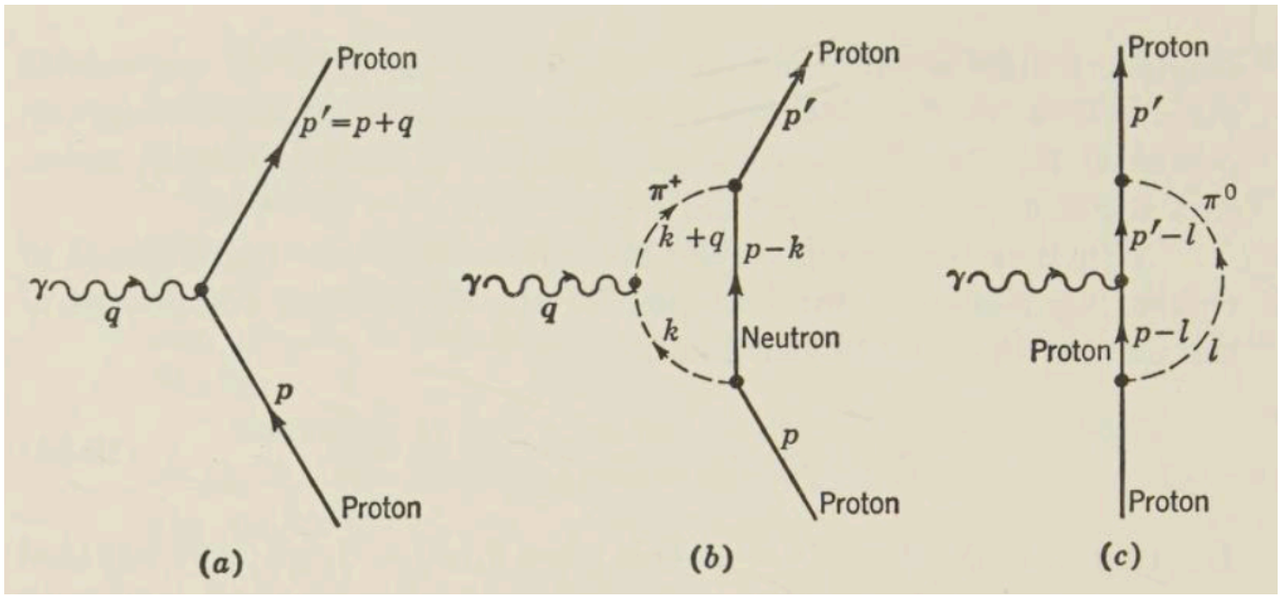

Fig. 10-13 Proton electromagnetic vertex and radiative corrections.

The corresponding modification of the proton current according to our rules is

$$\bar{u}(p')e\gamma_\mu u(p) \to \bar{u}(p')e\gamma_\mu u(p)$$

$$+ (-ig_0\sqrt{2})^2 \int \frac{d^4k}{(2\pi)^4}\, \bar{u}(p')i\gamma_5 \frac{i}{\not p - \not k - M} i\gamma_5 u(p)$$

$$\times \frac{i}{k^2 - \mu^2} e(2k_\mu + q_\mu) \frac{i}{(k+q)^2 - \mu^2}$$

$$+ (-ig_0)^2 \int \frac{d^4l}{(2\pi)^4}\, \bar{u}(p')i\gamma_5 \frac{i}{\not p' - \not l - M} e\gamma_\mu$$

$$\times \frac{i}{\not p - \not l - M} i\gamma_5 u(p) \frac{i}{l^2 - \mu^2} \equiv \bar{u}(p')e\Gamma_\mu(p',p)u(p) \qquad (10.86)$$

We again find, as was the case for the π^+ current, that the proton current transforms as a four-vector.

After the momentum integrations are done, the only vectors remaining to be sandwiched between the proton spinors are p_μ, p'_μ, and γ_μ.

Any other remaining y matrices must be of the form $\not p,\ \not p',$ or γ_5.

However, the γ_5's can be eliminated, since there are an even number of π-N vertices and hence of γ_5 factors which can be paired together with $\gamma_5^2 = 1$.

Furthermore, all factors of p' and p appearing in a product of γ matrices can be anticommuted to the left or right until they are adjacent to $\bar{u}(p')$ or $u(p)$, respectively, and set equal to M.

(Recall the concrete example of this procedure in the calculation of the electron electromagnetic vertex in Chap. 8.)

We conclude from these arguments that the general structure of (10.86) must be

$$\bar{u}(p')e\Gamma_\mu(p',p)u(p) = e\bar{u}(p')[p_\mu\Gamma_1(q^2) + p'_\mu\Gamma_2(q^2) + \gamma_\mu\Gamma_3(q^2)]u(p) \qquad (10.87)$$

with $\Gamma_i(q^2)$, i = 1, 2, 3, scalar functions of q^2.

Identical arguments lead to the same general form for the neutron current.

Current conservation gives a relation between the three form factors $\Gamma_i(q^2)$.

In analogy with (10.84),

$$q^\mu\bar{u}(p')\Gamma_\mu(p',p)u(p) = 0$$

As a consequence, $\Gamma_1(q^2) = \Gamma_2(q^2)$, and the nucleon current takes as its most general form

$$\bar{u}(p')e\Gamma_\mu(p',p)u(p) = e\bar{u}(p')[(p_\mu + p'_\mu)\Gamma_1(q^2) + \gamma_\mu\Gamma_3(q^2)]u(p)$$

In discussing the electromagnetic structure of the nucleons it is conventional to eliminate the vector $(p_\mu + p'_\mu)$ in favor of

$$\sigma_{\mu\nu}(p'^\nu - p^\nu) = \sigma_{\mu\nu}q^\nu$$

by means of the Gordon reduction of the current.

Since the spinor wave functions $\bar{u}(p')$ and $u(p)$ obey free Dirac equations, we may apply (3.26) directly to find the equivalent structure

$$\bar{u}(p')e\Gamma_\mu(p',p)u(p) = e\bar{u}(p')[\gamma_\mu F_1(q^2) + \frac{i\sigma_{\mu\nu}q^\nu}{2M}\kappa F_2(q^2)]u(p) \tag{10.88}$$

a form familiar from our discussion of the radiative corrections to the electron vertex (8.61).

With κ set equal to the anomalous part of the magnetic moment in units of the Bohr magneton ($\kappa_p = 1.79$ and $\kappa_n = -1.91$) and with e the physical proton charge, $F_2(0) = 1$ and $F_1(0) = 1$ for the proton and $F_2(0) = 0$ for the neutron.

Using the isotopic spin formalism we may combine the proton and neutron currents into the sum of an isotopic scalar and an isotopic vector part:

$$\begin{aligned} &e\chi^\dagger\bar{u}(p')\left\{\gamma_\mu[F_1^{(s)}(q^2) + \tau_3 F_1^{(v)}(q^2)] + \frac{i\sigma_{\mu\nu}q^\nu}{2M}[F_2^{(s)}(q^2) + \tau_3 F_2^{(v)}(q^2)]\right\}u(p)\chi \\ &\equiv \chi^\dagger[j_\mu^{(s)}(p',p) + \tau_3 j_\mu^{(v)}(p',p)]\chi \end{aligned} \tag{10.89}$$

where

$$\begin{aligned} F_1^{(s)} &= \tfrac{1}{2}(F_1^{(p)} + F_1^{(n)}) & F_1^{(s)}(0) &= \tfrac{1}{2} \\ F_1^{(v)} &= \tfrac{1}{2}(F_1^{(p)} - F_1^{(n)}) & F_1^{(v)}(0) &= \tfrac{1}{2} \\ F_2^{(s)} &= \tfrac{1}{2}(\kappa_p F_2^{(p)} + \kappa_n F_2^{(n)}) & F_2^{(s)}(0) &= -0.06 \\ F_2^{(v)} &= \tfrac{1}{2}(\kappa_p F_2^{(p)} - \kappa_n F_2^{(n)}) & F_2^{(v)}(0) &= +1.85 \end{aligned}$$

For a proton $\chi^\dagger\tau_3\chi = 1$ and (10.89) reduces to the proton current; for the neutron $\chi^\dagger\tau_3\chi = -1$ and (10.89) reduces to the neutron current.

The general form (10.88) or (10.89) again provides a severe limitation to the differential cross section for scattering of a proton or a neutron in an electromagnetic potential.

To the accuracy of the Born approximation in $\alpha = 1/137$, but to arbitrary order in the strong couplings, the cross section for the scattering of an electron by a physical proton or neutron is modified from that calculated in (7.46) according to

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 \left[\left(F_1^2 - \frac{\kappa^2 q^2}{4M^2} F_2^2 \right) \cos^2 \frac{\theta}{2} - \frac{q^2}{2M^2} (F_1 + \kappa F_2)^2 \sin^2 \frac{\theta}{2} \right]}{4E^2 [1 + (2E/M) \sin^2 (\theta/2)] \sin^4 (\theta/2)} \tag{10.90}$$

where θ is the laboratory scattering angle.

Individual determinations of F_1 and F_2 can be obtained by comparing measurements taken at different scattering angles and energies but the same q^2.

More than two observations at the same q^2 must yield a series of points all lying on the same straight line when

$$\left(\sin^4 \frac{\theta}{2} \right) E^2 \left(1 + \frac{2E}{M} \sin^2 \frac{\theta}{2} \right) \frac{d\sigma}{d\Omega}$$

is plotted against $\cos^2(\theta/2)$ for fixed q^2.

Any deviations from this cannot be attributed to our ignorance of the strong couplings or to our inability to calculate the form factors but must be blamed on a failure in the electrodynamic part of the calculation —perhaps to a failure in the approximation of keeping only the first term in the power series in a or to more profound reasons.

10.10 Weak Interactions

The weak interactions, of which β decay is the most familiar example, may be classified into two general groups: leptonic and nonleptonic.

The leptonic interactions involve μ mesons (μ^-), electrons (e^-), and two kinds of neutrinos (υ,υ') and inelude, along with the corresponding antiparticle interactions (μ^+, e^+ ,$\bar{\nu}, \bar{\nu}'$),

$$\beta \text{ decay} \qquad n \to p + e^- + \bar{\nu} \tag{10.91a}$$

$$\mu \text{ decay} \qquad \mu^- \to e^- + \nu' + \bar{\nu} \tag{10.91b}$$

$$\pi \text{ decay} \qquad \pi^- \to \begin{cases} \mu^- + \bar{\nu}' \\ e^- + \bar{\nu} \end{cases} \tag{10.91c}$$

$$\mu \text{ capture} \qquad \mu^- + p \to n + \nu' \tag{10.91d}$$

as well as a host of strangeness changing leptonic interactions which transmute the strange particles into nucleons, leptons, and possibly π mesons; for instance

$$K^- \to \begin{cases} \mu^- + \bar{\nu}' \\ \pi^0 + e^- + \bar{\nu} \end{cases}$$

$$\Lambda \to p + e^- + \bar{\nu}$$

Examples of the nonleptonic decays, which always involve the strange particles, are

$$\Lambda \to p + \pi^- \qquad K^+ \to \pi^+ + \pi^+ + \pi^-$$

We shall consider the leptonic interactions which do not involve strange particles; the weak interactions involving the strange particles are not well understood and will not be discussed here.

The problem we face in discussing (10.91a, b, and c) is that of deducing from available experimental observations the structure of the interaction vertices in the graphs for these processes.

Two of our main standbys from the discussion of the strong interactions are lost here.

Both parity and isotopic spin conservation are approximate symmetries and are violated by the weak interactions in nature.

10.11 Beta Decay

The fundamental process (10.91a) is responsible for the B decay in nuclei, and so we first study free neutron decay.

The general structure of the S matrix element describing this decay, illustrated in Fig. 10.14,

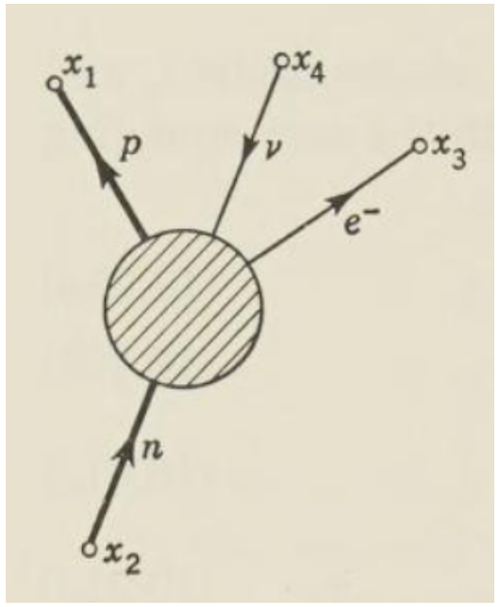

Fig. 10-14 β decay.

must be linear in the wave functions describing the incident neutron and outgoing particles, that is,

$$S_{fi}^{(e^-)} = -i \sum_{\alpha\beta\gamma\delta=1}^{4} \int d^4x_1 \cdots d^4x_4 \, \psi_\alpha^{\dagger(p)}(x_1)\psi_\beta^{(n)}(x_2)\psi_\gamma^{\dagger(e)}(x_3)\psi_\delta^{(\nu)}(x_4) \times F_{\alpha\beta\gamma\delta}(x_1 \cdots x_4) \tag{10.92}$$

As usual, the hermitian conjugate wave functions $\psi^\dagger$ represent emerging particles (p, e^-) or incident antiparticles, corresponding to the negative-energy solutions propagating backward in time, and the ψ represent incident particles (n) or emerging antiparticles $(\bar{\nu})$.

Thus all interactions such as

$$\nu + n \rightarrow p + e^-$$

are also included in (10.92) in addition to β decay.

To allow for the possibility of inverse β decay, or positron emission

$$p \rightarrow n + e^+ + \nu$$

as is observed in energetically allowed nuclear transitions, there must be an analogous matrix element to (10.92)

$$S_{fi}^{(e^+)} = -i \sum_{\alpha\beta\gamma\delta=1}^{4} \int d^4x_1 \cdots d^4x_4 \, \psi_\alpha^{\dagger(n)}(x_1)\psi_\beta^{(p)}(x_2)\psi_\gamma^{\dagger(\nu)}(x_3)\psi_\delta^{(e)}(x_4) \times \bar{F}_{\alpha\beta\gamma\delta}(x_1, \ldots, x_4) \tag{10.93}$$

The functions F and $\bar{F}$ in (10.92) and (10.93) must be determined from experiment.

On general theoretical grounds we shall make only one assumption here, namely, that

$$\bar{F}_{\alpha\beta\gamma\delta}(x_1,x_2,x_3,x_4) = F^*_{\beta\alpha\delta\gamma}(x_2,x_1,x_4,x_3) \tag{10.94}$$

This assures the principle of detailed balance for the weak interactions and tells us that, aside from phase-space factors, the reactions

$$n \rightleftarrows p + e^- + \bar{\nu} \qquad \nu + n \rightleftarrows p + e^-$$

proceed with the same rates from right to left and left to right.

That detailed balance (10.94) should apply follows from the weakness of the β-decay interaction and the unitarity of the S matrix, discussed in Chap. 8.

For no interaction S_{fi} reduces to a unit matrix δ_{fi} .

Writing

$$S_{fi} = \delta_{fi} - iT_{fi}$$

the unitary requirement (8.31) is for T_{fi},

$$i(T_{fi} - T_{fi}^{\dagger}) = \sum_n T_{fn}^{\dagger} T_{ni}$$

For β decay, the right-hand side is much smaller than the left, being second order in the weak coupling constant.

This leads to (10.94), since for $f \neq$ i.

$$S_{fi} = -iT_{fi}$$

The neutrino is a massless neutral Dirac particle, and we maintain the experimentally rigorously satisfied law of conservation of leptons as well as of nucleons by identifying as an antineutrino the neutral partner of the electron in β decay.

For process (10.91a) this is only a matter of definitions; lepton conservation has definite implications, however, for π and μ decay.

More recently the observation of the $\pi^- \rightarrow e^- + \bar{\nu}$ decay process has confirmed this.

Finally, the masslessness of the neutrino requires a change in our normalization convention for the neutrino amplitude relative to that for other fermions.

We write for a plane-wave solution with quantum numbers (k, s)

$$\psi^{(\nu)}(x) = \frac{1}{\sqrt{2E_k(2\pi)^3}} u^{(\nu)}(k,s)e^{-ik\cdot x} \tag{10.95}$$

with

$$u^{\dagger(\nu)}(k,s)u^{(\nu)}(k,s) = 2E_k$$

and therefore

$$\bar{u}^{(\nu)}(k,s)u^{(\nu)}(k,s) = 0$$

The neutrino projection operators are

$$\Lambda^{(+)}_{\alpha\beta}(k,s) = \sum_{s=1}^{2} u_\alpha(k,s)\bar{u}_\beta(k,s) = \not{k}$$

$$\Lambda^{(-)}_{\alpha\beta}(k,s) = \sum_{s=1}^{2} v_\alpha(k,s)\bar{v}_\beta(k,s) = -\not{k}$$

A convenient way to handle the normalization is to treat neutrinos in the identical fashion as electrons of finite mass, normalizing as in Chap. 3, and to subsequently take the limit $m_\upsilon \rightarrow 0$.

The rules (10.95) lead to identical results.

After 30 years of research, the function F in (10.94) has finally been completely determined at low energies, that is, for relatively large coordinate separations $x_i - x_j$.

Indeed, if the range of distances over which F is nonvanishing is small compared with $\hbar/ E_\beta c \sim 10^{-12}$ cm, the characteristic distance associated with β-decay energies $E_\beta \sim 1$ to 10 MeV, one may take F to a first approximation to be a local interaction, that is,

$$F_{\alpha\beta\gamma\delta}(x_1,x_2,x_3,x_4) \approx \mathfrak{F}_{\alpha\beta\gamma\delta}\delta^{(4)}(x_1 - x_2)\delta^{(4)}(x_1 - x_3)\delta^{(4)}(x_1 - x_4) \tag{10..96}$$

where $\mathfrak{F}_{\alpha\beta\gamma\delta}$ is a constant matrix tying together the spinor factors.

This turns out to be an extremely good approximation for all observations thus far.

Fourier-transforming (10.96) to momentum space gives

$$\begin{aligned} F_{\alpha\beta\gamma\delta}(k_1,k_2,k_3,k_4) &\equiv \int d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4 \\ &\times e^{i(k_1\cdot x_1+k_2\cdot x_2+k_3\cdot x_3+k_4\cdot x_4)} F_{\alpha\beta\gamma\delta}(x_1,x_2,x_3,x_4) \\ &= (2\pi)^4\delta^4(k_1 + k_2 + k_3 + k_4)\mathfrak{F}_{\alpha\beta\gamma\delta} \end{aligned} \tag{10..97}$$

The interaction is just a constant matrix times the usual δ function expressing energy-momentum conservation at the interaction vertex.

This is to be contrasted with the case of nucleon-nucleon scattering due to the exchange of π mesons, where

$$F \sim \frac{1}{q^2 - \mu^2} \tag{10..98}$$

corresponding to a potential of range $\sim\hbar/\mu c$.

If the mass of the π meson were allowed to become large, one would for small q^2 approach a situation similar to β decay, that is, an approximate point coupling of the four fermions.

Conversely, as energies involved in weak interaction processes increase, such as in the inverse β decay reaction

$$\bar{\nu} + p \rightarrow n + e^+$$

one may anticipate that the interaction will appear nonlocal; for instance, a possible heavy boson W^+ might be exchanged between the p-n and e-υ systems, as in Fig. 10.15, or some more complicated non-locality might occur.

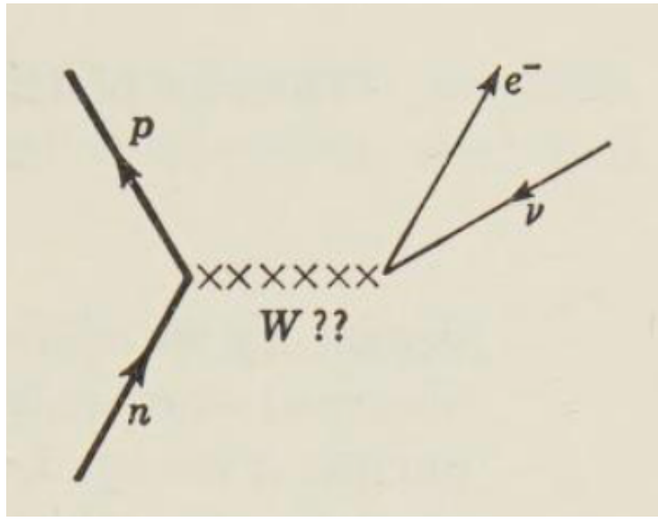

Fig. 10-15 A possible heavy W^+ boson exchange in β decay.

For our present discussion we remain with the approximation (10.96) and (10.97).

In the realm of low-energy β decay, nucleon recoil may also be neglected to within corrections of ~|q|/M, where q is the recoil momentum of the proton, and the neutron and proton wave functions are replaced by constant spinors in (10.92) and (10.93).

After squaring and doing the spin sums for unpolarized neutrons and final particles, we obtain

$$\frac{|S_{fi}{}^2|}{VT} \propto \frac{1}{(2E_e)(2E_{\bar{\nu}})} \sum_{A,B} \mathrm{Tr}\,(\not{p}_e + m)\Gamma^A \not{p}_{\bar{\nu}} \Gamma^B \times (2\pi)^4 \delta^4(p_e + p_{\bar{\nu}} + p_p - p_n) \tag{10.99}$$

where p_i represents the four-momentum of particle i = $(e, \bar{\nu}, p, n)$ and E_i is the corresponding energy; $E_p \approx M_p$ and $E_n \approx M_n$.

Γ_A and Γ_B are some complicated matrices which depend upon the structure of the matrix $\mathfrak{F}_{\alpha\beta\gamma\delta}$.

In the rest frame of the neutron, they are constant matrices, since they are functions of the nucleon variables only.

Therefore, after carrying out the trace (10.99), which must have the general form

$$AE_{\bar{\nu}} + BE_eE_{\bar{\nu}} + CE_eE_{\bar{\nu}}\beta_e \cdot \hat{\mathbf{p}}_{\bar{\nu}}$$

where A, B, and C are constants, $\beta_e \equiv p_e/E_e$, and $\hat{\mathbf{p}}_{\bar{\nu}} \equiv p_p/E_p$, we find for the transition rate to a given final state

$$\frac{|S_{fi}|^2}{VT} \propto \left(\frac{A}{E_e} + B + C\beta_e \cdot \hat{\mathbf{p}}_{\bar{\nu}}\right)(2\pi)^4 \delta^4(p_e + p_{\bar{\nu}} + p_p - p_n) \tag{10.100}$$

We may now multiply by phase-space factors d^3p_e, $d^3p_{\bar{\nu}}$, d^3p_p for the final state and integrate over all proton and neutrino momenta to find the electron spectrum in the neutron decay:

$$d\omega_e \propto d^3p_e \int d^3p_{\bar{\nu}}\, \delta(M_n - M_p - E_e - E_{\bar{\nu}}) \left(\frac{A}{E_e} + B + C\boldsymbol{\beta}_e \cdot \hat{\mathbf{p}}_{\bar{\nu}}\right)$$
$$\propto p_e E_e (M_n - M_p - E_e)^2 \left(\frac{A}{E_e} + B\right) dE_e \tag{10.101}$$

Thus by plotting

$$\frac{1}{p_e E_e (M_n - M_p - E_e)^2} \frac{d\omega_e}{dE_e} \tag{10.102}$$

versus E_e we should find an energy dependence $A/E_e + B$.

It is observed-not only in free neutron decay but also in a wide class of nuclear β decays known as the allowed transitions-that the electron spectrum fits this dependence and, further, that $A = 0$, that is, (10.102) is observed to be a constant in experiments.

A more familiar way of presenting the data is in terms of a Kurie plot, which displays

$$\left(\frac{1}{p_e E_e} \frac{d\omega_e}{dE_e}\right)^{1/2}$$

versus E_e as a straight line proportional to $(M_n - M_p - E_s)$.

The absence of a term α A/E_e in (10.100) is a consequence of the detailed nature of the β-decay interaction to which we come in the next paragraph; it is one of the so-called "Fierz interference" terms which are conspicuously absent from all observed β spectra, including the forbidden β-decay transitions which have matrix elements proportional to the nucleon velocities, but which again verify the approximate locality of the β-decay interaction.

It is amusing to note that the electron energy spectrum (10.101), with A = 0, is simply proportional to the phase-space volume ($E_\upsilon^2 p_e E_e dE_e$) and has the statistical shape predicted by a constant S-matrix element $S_{fi} \alpha (2\pi)^4 \delta^4(p_e+p_\upsilon+p_p - p_n)$, in place of (10.100).

We turn next to the structure of the matrix $\mathfrak{F}_{\alpha\beta\gamma\delta}$ in (10.97).

Holding the indices γ and δ fixed for the moment in (10.92), we can summarize the α, β dependence in complete generality in terms of the 16 γ-matrices discussed in Chap. 2:

$$\mathfrak{F}_{\alpha\beta\gamma\delta} = F^{(S)}{}_{\gamma\delta} 1_{\alpha\beta} + F^{(P)}{}_{\gamma\delta} \gamma^5{}_{\alpha\beta} + \sum_{\tau=0}^{3} F^{(V)\tau}{}_{\gamma\delta} (\gamma_\tau)_{\alpha\beta} + \sum_{\tau=0}^{3} F^{(A)\tau}{}_{\gamma\delta} (\gamma_5\gamma_\tau)_{\alpha\beta} + \sum_{\lambda\neq\tau=0} F^{(T)\lambda\tau}{}_{\gamma\delta} (\sigma_{\lambda\tau})_{\alpha\beta}$$

The matrices 1, γ_5, γ_τ, $\gamma_5\gamma_\tau$, and on, form scalars (S), pseudoscalars (P), vectors (V), axial vectors (A), and tensors of rank 2 (T), respectively, when sandwiched between Dirac solutions $\bar{\psi}(x) \cdots \psi(x)$, as seen in (2.38).

It is then clear from (10.92) and (10.96) that proper Lorentz covariance of the transition amplitude requires $F^{(S)}$ and $F^{(P)}$ to be linear combinations of 1 and γ_5, $F^{(V)\tau}$ and $F^{(A)\tau}$ to be linear combinations of γ^τ and $\gamma^\tau \gamma_5$, and $F^{(T)\tau}$ to be a linear combination of $\sigma^{\lambda\tau}$ and $\sigma^{\lambda\tau}\gamma_5$.

The S matrix then takes the form

$$S_{fi}^{(e^-)} = \frac{1}{(2\pi)^6}\sqrt{\frac{m_p m_n m_e}{2E_{\bar{\nu}}E_p E_n E_e}}\,(2\pi)^4\delta^4(p_e + p_{\bar{\nu}} + p_p - p_n)\mathfrak{M}$$

with

$$\mathfrak{M} = \sum_{i=S,P,V,A,T} C_i[\bar{u}_p(p_p)\Gamma_i u_n(p_n)]\{\bar{u}_e(p_e)[1 + \alpha_i\gamma_5]\Gamma^i v_{\bar{\nu}}(p_{\bar{\nu}})\} \qquad (10.103)$$

and

$$\Gamma_i = (1, \gamma_5, \gamma_\mu, \gamma_5\gamma_\mu, \sigma_{\mu\nu})$$
$$\Gamma^i = (1, \gamma_5, \gamma^\mu, \gamma^5\gamma^\mu, \sigma^{\mu\nu})$$

The S-matrix element for inverse B decay takes the same form, with the constants C_i and α_i replaced by their complex conjugates according to (10.94) and with the labels on the spinors interchanged according to

$$\bar{u}_p \cdots u_n \rightarrow \bar{u}_n \cdots u_p$$
$$\bar{u}_e \cdots v_{\bar{\nu}} \rightarrow \bar{u}_\nu \cdots v_{\bar{e}}$$

If we again neglect all momentum dependence in the nucleon spinors, thereby restricting our attention to "allowed" transitions, (10.103) simplifies in terms of two-component Pauli spinors for the nucleons to

$$\begin{aligned}\mathfrak{M} \approx\ & (u_p^\dagger u_n)\{C_S\bar{u}_e(p_e)[1 + \alpha_S\gamma_5]v_{\bar{\nu}}(p_{\bar{\nu}}) + C_V\bar{u}_e(p_e)[1 + \alpha_V\gamma_5]\gamma^0 v_{\bar{\nu}}(p_{\bar{\nu}})\} \\ & + (u_p^\dagger \boldsymbol{\sigma} u_n)\cdot\{2C_T\bar{u}_e(p_e)[1 + \alpha_T\gamma_5]\boldsymbol{\sigma}v_{\bar{\nu}}(p_{\bar{\nu}}) + C_A\bar{u}_e(p_e)[1 + \alpha_A\gamma_5]\gamma_5\boldsymbol{\gamma}v_{\bar{\nu}}(p_{\bar{\nu}})\}\end{aligned} \qquad (10.104)$$

The first line induces the allowed Fermi transitions (S,V) with $|\Delta S| = 0$ and the second line the allowed Gamow-Teller transitions (A,T) with $|\Delta S| = 1$ in the nucleon state.

One may separate Fermi from Gamow-Teller in nuclear β decay, where transitions occur between well-defined angular-momentum states; both contribute to the free neutron decay.

All the terms proportional to $\alpha_i\gamma_5$ in (10.104) violate parity and prior to 1956 and the work of Lee and Yang were abolished so as not to destroy the invariance of S_{fi} under the parity operation.

The experimental discovery of parity violation in weak decays following Lee and Yang' has now led to a complete determination of all the α_i and C_i in a series of key experiments.

The constants α_i of the parity-violating terms are determined by measuring the longitudinal polarization of the electron.

One method of determining this polarization is by measuring a left-right asymmetry in the scattering of the β-decay electron from an atom.

We define the polarization, as in (7.95), by

$$P = \frac{N_R - N_L}{N_R + N_L}$$

where N_R equals the number of "right-handed" electrons with spin in the direction of motion, $\langle \sigma \rangle \cdot \hat{\mathbf{p}} = +1$, and N_L is the number of "left-handed" ones with $\langle \sigma \rangle \cdot \hat{\mathbf{p}} = -1$.

Both in Fermi and Gamow-Teller transitions in nuclei as well as in free neutron decay, the electron polarization in the limit of no recoil and after integrating over neutrino angles is given to good accuracy by

$$P = -\frac{|\mathbf{p}_e|}{E_e} = -|\beta_e| \tag{10.105}$$

In the limit $\beta_e \rightarrow 1$ only left-handed electrons are emitted.

In the same limit the spin projection operator for a left-handed electron becomes, as in (7.107), $(1 - \gamma_5)/2$, so that the wave function for a left-handed electron is

$$\psi^{\mathrm{LH}} = \frac{1-\gamma_5}{2}\psi \qquad \bar{\psi}^{\mathrm{LH}} = \bar{\psi}\frac{1+\gamma_5}{2} \tag{10.106}$$

Therefore all the α_i's in (10.104) equal +1 in order that the correct sign and relativistic limit be reproduced.

The polarization for arbitrary β_e is found by introducing (3.19) and (7.94) into (10.99) and carrying out traces and the neutrino angular integral to obtain $- S^0 m_e/E_e = - |\beta_e|$ as claimed.

Equation (10.104) is now simplified to

$$\begin{aligned}\mathfrak{M} = {} & (u_p^\dagger u_n)\bar{u}_e(p_e)(1+\gamma_5)(C_S + C_V\gamma_0)v_{\bar{\nu}}(p_{\bar{\nu}}) \\ & + (u_p^\dagger \boldsymbol{\sigma} u_n)\cdot\bar{u}_e(p_e)(1+\gamma_5)(2C_T\boldsymbol{\sigma} + C_A\boldsymbol{\gamma})v_{\bar{\nu}}(p_{\bar{\nu}})\end{aligned} \tag{10.107}$$

and the relative magnitudes of the constants C_S, C_V, C_T, and C_A remain to be determined

It is a straightforward exercise to square $\mathfrak{M}$ and sum over spins.

For unpolarized nucleons there are no interference terms between the Fermi (S,V) transitions and the Gamow-Teller (A,T) transitions.

Moreover, there are no interference terms between S and V or A and T in (10.107), since these lead to different final antineutrino states.

When the $(1 + \gamma_5)$ factors are commuted to the right, we see that S and T transitions lead to the emission of left-handed antineutrinos

$$v_{\bar{\nu}}^{\mathrm{LH}}(p_{\bar{\nu}}) = \frac{1 + \gamma_5}{2} v_{\bar{\nu}}^{\mathrm{LH}}(p_{\bar{\nu}})$$

On the other hand, for transitions via V and A only right-handed anti-neutrinos are emitted.

In the absence of interference between S and V or A and T there are no Fierz terms; that is, the coefficient A in (10.100) and (10.101) vanishes.

The contributions to A come from terms of the form

$$m_e \,\mathrm{Tr}\, \Gamma_A \not{p}_{\bar{\nu}} \Gamma_B$$

If there are an even number of γ matrices in Γ_{A}, there must be an odd number in Γ_{B}, and vice versa; thus all contributions to the Fierz term are interference terms, which, as we have seen, vanish.

Were neutrinos emitted with less than 100 percent polarization, the absence of Fierz terms would require the coupling to contain either S or V and either A or T terms only.

All four terms may appear in (10.107), however, and we must consider experiments which measure the coefficient C in (10.100), that is, the electron-antineutrino angular correlation, for more information on C_{S}, C_{V}, C_{A}, and C_{T}.

Consider, for example, a pure Fermi transition involving the S and V contributions only.

Summing over electron and antineutrino spin variables in (10.107) gives for the angular distribution of the emerging antineutrino relative to the electron

$$\begin{aligned} N_{\text{Fermi}}(\theta) &\propto \text{Tr}\,(\not{p}_e + m)(1+\gamma_5)(C_S + \gamma_0 C_V)\not{p}_{\bar{\nu}}(C_S^* + \gamma_0 C_V^*)(1-\gamma_5) \\ &= 8E_e E_{\bar{\nu}}[|C_S|^2(1 - \beta_e \cos\theta) + |C_V|^2(1 + \beta_e \cos\theta)] \end{aligned} \tag{10.108}$$

where θ is the angle between them.

The same distribution is found for inverse β decay.

The experimental distribution, determined for the almost pure Fermi transition in A by measuring the direction of the positron and recoil nucleus, is found to be approximately $1 + \beta_e \cos\theta$, suggesting the transition is vector. A similar calculation for Gamow-Teller transitions gives

$$N_{\text{GT}}(\theta) \propto E_e E_{\bar{\nu}}[|C_A|^2(1 - \tfrac{1}{3}\beta_e \cos\theta) + 4|C_T|^2(1 + \tfrac{1}{3}\beta_e \cos\theta)] \tag{10.109}$$

and the measurements on a pure Gamow-Teller transition, as in Ne^{23}, together with other data on mixed Fermi and Gamow-Teller transitions, indicate $|C_T/C_A| < 1$.

Equation (10.107) may then be simplified to two terms in the limit $C_T/C_A = 0$:

$$\begin{aligned} \mathfrak{M} \approx\ & (u_p^\dagger u_n)\bar{u}_e(p_e)C_V\gamma^0(1-\gamma_5)v_{\bar{\nu}}(p_{\bar{\nu}}) \\ & + (u_p^\dagger \boldsymbol{\sigma} u_n)\cdot\bar{u}_e(p_e)(+C_A\boldsymbol{\gamma})(1-\gamma_5)v_{\bar{\nu}}(p_{\bar{\nu}}) \end{aligned} \tag{10.110}$$

which leads to the emission of right-handed antineutrinos only.

Now only the magnitudes of C_V and C_A and their relative phase remain to be determined.

The magnitudes are found from measurement of the decay rates of the neutron and of the pure Fermi transition in O^{14}.

The phase is determined from the measurement of the electron's angular distribution relative to the neutron's spin axis in the β decay of polarized neutrons, which is sensitive to the V-A mixture.

The final result is

$$\begin{aligned} \sqrt{2}\, C_V &= (1.015 \pm 0.03) \times 10^{-5} \frac{1}{M_p^2} \equiv G \\ C_A &= (+1.21 \pm 0.03) C_V \\ &\equiv +\alpha C_V \end{aligned} \tag{10.111}$$

Inserting the notation (10.111) into (10.110) and reverting to a relativistic notation, we have for the invariant amplitude of β decay

$$\mathfrak{M} = \frac{G}{\sqrt{2}} [\bar{u}_p \gamma_\mu (1 - \alpha\gamma_5) u_n][\bar{u}_e \gamma^\mu (1 - \gamma_5) v_{\bar{\nu}}] \tag{10.112}$$

It would be natural to regard $\mathfrak{M}$ as a first-order interaction amplitude and to study higher-order effects such as illustrated by the Feynman diagrams in Fig. 10.16.

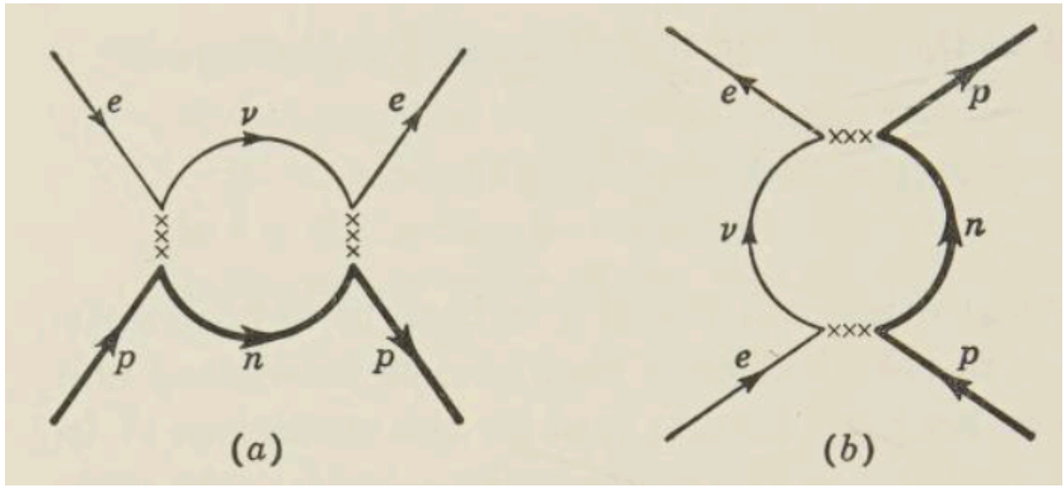

Fig. 10-16 Some higher order graphs for β decay.

However, in the approximation (10.96) of local interactions we do not know how to calculate these amplitudes.

The closed loops in these diagrams give rise to infinite contributions which cannot be separated and isolated into renormalization constants as they were in Chap. 8.

The difficulty here arises from the loss of boson propagators between the nucleon and lepton vertices which provide convergence at high momenta.

The local approximation (10.96) is equivalent to replacing the propagators by constants as in (10.98); this will not be valid in closed-loop calculations which require integration over all momenta.

Although we are dealing with weak interactions with a very small coupling constant (10.111), this question of higher-order contributions is not entirely academic, since the cross sections computed from (10.112) for scattering processes such as

$$\bar{\nu} + p \to n + e^+$$

increase with the square of the energy and are proportional to

$$G^2 E_{\text{c.m.}}^2 \sim \left(\frac{E_{\text{c.m.}}}{300 M_p}\right)^4 \frac{1}{E_{\text{c.m.}}^2} \tag{10.113}$$

By the time we reach energies $E_{c.m.} \sim 300M_p \sim 300$ GeV the weak interactions may well have grown into strong ones and the effects of nonlocalities and of higher-order graphs may become of major importance.

10.12 Two-component Neutrino Theory

We have already noted that only right-handed antineutrinos are emitted in β decay.

Correspondingly, only left-handed neutrinos appear in the inverse processes, according to (10.94), owing to the $(1 - \gamma_5)$ in (10.112).

Since the right-handed neutrinos and their left-handed antiparticles are absent from these as from all weak interactions, they represent an unnecessary extra degree of freedom in the Dirac equation for a massless particle which we may attempt to remove.

The Dirac equation for a massless particle

$$i\frac{\partial \psi_\nu}{\partial t} = -i\boldsymbol{\alpha} \cdot \nabla \psi_\nu \tag{10.114}$$

does not contain the β matrix, and the anticommutation relations (1.16) for the three matrices α_1, α_2, and α_3

$$\{\alpha_i, \alpha_k\} = 2\delta_{ik} \qquad \alpha_i^2 = 1 \tag{10.115}$$

may be satisfied with the 2 x 2 Pauli matrices; that is,

$$\boldsymbol{\alpha} = \boldsymbol{\sigma} \tag{10.116}$$

A positive-energy plane-wave solution of (10.114) and (10.116) has the form

$$\psi(x) = \frac{1}{\sqrt{2E(2\pi)^3}} u(p,s) e^{-i(Et - \mathbf{p}\cdot\mathbf{x})} \tag{10.117}$$

with E = |**p**| and the spinor u(p,s) satisfying the equation

$$Eu(p,s) = \boldsymbol{\sigma} \cdot \mathbf{p} u(p,s) \tag{10.118}$$

The solution of (10.118) in the usual representation of the Pauli matrices, with the direction of the z axis along **p**, is

$$u(p,+) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{10.119}$$

which describes right-handed neutrinos with spin along the direction of motion:

$$\frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E} u(p,+) = +u(p,+)$$

To obtain left-handed neutrinos, as observed in nature, we must choose the solution of (10.115) with

$$\boldsymbol{\alpha} = -\boldsymbol{\sigma} \tag{10.120}$$

instead of (10.116).

In this case (10.118) is replaced by

$$Eu(p,-) = -\boldsymbol{\sigma} \cdot \mathbf{p} u(p,-) \tag{10.121}$$

and we have left-handed neutrinos

$$u(p,-) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{10.122}$$

To understand better the relation of these two-component solutions to the by-now-familiar four-component electron spinors, we return to the Dirac equation for a particle with mass m and choose a representation for the α, β matrices

$$\alpha_i = \begin{bmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{bmatrix} \qquad \beta = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \tag{10.123}$$

which differs from (1.17) by a unitary transformation

$$\mathfrak{U} = \frac{1}{\sqrt{2}}(1 + \gamma_5\gamma_0)$$

In this representation the Dirac equation (1.13) may be expressed in a split two-component notation, with

$$\psi = \begin{bmatrix} u(+) \\ u(-) \end{bmatrix}$$

as

$$i\frac{\partial}{\partial t}u(+) = -i\mathfrak{d}\cdot\nabla u(+) - mu(-)$$

$$i\frac{\partial}{\partial t}u(-) = +i\mathfrak{d}\cdot\nabla u(-) - mu(+) \qquad (10.124)$$

The upper and lower components of ψ are mixed only by the mass terms in (10.124); and in the limit $m \to 0$ there result two uncoupled two-component equations corresponding to (10.114), with $\alpha = \sigma$ as in (10.116) and $\alpha = -\sigma$ as in (10.120).

In the representation (10.123)

$$\gamma_5 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and it is easy to see that the solution

$$\psi(+) = \begin{bmatrix} u(+) \\ 0 \end{bmatrix} \qquad \text{for } m \to 0$$

corresponds to right-handed neutrinos, since

Similarly for

$$\gamma_5\psi(+) = +\psi(+)$$

$$\psi(-) = \begin{bmatrix} 0 \\ u(-) \end{bmatrix} \qquad \gamma_5\psi(-) = -\psi(-)$$

showing that the solutions $\psi(-)$ represent left-handed neutrinos.

The possibility of describing massless Dirac particles by a two-component equation was first discussed by Weyl in 1929 but was not taken seriously, the reason being that the β matrix, and thus the parity operation P of (2.32), has been lost in reducing to two components.

After the downfall of parity in 1956, the Weyl equation was resurrected by Landau, Lee and Yang, and Salam, who observed that the charge conjugation symmetry C of (5.5) and (5.6) has also been lost, but that a combined invariance under CP remains.

According to (5.5), the charge conjugation operation consists of replacing $\psi(\mathbf{x},t)$ by $\psi_c(\mathbf{x},t) = C\beta\psi^*(\mathbf{x},t)$ with C, a matrix satisfying (5.4) and (5.6).

In the new representation the matrix

$$C = -i \begin{bmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{bmatrix}$$

satisfies these conditions.

For the case of the two-component neutrino, the presence of the β in the above charge conjugation transformation means that it is no longer a symmetry operation; however, the combined operation CP is.

For if $\psi(x)$ is a solution of (10.114), so is

$$\psi_{CP}(\mathbf{x},t) = C\psi^*(-\mathbf{x},t) = \mp i\sigma_2\psi^*(-\mathbf{x},t) \tag{10.125}$$

since

$$\left(i\frac{\partial}{\partial t} + i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}_x\right)\psi_{CP}(\mathbf{x},t) = \mp i\sigma_2\left(i\frac{\partial}{\partial t} - i\boldsymbol{\alpha}^*\cdot\boldsymbol{\nabla}_x\right)\psi^*(-\mathbf{x},t)$$

$$= \mp i\sigma_2\left[\left(-i\frac{\partial}{\partial t} - i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}_{x'}\right)\psi(\mathbf{x}',t)\right]^* = 0$$

with $\mathbf{x}' = -\mathbf{x}$.

To form the antineutrino wave function, we take—as in (5.7) and (5.8) for electrons—a negative-energy neutrino solution, complex-conjugate the wave function, and multiply by $i\sigma_2$.

For example, a negative-energy solution of (10.114) with $\alpha = -\sigma$ for left-handed neutrinos is, with $E = +|\mathbf{p}|$,

$$\psi(x) = \frac{1}{\sqrt{2E(2\pi)^3}}\,v(-p,-)e^{+i(Et+\mathbf{p}\cdot\mathbf{x})}$$

where

$$-E\,v(-p,-) = -\boldsymbol{\sigma}\cdot\mathbf{p}\,v(-p,-)$$

and

$$v(-p,-) = \begin{bmatrix}1\\0\end{bmatrix} \tag{10.126}$$

According to (10.125) the antineutrino wave function is then

$$\psi_{CP}(\mathbf{x}',t) = \begin{bmatrix}0\\1\end{bmatrix}\frac{1}{\sqrt{2E(2\pi)^3}}\,e^{-i(Et-\mathbf{p}\cdot\mathbf{x}')} \tag{10.127}$$

which evidently is also a solution of the left-handed Weyl equation.

Now (10.127) indeed represents a left-handed particle, but in the parity reflected system, with $\mathbf{x}' = -\mathbf{x}$; just as a right-handed person appears to be left-handed in a mirror, so the left-handed antineutrino in the primed system (10.127) is right-handed to us in the unprimed system.

10.13 Mu-meson Decay

The decay of a μ meson

$$\mu^- \rightarrow e^- + \nu' + \bar{\nu}$$

involves four fermions, including an $(e^-, \bar{\nu})$.

This similarity to the β decay of the neutron suggests that we try the same form of S-matrix element as in the β decay.

As in the inverse β-decay process, the amplitude for μ^+ decay

$$\mu^+ \rightarrow e^+ + \nu + \bar{\nu}'$$

will be given by detailed balance.

We again pair the $e^-\bar{\nu}$ wave functions as in (10.112) assuming the form of the coupling to be

$$\bar{u}_e(p)\gamma^\mu(1 - \gamma_5)v_{\bar{\nu}}(\bar{k}) \tag{10.128}$$

This implies right-handed antineutrinos and completely left-polarized electrons in u decay (and right-handed positrons in μ^+ decay).

The neutrino polarization is not observed, but experiments' indicate the complete left-handed polarization of the electrons predicted by (10.128).

The S-matrix element for μ decay then takes the form

$$S_{fi} = \frac{-i}{(2\pi)^6}\sqrt{\frac{m_\mu}{E_P}\frac{m_e}{E_p}\frac{1}{2E_k}\frac{1}{2E_{\bar{k}}}}\,(2\pi)^4\delta^4(P-p-k-\bar{k})\mathfrak{M}$$

$$\mathfrak{M} = \frac{1}{\sqrt{2}}\tilde{G}[\bar{u}_{\nu'}(k)\gamma^\mu(1-\lambda\gamma_5)u_\mu(P)][\bar{u}_e(p)\gamma_\mu(1-\gamma_5)v_{\bar{\nu}}(\bar{k})] \qquad (10.129)$$

with the kinematics illustrated in Fig. 10.17.

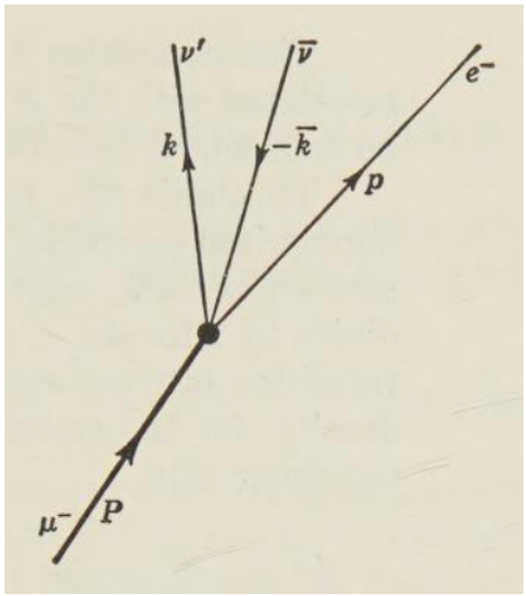

Fig. 10-17 μ decay.

The parameter λ which determines the polarization of the υ' associated with the μ^- meson and the coupling constant $\tilde{G}$ remain to be determined from the observed u-decay rate and spectrum.

To obtain the transition rate for μ^- decay from (10.129) for unpolarized particles, we take $|S_{fi}|^2$, multiply by final phase-space factors $d^3p\ d^3k\ d^3\bar{k}$, insert projection operators for the sum over spins, divide by 2 for the μ^- spin average, divide by $VT = (2\pi)^4\delta^4(0)$ for the transition rate per unit volume, and finally divide by $1/(2\pi)^3$, the μ^- density, for the transition rate per μ meson.

These steps give for the decay rate

$$d\omega = \frac{1}{2(2\pi)^5}\frac{1}{2E_P}\int \frac{d^3p}{2E_p}\frac{d^3k}{2E_k}\frac{d^3\bar{k}}{2E_{\bar{k}}}\,\delta^4(P-p-k-\bar{k})\sum_{\text{spins}}|\mathfrak{M}|^2$$

with

$$\sum_{\text{spins}}|\mathfrak{M}|^2 = \frac{\tilde{G}^2}{2}\,\mathrm{Tr}\,[\gamma^\mu(1-\lambda\gamma_5)(\not{P}+m_\mu)\gamma^\nu(1-\lambda^*\gamma_5)k] \times \mathrm{Tr}\,[(\not{p}+m_e)\gamma_\mu(1-\gamma_5)\not{\bar{k}}\gamma_\nu(1-\gamma_5)] \tag{10.130}$$

For the total integrated rate, everything to the right of $(2E_p)^{-1}$ in (10.130) is Lorentz invariant, and we see that the inverse decay rate is proportional to the energy E_p as required by special relativity.

In the rest frame of the μ meson this is the lifetime.

Carrying out the traces, we obtain

$$\sum_{\text{spins}}|\mathfrak{M}|^2 = 32\tilde{G}^2(1+|\lambda|^2)(k\cdot p\,P\cdot\bar{k}+k\cdot\bar{k}\,p\cdot P) + \tilde{G}^2(\lambda+\lambda^*)\,\mathrm{Tr}\,(k\gamma^\mu\not{P}\gamma^\nu\gamma_5)\,\mathrm{Tr}\,(\not{p}\gamma_\mu\not{\bar{k}}\gamma_\nu\gamma_5) \tag{10.131}$$

The last trace yields a scalar antisymmetric in k and P, antisymmetric in $\bar{k}$ and p, and linear in all four momentum variables.

It therefore has the form

$$\mathrm{Tr}\,(k\gamma^\mu\not{P}\gamma^\nu\gamma_5)\,\mathrm{Tr}\,(\not{p}\gamma_\mu\not{\bar{k}}\gamma_\nu\gamma_5) = a(k\cdot\bar{k}\,p\cdot P - k\cdot p\,P\cdot\bar{k}) \tag{10.132}$$

and there is only the number a to be determined.

We do this by temporarily choosing the vectors k $\bar{k}$, p, P at our convenience, requiring only that (10.132) not vanish.

For example, with $k_\mu = \bar{k}_\mu = (1,0,0,0)$ and $P_\mu = p_\mu = (0,1,0,0)$, (10.132) becomes

$$-a = \mathrm{Tr}\,(\gamma_0\gamma^\mu\gamma_1\gamma^\nu\gamma_5)\,\mathrm{Tr}\,(\gamma_1\gamma_\mu\gamma_0\gamma_\nu\gamma_5)$$

In summing indices u and v only the two combinations $\mu = 2, \upsilon = 3$ and $\mu = 3, \upsilon = 2$ survive and contribute equally; thus

$$-a = 2\,\mathrm{Tr}\,(\gamma_0\gamma_2\gamma_1\gamma_3\gamma_5)\,\mathrm{Tr}\,(\gamma_1\gamma_2\gamma_0\gamma_3\gamma_5) = +32$$

and the spin sum in (10.131) becomes

$$\sum_{\text{spins}} |\mathfrak{M}|^2 = 32\tilde{G}^2[|1-\lambda|^2 k\cdot\bar{k}\; p\cdot P + |1+\lambda|^2 k\cdot p\; \bar{k}\cdot P] \qquad (10.133)$$

To proceed further, we integrate over the variables of the unobserved neutrinos in (10.130).

Since $|\mathfrak{M}|^2$ is linear in k and $\bar{k}$, we are led to evaluate the integral

$$I^{\alpha\beta} \equiv \int \frac{d^3k}{2E_k}\frac{d^3\bar{k}}{2E_{\bar{k}}}\, k^\alpha \bar{k}^\beta \delta^4(Q-k-\bar{k}) \qquad (10.134)$$

with Q = P - p.

This integral transforms as a second-rank Lorentz tensor and is best evaluated in the center-of-mass system of the two neutrinos, in which Q is pure time-like.

In this system we find after a short calculation

$$I_{(0)}^{\alpha\beta} = \frac{\pi}{24} Q_0^2 [g^{\alpha\beta} + 2g^{\alpha 0} g^{\beta 0}]$$

Knowing that (10.134) is a tensor of rank 2, we can now express the answer in an arbitrary Lorentz frame as

$$I^{\alpha\beta} = \frac{\pi}{24} [g^{\alpha\beta} Q^2 + 2Q^\alpha Q^\beta] \qquad (10.135)$$

$\bar{k}$

Putting (10.130), (10.133), and (10.135) together gives

$$d\omega = \frac{\tilde{G}^2}{192\pi^4} \frac{p\, dE_p\, d\Omega_p}{E_P} \{|1 - \lambda|^2 6p\cdot P(m_\mu{}^2 + m_e{}^2 - 2p\cdot P) + |1 + \lambda|^2[-4(p\cdot P)^2 + 3p\cdot P(m_\mu{}^2 + m_e{}^2) - 2m_\mu{}^2 m_e{}^2]\}$$

Integrating over electron angles and neglecting the electron rest mass, that is, $m_e/E_p \to 0$, gives for the energy distribution of electrons in the μ^- rest frame

$$\left(\frac{d\omega}{dE_p}\right)^0 = \frac{\tilde{G}^2 m_\mu{}^2 E_p{}^2}{48\pi^3} \left[|1 - \lambda|^2 6\left(1 - \frac{2E_p}{m_\mu}\right) + |1 + \lambda|^2 \left(3 - \frac{4E_p}{m_\mu}\right)\right] \qquad (10.136)$$

The observed energy distribution favors $\lambda = +1$ to good accuracy, that is, the neutrino associated with the μ^- meson in (10.129) is also left-handed.

Setting $\lambda = 1$ in (10.136) and integrating over all electron energies $0 < E_p \leq \frac{1}{2}m_\mu$, gives for the total μ-decay rate

$$\omega^0 = \frac{1}{\tau_\mu} = \frac{\tilde{G}^2 m_\mu{}^5}{192\pi^3} \qquad (10.137)$$

The value of $\tilde{G}$ deduced from the observed

$$\tau_\mu = (2.21 \pm 0.003) \times 10^{-6} \text{ sec}$$

is equal to the vector coupling constant G = C_v in nuclear β decay (10.111), to within 2 per cent!

From this strong indication of a universality in the weak β-decay couplings between fermions we draw additional support for the description of the interactions given here and look to other processes involving pairs of leptons (e^-,υ) or (μ^-,υ') to see if they too indicate a V-A coupling form (10.128).

10.14 Pi-meson Decay

In writing the S-matrix elements for the decay of the π meson (10.91c), we may optimistically start off again with the V-A coupling in (10.128) which has done so well in β decay and μ decay.

We are strongly aided in this approach by two experimental findings.

First, the μ^- meson in the $\pi^- \rightarrow \mu^- + \bar{\nu}'$ decay is observed to be longitudinally polarized and right-handed.

Then the antineutrino must also be emitted with a right-handed polarization, since the π^- has spin zero.

In the π^- rest frame the μ^- and $\bar{\nu}'$ emerge with equal and opposite momenta, and both must be right- or left-handed if their angular momenta along the direction of emission are to cancel as required by angular-momentum conservation.

This observation agrees with (10.128), which predicts that only right-handed antineutrinos (or left-handed neutrinos in it decay) are emitted.

The second experimental finding in support of (10.128) is the observed very small value of the branching ratio

$$R\left(\frac{\pi \to e + \nu}{\pi \to \mu + \nu'}\right) = 1.3 \times 10^{-4} \tag{10.138}$$

A very strong depression in the electron emission rate is predicted by (10.128) owing to the factor $(1 - \gamma_5)$, which leads to completely polarized left-handed neutrinos and right-handed antineutrinos.

Anticommuting it to the left gives $\bar{u}(p)(1 + \gamma_5)$, corresponding in the limit $v_e/c \to 1$ to the emission of left-handed electrons (or right-handed positrons) only.

However, angular-momentum conservation requires, as already remarked above and illustrated in Fig. 10.18, that right-handed electrons accompany the right-handed antineutrinos in π^- decay.

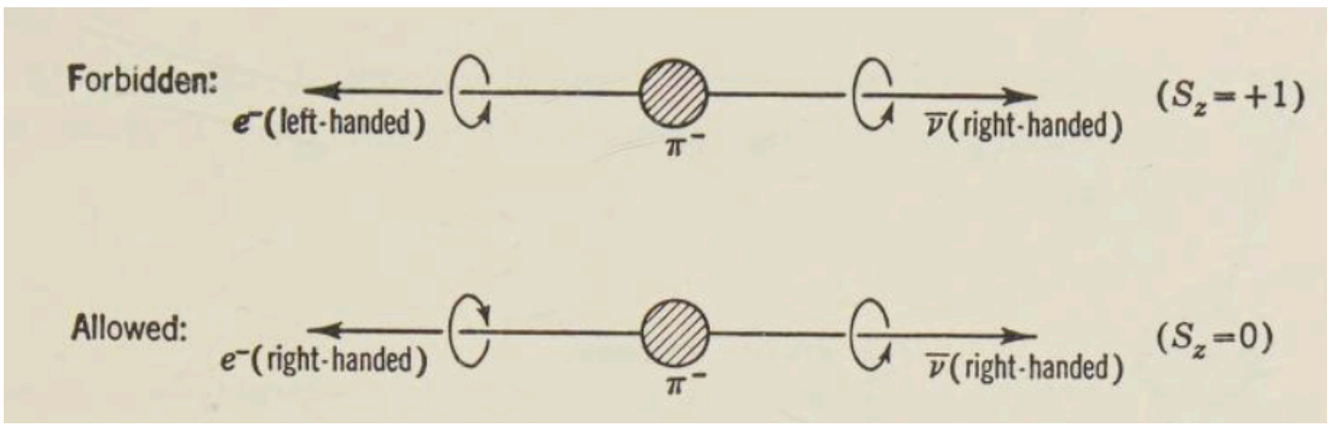

Fig. 10-18 Spin conservation in π decay showing both e^- and $\bar{\nu}$ as right handed.

Therefore, the transition rate calculated from (10.128) is depressed by a factor

$$1 - \left(\frac{v_e}{c}\right)^2 \cong \left(\frac{2m_e}{\mu}\right)^2$$

where μ is the π-meson mass, representing the probability for a right-handed electron to be emitted.

For the heavier μ meson which is emitted nonrelativistically with energy

$$E_\mu = \frac{\mu^2 + m_\mu{}^2}{2\mu} = 1.04 m_\mu$$

the spin projection operator differs considerably from $(1 \pm \gamma_5)$ and there is no appreciable depression factor.

We adopt, then, (10.128) for the lepton term in the π-decay S-matrix element and seek a four-vector or axial vector with which to multiply it in forming the invariant transition amplitude.

Since the π meson has no spin, this vector must be formed from the two independent momentum four-vectors in the decay, P_μ for the π and $\bar{k}_\mu$ for the $\bar{\nu}$ as illustrated in Fig. 10.19.

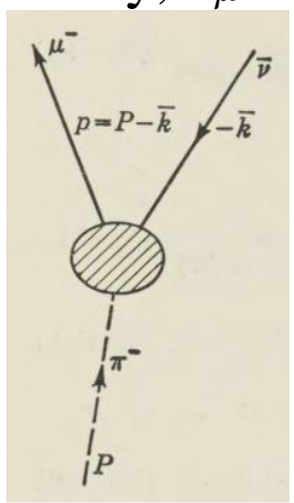

Fig 10-19 π decay.

The $\bar{k}_\mu$ will not contribute, since $\bar{k} v_{\bar{\nu}}(\bar{k}) = 0$.

Therefore, the structure of the S-matrix element for π^- decay is unique and may be written

$$S_{fi}^{(\pi)} = \frac{-i}{(2\pi)^{9/2}} \sqrt{\left(\frac{1}{2E_P}\right)\left(\frac{m_e}{E_p}\right)\left(\frac{1}{2E_{\bar{k}}}\right)} \frac{Ga}{\sqrt{2}} iP^\mu [\bar{u}(p)\gamma_\mu(1 - \gamma_5)v_{\bar{\nu}}(\bar{k})] \times (2\pi)^4 \delta^4(P - p - \bar{k}) \tag{10.139}$$

The constant G is the β-decay constant (10.111), and the constant a which determines the overall decay rate of the meson may differ for the μ^- - and e^- -decay modes.

Repeating the similar (but simpler) steps (10.130) to (10.137), we find for the decay rate

$$\frac{1}{\tau_\pi} = \frac{(2\pi)^3}{(2\pi)^5}\left(\frac{1}{2\mu}\right)\frac{G^2|a|^2}{2}\,8\int \frac{d^3\bar{k}}{2E_{\bar{k}}}\frac{d^3p}{2E_p}\,[2p\cdot P\;\bar{k}\cdot P - \bar{k}\cdot p\;P^2]\delta^4(P-\bar{k}-p)$$

$$= \frac{G^2|a|^2}{8\pi}\,\mu^3\left(\frac{m}{\mu}\right)^2\left(1-\frac{m^2}{\mu^2}\right)^2 \qquad (10.140)$$

where m is the mass of the electron or μ meson emitted.

If the constant a is the same for both the electron and μ-meson decay modes of the π meson, (10.140) predicts a branching ratio

$$R\left(\frac{\pi\to e+\nu}{\pi\to\mu+\nu'}\right) = \left(\frac{m_e}{m_\mu}\right)^2\frac{(\mu^2-m_e{}^2)^2}{(\mu^2-m_\mu{}^2)^2} = 1.23\times10^{-4}$$

in agreement with the observed value (10.138), within errors of the order of 5 per cent.

This is further strong support for the coupling (10.128) which applies universally for all lepton decays.

Without the strict selection rule of the V-A interaction for left-handed neutrinos and right-handed antineutrinos this dramatically small branching ratio would be replaced by a much larger one lying closer to the phase-space ratio

$$\frac{(\mu^2-m_e{}^2)^2}{(\mu^2-m_\mu{}^2)^2} \approx 5.5$$

From the observed lifetime of $\tau = (2.55 \pm 03) \times 10^{-8}$ sec for the π meson, we compute a value

$$|a| \cong 0.93\mu \tag{10.141}$$

for the constant in (10.140).

With the assumption of a universal coupling, (10.128), for all lepton decays the polarization of the μ meson from π-meson decay is determined.

This leads to a unique prediction of an asymmetry parameter in the μ-decay spectrum correlating the direction of the decay electron with that of the μ-meson spin in the decay chain

$$\begin{array}{l} \pi^- \to \mu^- + \bar{\nu}' \\ \qquad\quad \big\lfloor \to e^- + \bar{\nu} + \nu' \end{array}$$

We calculate this by first computing the polarization of the μ meson spectrum of decay electrons from these μ mesons.

To compute the μ-meson polarization in π decay, we return to (10.139) and compute the differential decay rate to a state of given spin polarization s.

The spin projection operator $(1 + \gamma_5 \$)/2$ is introduced as in (7.89) and (7.90) to enable us to use trace techniques, and we obtain in place of (10.140)

$$d\omega_{\pi,s} = \frac{1}{4\pi^2}\frac{1}{2E_P}\frac{G^2|a|^2}{2}\int \frac{d^3\bar{k}}{2E_{\bar{k}}}\frac{d^3p}{2E_p} \times 2\,\mathrm{Tr}\left[(\not{p} + m_\mu)\left(\frac{1+\gamma_5 \$}{2}\right)\not{P}(1-\gamma_5)\not{\bar{k}}\not{P}\right]\delta^4(P - p - \bar{k})$$

$$= \frac{G^2|a|^2}{4\pi^2}\left(\frac{\mu}{E_P}\right)\mu^3\left(\frac{m_\mu}{\mu}\right)^2\int\frac{d^3p}{2E_p}\,\delta[(P-p)^2] \times\left\{\frac{1}{2}\left(1-\frac{{m_\mu}^2}{\mu^2}\right)+\frac{m_\mu\, s\cdot P}{\mu^2}\right\} \tag{10.142}$$

P and pare the π and μ four-momenta, respectively.

The decay rate is a maximum for a right-handed u of positive helicity, that is, for

$$s_R\cdot P = \frac{1}{2}\frac{\mu^2}{m_\mu}\left(1-\frac{m_\mu^2}{\mu^2}\right)$$

by Eq. (7.94).

It vanishes for a negative-helicity u meson with $s_L = - s_R$.

For the decay rate of the μ meson with a given spin polarization s we return to (10.129) and repeat that calculation, inserting $\lambda = +1$ as measured and neglecting the electron rest mass, $m_e/ E_p \rightarrow 0$.

The spin projection operator $(1 + \gamma_5\$)/2$ again enables us to use trace techniques in evaluating the squared matrix element.

Retracing the steps to (10.133), we find

$$\sum_{\substack{\text{electron}\\ \text{spin}}} |\mathfrak{M}|^2 = \frac{\tilde{G}^2}{2}\,\mathrm{Tr}\left[\gamma^\mu(1-\gamma_5)\left(\frac{1+\gamma_5\$}{2}\right)(\not{P}+m_\mu)\gamma^\nu(1-\gamma_5)k\right] \times \mathrm{Tr}\,[\not{p}\gamma_\mu(1-\gamma_5)\bar{k}\gamma_\nu(1-\gamma_5)]$$

$$= 64\tilde{G}^2\, k\cdot p\;\bar{k}\cdot(P-m_\mu s)$$

Integration over the variables of the unobserved neutrinos is carried out precisely as in (10.135) to give

$$d\omega_s = \frac{\tilde{G}^2}{48\pi^4}\frac{p\,dE_p\,d\Omega_p}{E_P}\{-4(p\cdot P)^2 + 3m_\mu^2(p\cdot P) - m_\mu\, s\cdot p[m_\mu^2 - 4(p\cdot P)]\}$$

In the μ rest system this reduces to

$$d\omega_s^0 = \frac{\tilde{G}^2}{48\pi^4}\, m_\mu^2 E_p{}^2\, dE_p\, d\Omega\left[\left(3 - 4\frac{E_p}{m_\mu}\right) + \frac{s\cdot p}{E_p}\left(4\frac{E_p}{m_\mu} - 1\right)\right]$$

As we saw in Eq. (10.142), the μ^- meson is produced with positive helicity in the π^- decay.

Therefore, by Eq. (7.94),

$$\left\langle\frac{s\cdot p}{E_p}\right\rangle = -\hat{\mathbf{s}}\cdot\hat{\mathbf{p}} \equiv -\cos\theta$$

where θ is the angle between the μ^- spin and the direction of the decay electron.

This gives

$$\langle d\omega^0\rangle = \frac{\tilde{G}^2}{24\pi^3}\, m_\mu^2 E_p{}^2\, dE_p\, d(\cos\theta)\left[3 - 4\frac{E_p}{m_\mu}\right](1 - \alpha\cos\theta)$$

where

$$\alpha \equiv \frac{4E_p - m_\mu}{3m_\mu - 4E_p}$$

is the asymmetry parameter as observed.

10.15 Two Neutrinos

All the leptonic interactions we have studied have led to the same V-A structure for the leptonic matrix elements given in (10.128).

That is, μ^- and e^- transform into two-component left-handed neutrinos in weak interactions.

But nature, after being so economical in providing parity nonconservation as a way of eliminating the need for an extra neutrino degree of freedom, has been inexplicably generous in giving us two such objects, υ and υ', nearly alike but yet very different.

The υ associated at a vertex with an electron line is left-handed and has zero (or very small) mass, just like the υ' associated with the μ.

But they are different.

The first hint that they might be different came from comparison of unreliable theoretical estimates of the radiative decay rate,

$$\mu^- \to e^- + \gamma$$

with experiments which showed a branching ratio

$$\frac{R(\mu^- \to e^- + \gamma)}{R(\mu^- \to e^- + \bar{\nu} + \nu')} < 10^{-7}$$

This process does not occur to first order in the weak interactions by any known mechanism unless we invoke the hypothesis of an intermediate charged vector meson to mediate the weak interactions.

With this hypothesis calculations can be made with models along the lines shown in Fig. 10.20.

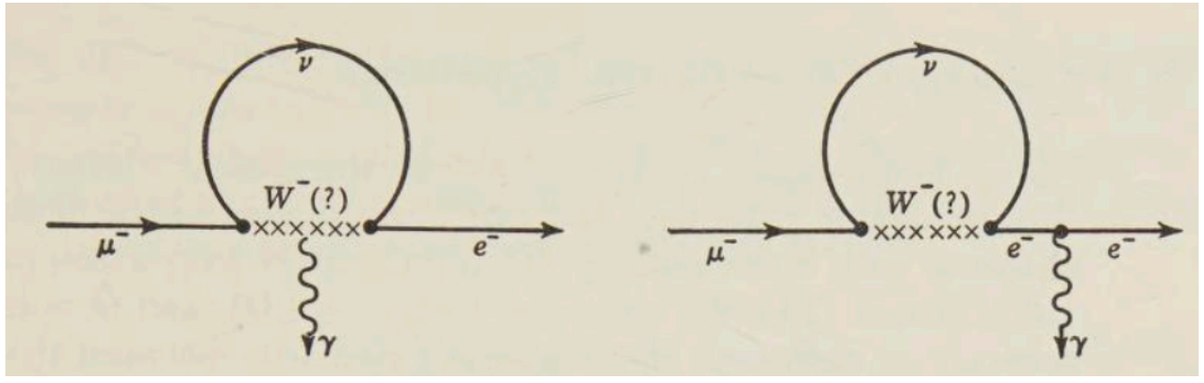

Fig. 10-20 Possible $\mu \rightarrow e + \gamma$ decay (for identical υ and υ').

Although the answers from these calculations diverge and are not to be taken too seriously, it was hard to push see the above rat below 10^{-4} to 10^{-5}.

These graphs evidently vanish, however, along with all others if the υ' associated with the μ^- is different from the υ associated with e^-.

A more reliable test for two neutrinos, proposed by Pontecorvo and Schwartz, involves initiating inverse β-decay reactions by a high-energy υ' beam from π decays.

In particular, one looks for high-energy electrons or μ mesons produced in the reactions

$$\nu' + \mathfrak{N} \rightarrow \mathfrak{N} + \mu \text{ or } e$$

Since μ production has been observed with certainty whereas no e events were identified, one now has positive evidence in favor of two neutrinos.

To the difficult question "What is the difference between μ and e other than rest mass and why did nature provide two charged leptons?" is now to be added "Why did nature bother with two neutrinos?"

10.16 Conserved Vector Current Hypothesis

The π^+ and μ^- meson decays, as well as the nuclear β decay, indicate that the lepton pairs (e,υ) and (u,υ') are coupled in the decay matrix elements with a common (V-A) form.

Moreover, a comparison of (10.112) and (10.129), with $\alpha = 1.21$, $\lambda = 1.00$, and $\tilde{G} \simeq G$ to high accuracy as discussed, shows a strong similarity between the nucleon and lepton couplings.

It would be natural to expect the cloud of strongly interacting π mesons which surrounds the physical nucleons, but not the leptons, to modify the strengths of the vector and axial vector parts of a coupling that is introduced for "bare" nucleons in the absence of such strong couplings.

It seems quite remarkable, then, that the strengths of the vector part of the interaction in β decay and μ decay are equal within 2 per cent.

Even the constant $\alpha = 1.21$ which gives the ratio of the axial part of the interaction for nucleons relative to leptons is sufficiently close to unity to encourage interesting speculations.

Our earlier discussions of the electromagnetic interactions of electrons in Chap. 8 and of protons in Sec. 10.9 provide an important clue for understanding the equality of the vector part of the interaction current for the nucleons and leptons.

In (8.50) and (8.57) we saw that the vertex function of an electron was modified by a factor Z_1^{-1} that was due to electromagnetic radiative corrections.

There also appears in the S-matrix elements, according to (8.46) and (8.57), an additional factor of Z_2 coming from renormalization of the electron wave functions owing to the self-energy insertions.

To lowest order in a, we found, from (8.54), that $Z_1 = Z_2$ so that these effects canceled; as remarked there, the identity $Z_1 = Z_2$ is valid to all orders and is a consequence of the Ward identity (8.51).1

A similar situation holds for the electromagnetic interaction of a proton.

The mesonic radiative corrections, as illustrated in Fig. 10.13 and Eq. (10.86), modify the electromagnetic vertex of a proton by a factor (infinite in perturbation theory) analogous to Z_1 which is again canceled by the mesonic renormalization of the proton wave functions.

This assertion can be verified to second order in the meson-nucleon coupling by showing that (8.51) remains valid in the presence of the meson-nucleon coupling.

From this identity of renormalization constants we come to the very important conclusion: the physical, observed charges of the electron and proton are equal in magnitude if their bare charges are equal, since the vacuum polarization (8.22) as computed for the photon propagator affects both the electron and proton in the same way.

This was already implicit in the discussion of the electromagnetic structure of the proton, where $F_1^{(p)}(0)$ was set equal to 1 in (10.89).

In β decay the form of the S-matrix element

$$\frac{G}{\sqrt{2}}\bar{\psi}_p\gamma_\mu\psi_n = \frac{G}{\sqrt{2}}\bar{\Psi}\gamma_\mu\tau_+\Psi \qquad (10.143)$$

with

$$\tau_+ = \tfrac{1}{2}(\tau_1 + i\tau_2)$$

used for the vector part of the interaction bears a strong resemblance to the electromagnetic vertex of a proton.

The fact that the mesonic radiative corrections do not alter the coefficient of γ_μ at zero momentum transfer makes this similarity even more striking.

The main difference is that the β-decay transition changes the charge by one unit.

Gerschtein and Zel'dovich' and later Feynman and Gell-Mann proposed that the vector "current" in β decay could be obtained from the electromagnetic current just by an isotopic rotation.

This is possible because of charge independence of the strong interactions.

Recalling from (10.89) that the electromagnetic current of nucleons may be written as a sum of an isotopic scalar and an isotopic vector term, the rule of Feynman and Gell-Mann is to take the isotopic vector part is and to replace it for β decay by

$$j_\mu^3 \to j_\mu^+ = \frac{G}{e\sqrt{2}}(j_\mu^1 + ij_\mu^2)$$

This rule is known as the "conserved vector current" hypothesis.

If there no radiative corrections, it reproduces the interaction (10.143).

With radiative corrections, the β-decay vector transition amplitude is obtained directly from (10.89), namely:

$$j_\mu^+ = 2\frac{G}{\sqrt{2}}\bar{\Psi}(p')\tau_+\left[\gamma_\mu F_1^{(v)}(q^2) + \frac{i\sigma_{\mu\nu}q^\nu}{2M}F_2^{(v)}(q^2)\right]\Psi(p) \qquad (10..144)$$

Since the form factors have been measured in electron-proton scattering and are essentially constant for $-q^2 << \mu^2$, they may be replaced by their values at $q^2 = 0$.

Equation (10.144) then becomes

$$j_\mu^+ = \frac{G}{\sqrt{2}}\,\bar{u}_p(p')\left[\gamma_\mu + \frac{3.70 i\sigma_{\mu\nu}q^\nu}{2M}\right]u_n(p) \tag{10.145}$$

We notice that this current is conserved, that is, by construction

$$(p' - p)^\mu j_\mu^+(q) = 0$$

In addition to ensuring the lack of renormalization of the coefficient of γ_μ in (10.145), we find an experimental consequence of the conserved vector current hypothesis in the second "weak magnetism" term.

It is, unfortunately, of the same order of magnitude as the first-order nucleon recoil corrections to β decay and is very difficult to observe.

However, a careful and beautiful experiment which analyzes the β spectra in the decays of B^{12} and N^{12} into C^{12} has established the existence of the weak magnetism term.

The rules for Feynman graphs according to the conserved current hypothesis may be inferred from those for electromagnetism.

First of all, there is a one-to-one correspondence between graphs in β decay and electromagnetic interactions.

For instance, the graphs of Fig. 10.21 replace those of Fig. 10.13 for the electromagnetic structure.

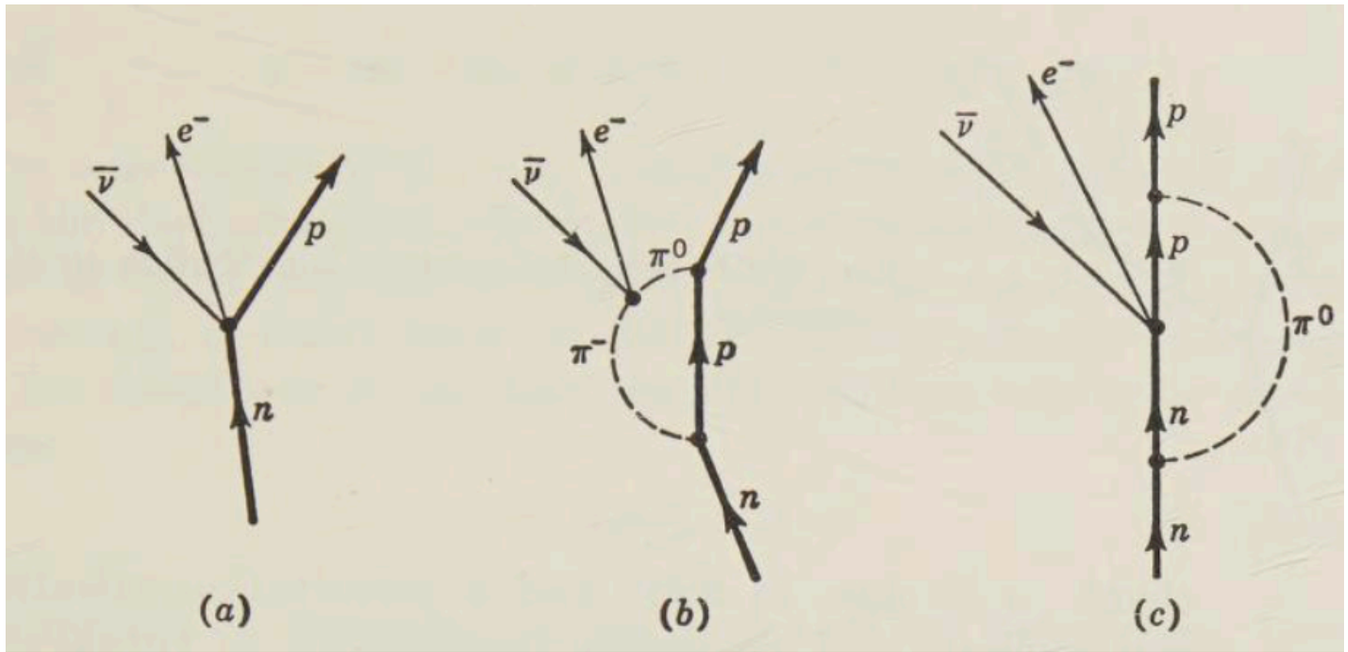

Fig. 10-21 Vector part of the weak-interaction vertex according to the conserved vector current hypothesis.

In Fig. 10.21a and 10.21c, the factor $e(1 + \tau_3)/2$ for electromagnetism is replaced by $(G/\sqrt{2})\tau_+$ according to our rule.

Figure 10.21b is an additional direct "pion weak current" term which is again determined by rotating the pion electromagnetic current in isotopic spin space.

The pion electromagnetic vertex is found by regarding the current in (10.37) as a transition matrix element; the vertex in momentum space is then

$$j_\mu{}^3 = -ie(\hat{\boldsymbol{\phi}}' \times \hat{\boldsymbol{\phi}})_3(p'_\mu + p_\mu) \tag{10.146}$$

where $\hat{\boldsymbol{\phi}}$ and $\hat{\boldsymbol{\phi}}'$, p and p' are the isotopic wave functions and momenta of initial and final mesons.

The apparent factor 2 difference between (10.146) and (10.37) arises because of the two ways of associating the wave functions $\boldsymbol{\phi}(x)$ in (10.37) with initial and final particles.

To obtain the vertex for β decay, we again replace the three-component by the (+) component and e by $G/\sqrt{2}$; ; the pion vertex in Fig. 10.21b will consequently have the form

$$j_\mu^+ = -\frac{iG}{\sqrt{2}}[(\hat{\phi}' \times \hat{\phi})_1 + i(\hat{\phi}' \times \hat{\phi})_2](p'_\mu + p_\mu) \tag{10.147}$$

The existence of this interaction term leads to another experimental consequence, namely, the existence of the reaction

$$\pi^+ \rightarrow \pi^0 + e^- + \bar{\nu} \tag{10.148}$$

The matrix element in the presence of strong interactions is again related to the electromagnetic current of the pion (10.85) by an isotopic rotation.

Therefore, mesonic radiative corrections modify (10.147) only by the inclusion of the pion form factor $F_\pi(q^2)$, which for the small values of q^2 present in the reaction (10.148) can be set equal to 1.

The rate of this reaction can then be calculated from the vertex (10.147), which leads to the unique, but difficult to measure,' prediction

$$\frac{R(\pi^- \rightarrow \pi^0 + e^- + \bar{\nu})}{R(\pi^- \rightarrow \mu^- + \bar{\nu})} = 1.0 \times 10^{-8} \tag{10.149}$$

The conserved vector current hypothesis may then be summarized as follows.

The lepton current $\bar{\psi}_e \gamma_\mu (1 - \gamma_5) \psi_\nu$, is to interact with the (+) component of the conserved isotopic spin current (10.43), treated as a transition current just as it was for electromagnetism.

Experimental consequences result because this (+) component is determined from the third component by the charge independence of the strong interactions; the third component is measurable by means of electromagnetic interactions.

10.17 "Partially Conserved" Axial Vector Coupling

The meson cloud about the nucleons will also affect the axial vector, or Gamow-Teller, part of the β-decay interaction.

We may interpret the number $\alpha = 1.21$ relating the strength of the axial vector to the vector coupling constant as coming from the effects of this meson cloud.

α is close to unity and a perturbation calculation of the renormalization of a coupling strength due to virtual cloud effects gives a logarithmically diverging number.

This suggests that perhaps an approximate conservation law is operating for the axial vector β-decay coupling.

At this time little progress has been made in explaining the magnitude of α on the basis of this vague idea or any other idea, and we shall not consider a further.

However, the idea that the leptons couple to a "partially" conserved axial vector current of the nucleons in the weak decay amplitude has had some success in calculating the observed lifetime of π^+ decay, as we now discuss.

The simplest radiative correction to the axial vector current is that involving a single a meson, as shown in Fig. 10.22.

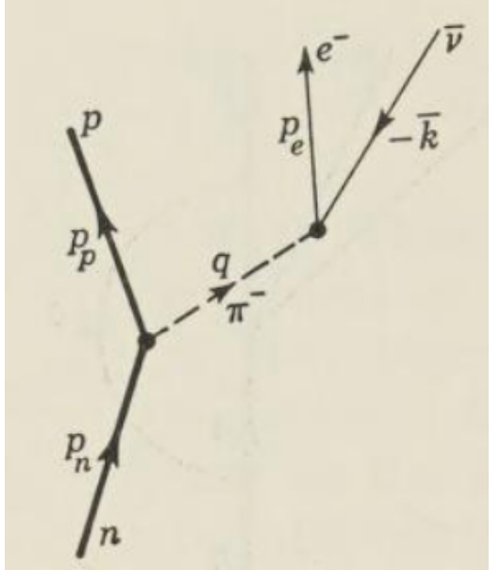

Fig. 10-22 Single π-exchange contribution to the weak axial interaction.

According to our rules, it contributes a term to the invariant amplitude for β decay

$$\mathfrak{M}_{1\pi} = \frac{Ga}{\sqrt{2}}(-ig\sqrt{2})[\bar{u}(p_p)i\gamma_5 u(p_n)]\frac{i}{q^2-\mu^2} \times (iq_\mu)[\bar{u}(p_e)\gamma^\mu(1-\gamma_5)v(\bar{k})] \tag{10.150}$$

where $Ga/\sqrt{2}$ is the coupling constant in (10.139) for π^+ decay and g is the π-N strong interaction coupling constant.

The additional $\sqrt{2}$ comes from the isotopic matrix for charged i emission.

The kinematics is illustrated in Fig. 10.22.

There are many additional contributions to first order in the weak couplings coming from such diagrams as shown in Fig. 10.23.

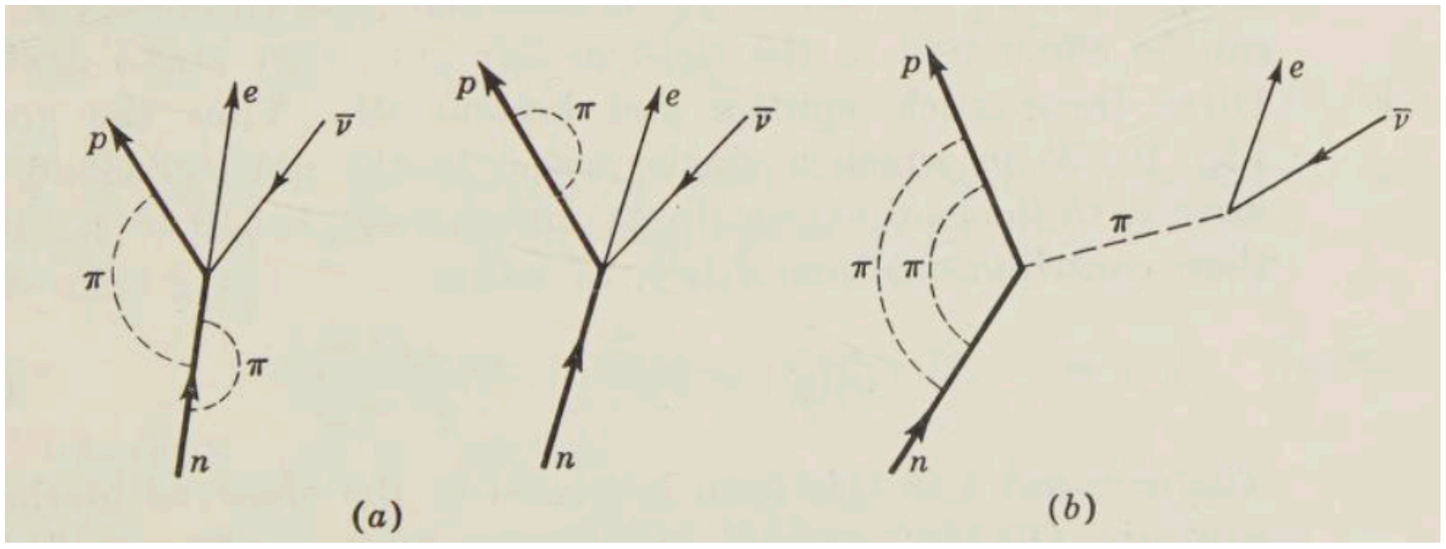

Fig. 10-23 Weak axial contributions to β decay.

All contributions from the diagrams of Fig 10.23 can be written in the form

$$\mathfrak{M} = \mathcal{J}_\mu^+(p_p,p_n)\bar{u}(p_e)\gamma^\mu(1-\gamma_5)v(\bar{k})$$

with

$$\mathcal{J}_\mu^+(p_p,p_n) = \frac{G}{\sqrt{2}}\bar{u}(p_p)[\gamma_\mu\gamma_5\mathcal{F}_1(q^2) + q_\mu\gamma_5\mathcal{F}_2(q^2) + P_\mu\gamma_5\mathcal{F}_3(q^2)]u(p_n) \qquad (10.151)$$

and

$$q = p_n - p_p = p_e + \bar{k} \qquad P = p_p + p_n$$

The construction of this form is similar to the construction of (10.87) for the electromagnetic current, the only difference being the insertion here of the γ_5 to form an axial vector.

If we assume that all contributions to $\mathcal{J}_\mu^+$, as is the case in Fig. 10.23, transform as the (+) component of an isotopic vector, we may simplify this form further by showing that

$$\mathcal{F}_3(q^2) = 0 \qquad (10.152)$$

on grounds of charge conjugation invariance and isotopic spin invariance of the strong interactions.

To show this, we first rotate $\mathcal{J}_\mu^+$ in isotopic space by changing the τ_+ in the weak vertices of the diagrams of Fig. 10.23a to τ_3 and, in Fig. 10.236, changing the τ_+ at the vertex for emission of the π^- meson which couples to the leptons to a τ_3.

Owing to the charge independence of the strong interactions, this transforms $\mathcal{J}_\mu^+$ into the third component of an isotopic vector; in particular for a proton

$$\mathcal{J}_\mu^3(p',p) = \frac{G}{\sqrt{2}}\bar{u}(p')[\gamma_\mu\gamma_5\mathcal{F}_1(q^2) + q_\mu\gamma_5\mathcal{F}_2(q^2) + P_\mu\gamma_5\mathcal{F}_3(q^2)]u(p) \qquad (10.153)$$

with

$$q_\mu = p_\mu - p'_\mu \qquad P_\mu = p'_\mu + p_\mu$$

According to charge conjugation invariance of the strong couplings, the additional contributions to (10.151) coming from the diagrams of Fig. 10.23 must lead to a $\mathcal{J}_\mu^3(p',p)$ which transforms, if the proton is replaced by an antiproton, exactly as the current of the "bare" proton

$$\bar{u}(p')\gamma_\mu\gamma_5 u(p) \tag{10.154}$$

transforms.

For the charge conjugate transition of an antiproton scattering from momentum p to p' we have the diagram of Fig. 10.24 corresponding to a negative-energy proton running backward in time to be scattered from -p' to -p as discussed in Chap. 6; its axial vector current is

$$\begin{aligned}\bar{v}(p)\gamma_\mu\gamma_5 v(p') &= -u^T(p)C^{-1}\gamma_\mu\gamma_5 C\bar{u}^T(p')e^{i\phi} \\ &= -\bar{u}(p')\gamma_\mu\gamma_5 u(p)e^{i\phi}\end{aligned} \tag{10.155}$$

where we have used (5.8).

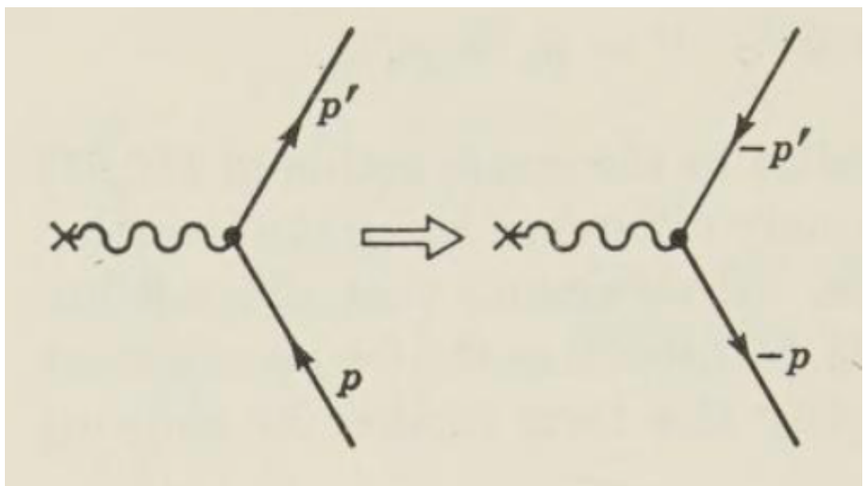

Fig. 10-24 Weak-interaction vertex of an antiproton.

The phase factor is determined by external momenta and spins as we saw before (5.8).

We must retain in (10.153) only those terms with the same transformation properties, that is,

$$\mathcal{J}_\mu^{3c}(-p,-p') = -\mathcal{J}_\mu^{3}(p',p)e^{i\phi} \tag{10.156}$$

From this it is easy to show that (10.152) follows.

The contributions to the axial vector part of the β-decay amplitude coming from diagrams like Fig. 10.23b can all be written in the same form as (10.150) multiplied by a scalar function of q^2.

All modifications of the π-nucleon vertex will lead to an interaction of the form

$$\bar{u}(p_p)\gamma_5\mathfrak{F}(q^2)u(p_n)$$

with $q \equiv p_n - p_p$ and $\mathfrak{F}(q^2)$ an invariant function of the invariant momentum transfer q^2.

This follows from the fact that there are always an odd number of γ_5 vertices and that all factors $\not{p}_p$ and $\not{p}_n$ can be commuted to the right or left until they stand next to the Dirac free-particle spinors and become M.

Thus the graphs of Fig. 10.23b in which a single meson in the nucleon cloud couples directly to the leptons contribute only to $\mathfrak{F}_2(q^2)$ in (10.151). Isolating these contributions from $\mathfrak{F}_2(q^2)$, we write

$$\mathfrak{F}_2(q^2) = \tilde{\mathfrak{F}}_2(q^2) - \frac{ag\sqrt{2}\,\mathfrak{F}(q^2)}{q^2-\mu^2} \tag{10.157}$$

The constant α in this term is related to the observed lifetime of decay by (10.140), and the form factor $\mathfrak{F}(q^2)$ can be specified at the point $q^2 = \mu^2$ in terms of the observed strength of the π-nucleon coupling constant g as discussed for pion-nucleon scattering in Sec.10.8.

With the g in (10.157) taken to be the physically observed π-N coupling constant $g^2/4\pi \simeq 14$, $\mathfrak{F}(q^2)$ is normalized at $q^2 = \mu^2$ to 1:

$$\mathfrak{F}(\mu^2) = 1 \tag{10.158}$$

Nothing is known about $\tilde{\mathfrak{F}}_2$. However, because $\mathfrak{F}_2$ is the coefficient of $q_\mu \gamma_5$, which is a recoil correction $\sim(\mathbf{q}/M)$, it has not been observed in β decay.

From our previous discussion we know that

$$\mathfrak{F}_1(0) = -\alpha = -1.21 \tag{10.159}$$

With these preliminaries, we may attack the question of the partially conserved axial vector current.

If the axial vector current were precisely conserved, the condition

$$q^\mu \mathcal{J}_\mu^+(p_p, p_n) = 0$$

applied to (10.151) leads to the relation

$$2M\mathfrak{F}_1(q^2) - q^2\mathfrak{F}_2(q^2) = 0$$

or

$$\mathfrak{F}_2(q^2) = +\frac{2M\mathfrak{F}_1(q^2)}{q^2}$$

Since $\mathfrak{F}_1(0) \neq 0$, this means that $\mathfrak{F}_2$ would have a pole at $q^2 = 0$, corresponding to the exchange of a massless pseudoscalar particle.

It is tempting to associate this pole with the π-meson pole in (10.157) and in addition to associate the breakdown of exact current conservation with the existence of a mass for the π meson.

We are thus led, using (10.151) and (10.157), to the modified hypothesis

$$0 = \lim_{\mu\to 0} q^\mu \mathcal{J}_\mu^+(p_p,p_n)$$
$$= \lim_{\mu\to 0} \bar{u}(p_p)\gamma_5 \left[-2M\mathcal{F}_1(q^2) + q^2\tilde{\mathcal{F}}_2(q^2) - \frac{ag\sqrt{2}\, q^2\mathcal{F}(q^2)}{q^2-\mu^2} \right] u(p_n) \tag{10.160}$$

With the additional assumption that the invariant form factors are changed little from their physical values by the limit $\mu^2 \to 0$, we find from (10.160)

$$-2M\mathcal{F}_1(0) = +2M\alpha = +2M(1.21) \approx ag\sqrt{2} \tag{10.161}$$

Numerically (10.161) predicts

$$|a| \approx 0.87\mu$$

which agrees with that obtained from the observed $\pi^\pm$ lifetime, Eq.(10.141), to better than 10 per cent.

Such a relation between the decay rate of a m meson, the Fermi constant G, and the π-nucleon coupling constant for the strong interactions was derived first by Goldberger and Treiman from an approximate dispersion theory calculation.

It was subsequently discussed as a consequence of a "partially conserved" axial vector current by Nambu and Bernstein et al.