Introductory Mechanics

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1. Getting Started

We now begin your journey along the path to becoming a physicist, astrophysicist or astronomer. I am a theoretical physicist and that will be reflected in my approach to teaching this course and to helping you understand the physics of classical particle mechanics.

So that we can talk to each other about this subject, we must first spend some time discussing the **language** of particle mechanics and of physics in general. It is important to realize that **the language of physics is mathematics**.

It might be possible to have some understanding of early topics in your study of physics using models and everyday experience along with using a non-mathematical language (words). But this approach will fail miserably as soon as you start to investigate modern areas of physics involving quantum mechanics, statistical mechanics and non-linear behavior. As we delve deeper into physical phenomena we will find that making models of any kind will fail to bring any understanding at all. So we need to get into the habit of using mathematics from the beginning.

In my view, the **only way** to understand the universe that physicists are presently studying is using mathematics.

In this course we will use differential and integral calculus, simple differential equations, vector analysis, Cartesian and plane-polar coordinate systems, difference equations, computational numerical methods and simulation techniques. In later courses, you will use techniques from linear algebra, matrix theory, multivariable calculus, Fourier theory, partial differential equations, modern algebra, group theory, complex analysis, tensor analysis, geometry and topology. The physicist, especially, the theoretical physicist, must be equally adept at mathematics and physics. There is no way to separate the two subjects if we want to understand the universe at a fundamental level. I will expect that you know how to do derivative, and integrals. For any other mathematics that I intend to use in my lectures, I will discuss it in class first.

I will start the physics from square zero. I do this so that we are sure that the basic material is understood by everyone in the same way. At the beginning, we will often be discussing or defining what seems to be obvious quantities. This is very important, however, since if we do not understand and properly define the fundamental quantities in the theory we have no hope of using them properly to understand physical systems. As I said, I am a theoretical physicist and my bias about the importance of understanding the basis of the theory at every stage will be reflected in the way we pursue the physics involved in this course.

1.1. Units and Dimensions

Theoretical physics cannot be done in a room without windows, that is, theoretical physics is based on the observations carried out by experimental physicists. For example, without any observations of the universe I might, by accident, think up a theory like special relativity, but there is no way that I could set the scale associated with the theory in this universe without actually measuring the speed of light c. In the same manner we could not understand the details of gravity without measuring the gravitational constant G, of quantum mechanics without measuring Planck's constant $\hbar$, or of electrodynamics without measuring the charge of the electron e.

The standard set of units used in physics is the M(eter) K(ilogram) S(econd) system or as it is now called the S(ysteme)I(nternational) d?Unites. It is based on three quantities

$$\begin{aligned} \text{meter} &= \text{unit of the dimension length} \\ \text{kilogram} &= \text{unit of the dimension mass} \\ \text{second} &= \text{unit of the dimension time} \end{aligned}$$

In addition, we need to define an unit of charge

$$\text{coulomb} = \text{unit of the dimension charge}$$

All other units for physical quantities are derived from this basic set of four definitions.

1.2. Some History of Units

Time

???? 1 second $= \dfrac{1}{86400}$ of the mean solar day

1956 1 second $= \dfrac{1}{31566926}$ of the year 1900

1967 1 second = 9192631770 oscillations or periods of the radiation corresponding to atomic transition between two hyperfine levels of Cs^{133} atom.

These first two entries are accurate standards only if the rotation of the earth around its axis or about the sun is constant and uniformly circular. It is neither!!

The last entry indicates that the second is now defined in terms of the oscillations of a Cs^{133} atomic clock.

Length

1791 1 meter $= \dfrac{1}{10000000}$ of the mean circumference of the earth

1999 1 meter = length of standard platinum-iridium bar supported properly in vacuum at a definite temperature and pressure

1960 1 meter = 1650763.73 wavelengths of orange-red light emitted by a Kr^{86} lamp

1983 speed of light defined to be $c = 299792458\ m/sec$

1 meter = distance traveled by light in $\dfrac{1}{299792458}$ second

The standard of length now **depends** on the standard of time.

Mass

1 kilogram = amount of stuff in standard object in a vacuum at definite temperature and pressure

Charge

1 coulomb = defined by force between two wires each carrying one ampere of current

e = charge on the electron(proton) $= 1.602 \times 10^{-19}$ coulomb

Standard Units

Quantity	MKS(SI)	cgs	English
time	second	second	second
length	meter	centimeter:100 cm = m	foot:3.281 f = m
mass	kilogram	gram:1000 gm = kg	pound:2.205 lb = kg
charge	coulomb	statcoulomb	-

Metric Prefixes

Metric Prefix Table

Prefix	Symbol	Multiplier	Exponential
yotta	Y	1,000,000,000,000,000,000,000,000	10^{24}
zetta	Z	1,000,000,000,000,000,000,000	10^{21}
exa	E	1,000,000,000,000,000,000	10^{18}
peta	P	1,000,000,000,000,000	10^{15}
tera	T	1,000,000,000,000	10^{12}
giga	G	1,000,000,000	10^{9}
mega	M	1,000,000	10^{6}
kilo	k	1,000	10^{3}
hecto	h	100	10^{2}
deca	da	10	10^{1}
		1	10^{0}
deci	d	0.1	10^{-1}
centi	c	0.01	10^{-2}
milli	m	0.001	10^{-3}
micro	µ	0.000001	10^{-6}
nano	n	0.000000001	10^{-9}
pico	p	0.000000000001	10^{-12}
femto	f	0.000000000000001	10^{-15}
atto	a	0.000000000000000001	10^{-18}
zepto	z	0.000000000000000000001	10^{-21}
yocto	y	0.000000000000000000000001	10^{-24}

Greek Letters

α = *alpha*

β = *beta*

γ, Γ = *gamma*

δ, Δ = *delta*

π = *pi*

ε = *epsilon*

ρ = *rho*

ν = *nu*

ζ = *zeta*

σ, Σ = *sigma*

η = *eta*

τ = *tau*

θ, Θ = *theta*

ι = *iota*

ϕ, Φ = *phi*

κ = *kappa*

χ = *chi*

λ, Λ = *lambda*

ψ, Ψ = *psi*

μ = *mu*

ω, Ω = *omega*

ξ, Ξ = *xi*

o = *omnicron*

υ = *upsilon*

2. Mathematical Preliminaries

2.1. Coordinate Systems or Reference Frames

The first thing we must do when attempting to study any physical system is to define a coordinate system or reference frame, which allows us to relate theoretical concepts to experimental measurements.

The first object we choose in a coordinate system is the **origin**. The location of the origin is arbitrary and is determined, in general, by the system under consideration. We generally choose the origin so that our equations are as simple as possible and extraneous complications not relevant to the physics involved are eliminated.

Experimental evidence says that no physics can depend on the choice of the origin.

Next we choose **coordinate axes**. The number of coordinates needed to describe a physical system and hence the number of coordinate axes that will be needed is determined by the **physical dimension** of the space associated with the system. During this course we will be able to do most problems in 1 or 2 dimensions with the occasional excursion into 3 or 4 dimensions. More exotic physical systems generally have more physical dimensions than this(up to 26 dimensions in modern particle physics).

2.2. Cartesian Coordinate Axes

1 Dimension: 1 coordinate axis = origin + line through origin in direction of motion + scale(arbitrary) = x-axis as shown in figure 1:

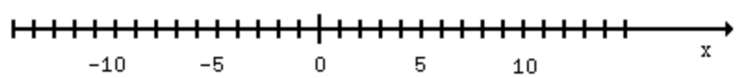

Figure 1:

No general rules follow from the 1 dimensional case.

2 Dimensions: Each coordinate axis is the line defined by setting **all** other coordinates = 0.

$$x\text{-axis} = \text{the line } y = 0 \quad , \quad y\text{-axis} = \text{the line } x = 0$$

In general, the axes are chosen to be orthogonal(perpendicular). This does not affect the physics involved, but does lead to great simplification in the algebra involving vectors, as we shall see. We choose the so-called **right-handed** arrangement of the axes.

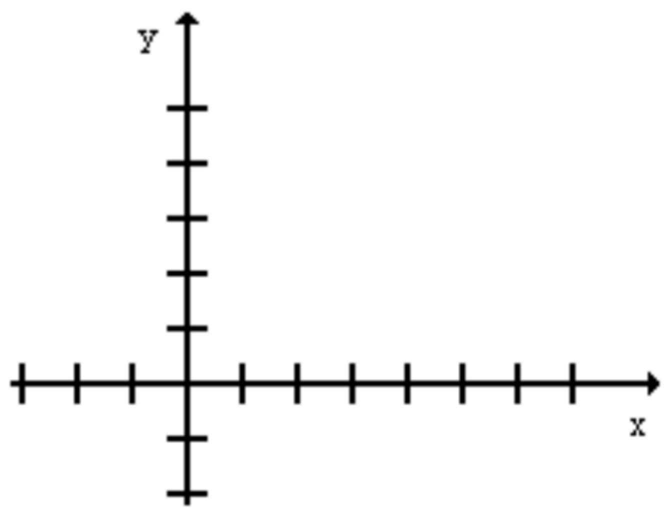

Figure 2:

3 dimensions: Again, each coordinate axis is the line defined by setting **all** other coordinates = 0.

$$\begin{aligned} x\text{-axis} &= \text{the line } y = 0 \textbf{ and } z = 0 \\ y\text{-axis} &= \text{the line } x = 0 \textbf{ and } z = 0 \\ z\text{-axis} &= \text{the line } x = 0 \textbf{ and } y = 0 \end{aligned}$$

Again, we choose the so-called **right-handed** arrangement

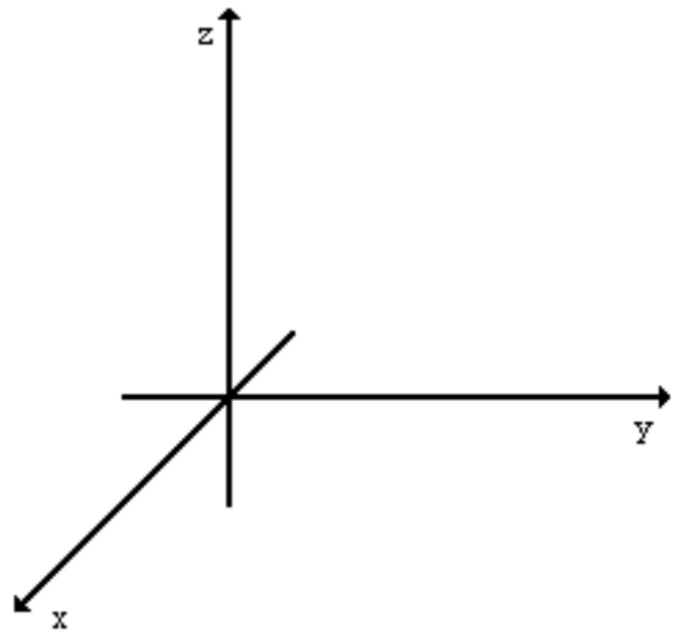

Figure 3:

2.3. Vectors

We use the arrow notation $\vec{V}$ to designate vectors.

A vector can be defined in several equivalent ways:

Length/Direction

Geometrically, a vector is a **directed line segment**. It is thus, a quantity that has a **magnitude** and a **direction**. We often represent it as an **arrow**.

In order to completely specify a vector we require n pieces of information where n = dimension of the space.

In particular, we have

1 dimensional vector = **(± length of vector)** where ± depends on direction = **1 number**.

2 dimensional vector = **(length, angle)** where angle is measured from the x-axis = **2 numbers**.

3 dimensional vector = **(length, 2 angles)** = **3 numbers**.

We now discuss the **algebra** of vectors.

If 2 vectors have the same length and direction they are **equal** (as shown in figure 4):

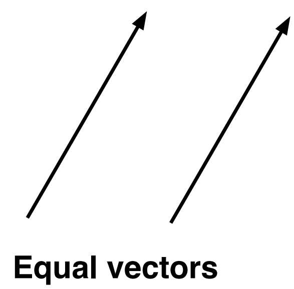

Figure 4:

A very important question that we will discuss in class is : How would you actually compare the two vectors to see that they are equal? It is not as obvious as you might think!

The **inverse** of a vector has the same length and opposite direction (as shown below) and a vector **multiplied** by a number has its length multiplied by the number and the same direction(as shown in figure 5).

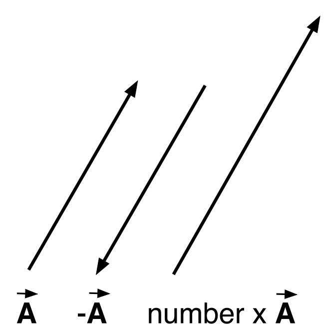

Figure 5:

Addition of two vectors is defined by the **geometric rule** shown in figure 6:

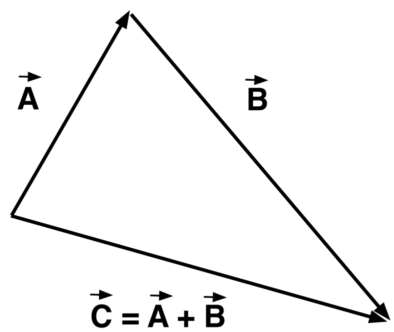

Figure 6:

Difference(subtraction) of two vectors is defined by the **geometric rule** shown in figure 7:

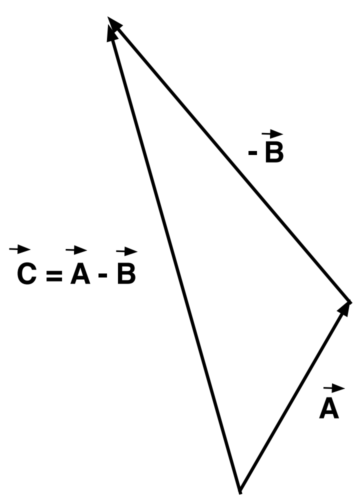

Figure 7:

Alternatively, we can simply define a vector as a **mathematical object** represented by a set of numbers.

A 1-dimensional vector is a single number with its sign indicating the direction of the vector(there are only two directions in 1 dimension) and its magnitude designating the length of the vector as shown figure 8.

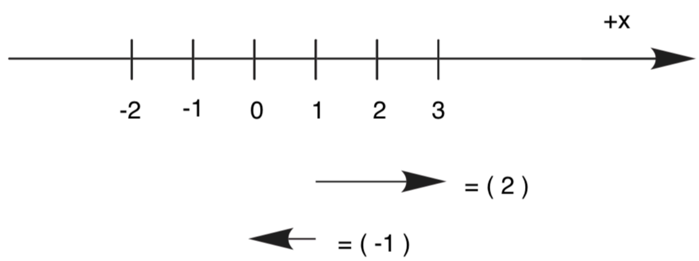

Figure 8:

Definition: n-tuple - An **n-tuple** is a set of n numbers, i.e., a 4-tuple = (2,3,6,1).

1 dimensional vector = (± **length of vector**) where ± depends on direction = **1-tuple**

A 2-dimensional vector is a set of two numbers. The first number represents a vector in the 1-direction or the x-direction and the second number representing a vector in the 2-direction or the y-direction. These could be any two perpendicular directions. The vector represented by the two numbers is the vector sum of the two vectors along the axes as shown in figure 9.

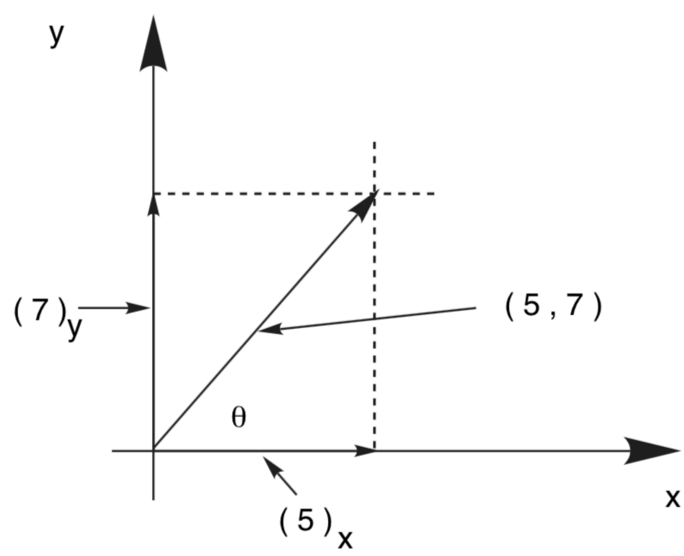

Figure 9:

2 dimensional vector = (**length, angle**) where angle is measured from the x -axis
= **2-tuple** = $(\text{amount}_1, \text{amount}_2)$ - these are equivalent

The length of the vector is defined to be the square root of the sum of the squares of the two numbers ($\sqrt{7^2 + 5^2}$) and the direction is specified by the angle θ where $\tan\theta = \frac{7}{5}$. These simple results only hold for Cartesian axes that are orthogonal(perpendicular).

Finally, a **3 dimensional vector = (length, 2 angles) = 3-tuple**.

Although representing the mathematical essence of the vector, the sets of numbers or n-tuples are not convenient for the physics of particle mechanics (they will be more appropriate when one studies quantum theory). Instead we will develop general vector properties and equations in terms of special vectors called **unit vectors**.

Unit Vectors

Unit vectors are vectors with length = 1 (we use the notation ˆ to designate unit vectors). An arbitrary vector $\vec{A}$ of length A in the direction of a particular unit vector $\hat{u}$ is designated by $\vec{A} = A\hat{u}$.

We can construct a unit vector in the direction of any given vector $\vec{A}$ by the construction $\hat{u} = \frac{\vec{A}}{A}$, where we have defined $A = |\vec{A}|$ = length of the vector $\vec{A}$.

In order to proceed deeper we must define the concept of **basis vectors**.

Basis vectors definition

A set of unit vectors is called a **basis** if **any other vector** in the space can be written in terms of the basis set. In particular, we mean that any other vector in the space can be written as the sum vectors which are multiples of the basis vectors.

Determination of a General Vector

It is clear that we can always do this with a set of unit vectors aligned along the Cartesian axes.

We define the set of 3 Cartesian basis vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3 = \hat{i}, \hat{j}, \hat{k} = \hat{x}, \hat{y}, \hat{z}$, which are oriented as shown in figure 10:

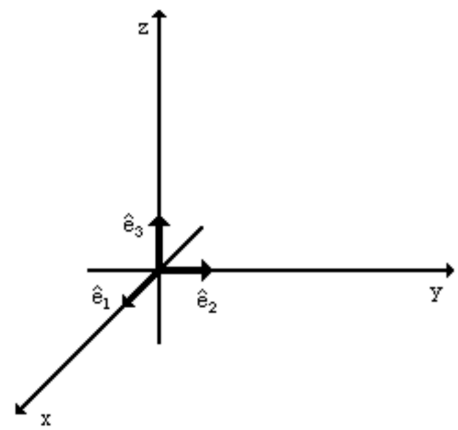

Figure 10:

Then any arbitrary vector $\vec{r}$ can be constructed from these three unit vectors shown in figure 11

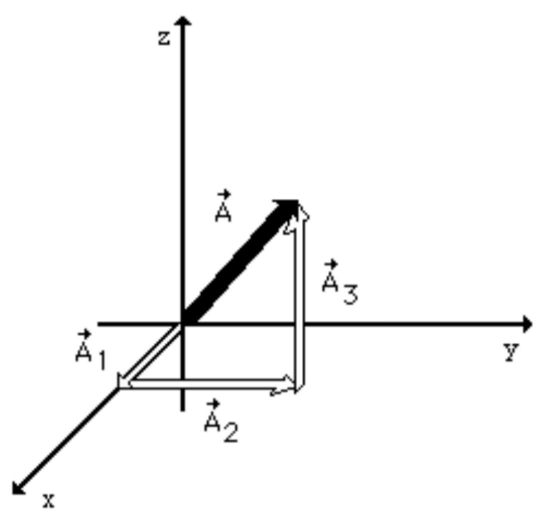

Figure 11:

It is clear from this direct construction that

$$\vec{A} = \vec{A}_1 + \vec{A}_2 + \vec{A}_3 = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3 = \sum_{i=1}^{3} A_i\hat{e}_i \tag{2.1}$$

where the quantities A_1, A_2, A_3 are the lengths of the vectors $\vec{A}_1, \vec{A}_2, \vec{A}_3$ and that given this set of Cartesian basis vectors we can easily construct any other vector in the space as shown in figure 12.

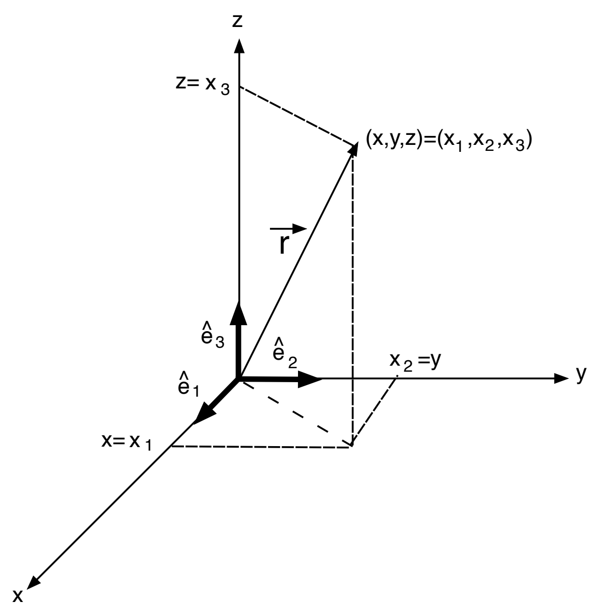

Figure 12:

Components - The three numbers (A_1, A_2, A_3) are called the **components** of the vector $\vec{A}$ along the respective Cartesian directions..

Vector Algebra

In terms of components we can easily do **vector algebra**:

$$\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3 = \sum_{i=1}^{3} A_i\hat{e}_i = (A_1, A_2, A_3)$$
$$\vec{B} = B_1\hat{e}_1 + B_2\hat{e}_2 + B_3\hat{e}_3 = \sum_{i=1}^{3} B_i\hat{e}_i = (B_1, B_2, B_3)$$
$$-\vec{A} = -A_1\hat{e}_1 - A_2\hat{e}_2 - A_3\hat{e}_3 = -\sum_{i=1}^{3} A_i\hat{e}_i = (-A_1, -A_2, -A_3)$$
$$c\vec{A} = cA_1\hat{e}_1 + cA_2\hat{e}_2 + cA_3\hat{e}_3 = c\sum_{i=1}^{3} A_i\hat{e}_i = (cA_1, cA_2, cA_3)$$

All the forms used above are equivalent. Then we have

$$\begin{aligned}\vec{A} + \vec{B} &= A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3 + B_1\hat{e}_1 + B_2\hat{e}_2 + B_3\hat{e}_3 \\ &= (A_1 + B_1)\hat{e}_1 + (A_2 + B_2)\hat{e}_2 + (A_3 + B_3)\hat{e}_3 \\ &= \sum_{i=1}^{2}(A_i + B_i)\hat{e}_i = (A_1 + B_1, A_2 + B_2, A_3 + B_3) \end{aligned} \tag{2.2}$$

$$\begin{aligned}\vec{A} - \vec{B} &= A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3 - B_1\hat{e}_1 - B_2\hat{e}_2 - B_3\hat{e}_3 \\ &= (A_1 - B_1)\hat{e}_1 + (A_2 - B_2)\hat{e}_2 + (A_3 - B_3)\hat{e}_3 \\ &= \sum_{i=1}^{2}(A_i - B_i)\hat{e}_i = (A_1 - B_1, A_2 - B_2, A_3 - B_3) \end{aligned} \tag{2.3}$$

Thus, vector addition and subtraction are easily done in terms of components.

We now define two distinct and very useful **products of vectors**. We will define these new products in terms of their effects on the basis vectors of the space. All properties of general vectors can then be derived from these defining relations as we will see.

Dot or Scalar Product $(\cdot)$

In terms of basis vectors we define the scalar product operation(symbol = $\cdot$) by the relations:

$$\hat{e}_1 \cdot \hat{e}_1 = \hat{e}_2 \cdot \hat{e}_2 = \hat{e}_3 \cdot \hat{e}_3 = 1 \tag{2.4}$$

and

$$\hat{e}_1 \cdot \hat{e}_2 = \hat{e}_2 \cdot \hat{e}_1 = \hat{e}_3 \cdot \hat{e}_1 = \hat{e}_1 \cdot \hat{e}_3 = \hat{e}_2 \cdot \hat{e}_3 = \hat{e}_3 \cdot \hat{e}_2 = 0 \tag{2.5}$$

These are cumbersome relations. We can compact the relations by defining a new mathematical object.

Kronecker Delta (δ_{ij})

If we define a new mathematical object called the **Kronecker delta** by the relationship

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{2.6}$$

then the set of relations defining the scalar product is written concisely as

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij} \text{ for } i, j = 1, 2, 3 \tag{2.7}$$

This new mathematical object has some very useful properties. In particular,

$$\sum_j A_j \delta_{ij} = A_i \leftrightarrow \sum_j A_j \delta_{2j} = \delta_{21} A_1 + \delta_{22} A_2 + \delta_{23} A_3 = A_2$$
$$\sum_j A_j B_k \delta_{jk} = A_k B_k \leftrightarrow \sum_j A_j B_3 \delta_{j3} = A_1 B_3 \delta_{13} + A_2 B_3 \delta_{23} + A_3 B_3 \delta_{33} = A_3 B_3$$

Relations for General Vectors

Now consider two general vectors defined by

$$\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3 = \sum_{i=1}^{3} A_i \hat{e}_i = (A_1, A_2, A_3)$$
$$\vec{B} = B_1 \hat{e}_1 + B_2 \hat{e}_2 + B_3 \hat{e}_3 = \sum_{i=1}^{3} B_i \hat{e}_i = (B_1, B_2, B_3)$$

Note the **dummy** indices, i.e.,

$$\sum_{i=1}^{3} B_i \hat{e}_i = \sum_{j=1}^{3} B_j i \hat{e}_j = \sum_{m=1}^{3} B_m \hat{e}_m$$

All of these expression are the same formula!

We then obtain using the defining relations

$$\begin{aligned}\vec{A}\cdot\vec{B} =&(A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3)\cdot(B_1\hat{e}_1 + B_2\hat{e}_2 + B_3\hat{e}_3)\\ =&A_1B_1\hat{e}_1\cdot\hat{e}_1 + A_2B_1\hat{e}_2\cdot\hat{e}_1 + A_3B_1\hat{e}_3\cdot\hat{e}_1\\ &+ A_1B_2\hat{e}_1\cdot\hat{e}_2 + A_2B_2\hat{e}_2\cdot\hat{e}_2 + A_3B_2\hat{e}_3\cdot\hat{e}_2\\ &+ A_1B_3\hat{e}_1\cdot\hat{e}_3 + A_2B_3\hat{e}_2\cdot\hat{e}_3 + A_3B_3\hat{e}_3\cdot\hat{e}_3\\ =&A_1B_1 + A_2B_2 + A_3B_3\end{aligned} \tag{2.8}$$

or using the full power of the new mathematical notation(hold onto your hats!!)

$$\vec{A}\cdot\vec{B} = \left(\sum_{i=1}^{3} A_i\hat{e}_i\right)\cdot\left(\sum_{j=1}^{3} B_j\hat{e}_j\right) = \sum_{i=1}^{3}\sum_{j=1}^{3} A_iB_j\hat{e}_i\cdot\hat{e}_j = \sum_{i=1}^{3}\sum_{j=1}^{3} A_iB_j\delta_{ij} = \sum_{j=1}^{3} A_jB_j \tag{2.9}$$

Note that

$$\vec{A}\cdot\vec{A} = \sum_{j=1}^{3} A_jA_j = A_1^2 + A_2^2 + A_3^2 = A^2 = \left(\text{length of } \vec{A}\right)^2 \tag{2.10}$$

or

$$\text{length of } \vec{A} = |\vec{A}| = A = \sqrt{\vec{A}\cdot\vec{A}} = \sqrt{A_1^2 + A_2^2 + A_3^2} \tag{2.11}$$

as usual.

Using the scalar product we can give a **general definition for the component** of a vector along a particular direction.

$$\vec{A}\cdot\hat{e}_j = \left(\sum_{i=1}^{3} A_i\hat{e}_i\right)\cdot\hat{e}_j = \sum_{i=1}^{3} A_i(\hat{e}_i\cdot\hat{e}_j) = \sum_{i=1}^{3} A_i\delta_{ij} = A_j \tag{2.12}$$

that is, the component of a vector in a given direction is the scalar product of the vector with the unit vector in that direction.

Other Properties

Since any two vectors define a single plane, we can restrict our attention to only that plane and write both vectors as 2-dimensional vectors in the plane in that case. Assume that we have chosen the x- and y-axes to lie in this plane. In this case, we have the situation as shown in figure 13:

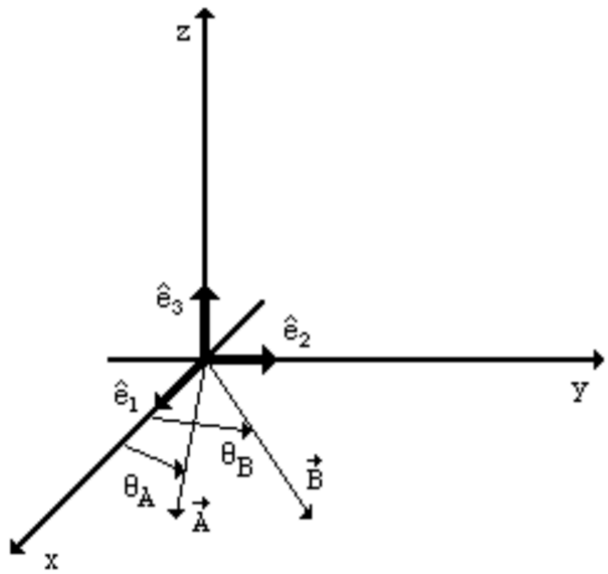

Figure 13:

We then have

$$\vec{A} \cdot \vec{B} = \sum_{j=1}^{2} A_j B_j = A_1 B_1 + A_2 B_2$$

Another (geometrical) way to define the component of a vector along a particular direction is to find the **projection** of the vector on that direction as shown below:

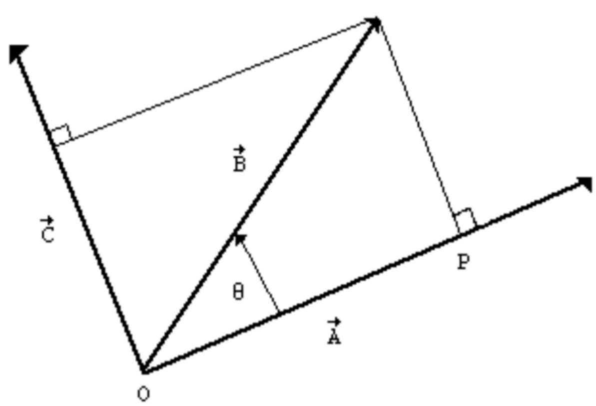

Figure 14:

The component of $\vec{B}$ in the direction of $\vec{A}$ is the projection of $\vec{B}$ onto $\vec{A}$ and as can be seen this is given by $|\vec{B}|\cos\theta$ and the component of $\vec{B}$ in the direction of $\vec{C}$ is the projection of $\vec{B}$ onto $\vec{C}$ and as can be seen this is given by $|\vec{B}|\sin\theta$.

Using this result we have

$$\begin{aligned}\vec{A}\cdot\vec{B} &= \sum_{j=1}^{2} A_j B_j = A_1 B_1 + A_2 B_2 = |\vec{A}|\cos\theta_A|\vec{B}|\cos\theta_B + |\vec{A}|\sin\theta_A|\vec{B}|\sin\theta_B \\ &= |\vec{A}||\vec{B}|(\cos\theta_A\cos\theta_B + \sin\theta_A\sin\theta_B) \\ &= |\vec{A}||\vec{B}|\cos(\theta_A - \theta_B) = |\vec{A}||\vec{B}|\cos\theta \end{aligned} \tag{2.13}$$

where we have defined $(\theta_B - \theta_A) = \theta$ angle between the vectors .

This is a very useful relationship since we can now state that

$$\begin{aligned}&\text{If } \vec{A}\cdot\vec{B} = 0, \text{ then } \vec{A} \text{ is orthogonal } (\theta = 90^\circ = \pi/2 \text{ rad}) \text{ to } \vec{B} \\ &\text{or if } \vec{A} \text{ is orthogonal } \vec{B}, \text{ then } \vec{A}\cdot\vec{B} = 0\end{aligned} \tag{2.14}$$

We now define the **Einstein summation convention**, which assumes a summation anytime that a **repeated index** appears in the same term. Using this convention the above relations look like

$$\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3 = \sum_{i=1}^{3} A_i\hat{e}_i = A_i\hat{e}_i \tag{2.15}$$

$$\vec{B} = B_1\hat{e}_1 + B_2\hat{e}_2 + B_3\hat{e}_3 = \sum_{i=1}^{3} B_i\hat{e}_i = B_i\hat{e}_i \tag{2.16}$$

$$\vec{A}\cdot\vec{B} = \sum_{j=1}^{3} A_j B_j = A_j B_j \tag{2.17}$$

$$\vec{A}\cdot\vec{A} = \sum_{j=1}^{3} A_j A_j = A_1^2 + A_2^2 + A_3^2 = A^2 = \left(\text{length of } \vec{A}\right)^2 = A_j A_j \tag{2.18}$$

$$\vec{A}\cdot\hat{e}_j = \left(\sum_{i=1}^{3} A_i\hat{e}_i\right)\cdot\hat{e}_j = \sum_{i=1}^{3} A_i(\hat{e}_i\cdot\hat{e}_j) = \sum_{i=1}^{3} A_i\delta_{ij} = A_j \tag{2.19}$$

becomes

$$\vec{A}\cdot\hat{e}_j = A_i\hat{e}_i\cdot\hat{e}_j = A_i\delta_{ij} = A_j \tag{2.20}$$

which is clearly much cleaner. In very complicated vector algebra, the Einstein summation convention dramatically educes the amount of writing needed! Using this convention is standard at the advanced level. For this class we will continue to use the summation symbols.

Geometry of Space

For general vectors $\vec{A}$ and $\vec{B}$, the scalar product gives

$$\vec{A}\cdot\vec{B} = \sum_{i,j=1}^{3} A_i B_j \hat{e}_i\cdot\hat{e}_j = \sum_{i,j=1}^{3} A_i B_j \delta_{ij} = \sum_{i=1}^{2} A_i B_i = A_1B_1 + A_2B_2 + A_3B_3 \quad (2.21)$$

and the length-squared of a vector is $\vec{A}\cdot\vec{A} = A^2 = |\vec{A}|^2 = \sum_{i=1}^{3} A_i^2$.

In 2-dimensions (this is a completely general result because any two vectors form a single 2-dimensional plane) we have (see figure 15)

$$\hat{e}(\vec{A}) = \cos\theta_{A_x}\hat{e}_x + \sin\theta_{A_x}\hat{e}_y = \cos\theta_{A_x}\hat{e}_x + \cos\theta_{A_y}\hat{e}_y \quad (2.22)$$

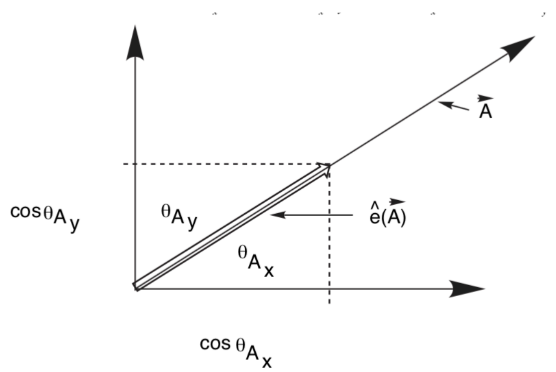

Figure 15:

These components of the unit vector $\hat{e}(\vec{A})$ are called **direction cosines**.

In general, $\hat{e}(\vec{A})\cdot\hat{e}_x = \cos\theta_{A_x}$, and so on, so that

$$\hat{e}(\vec{A}) = \cos\theta_{A_x}\hat{e}_x + \cos\theta_{A_y}\hat{e}_y + \cos\theta_{A_y}\hat{e}_z = \sum_{i=1}^{3}\left(\hat{e}(\vec{A})\cdot\hat{e}_i\right)\hat{e}_i \quad (2.23)$$

For a general vector

$$\vec{A} = A\hat{e}(\vec{A}) \rightarrow \vec{A}\cdot\hat{e}_i = i^{th} \text{ component } = A\hat{e}(A)\cdot\hat{e}_i = A\cos\theta_{A_i} \tag{2.24}$$

This leads to the general rule

$$\vec{A}\cdot\vec{B} = AB\hat{e}(\vec{A})\cdot\hat{e}(\vec{B}) = AB\cos\theta_{AB} \tag{2.25}$$

where

$$\theta_{AB} = \text{ angle between } \vec{A} \text{ and } \vec{B} \tag{2.26}$$

as we derived earlier.

Example

Given the two vectors

$$\vec{A} = 3\hat{i} + 2\hat{j} \quad , \quad \vec{B} = \hat{i} - \hat{j}$$

(a) Plot the vectors

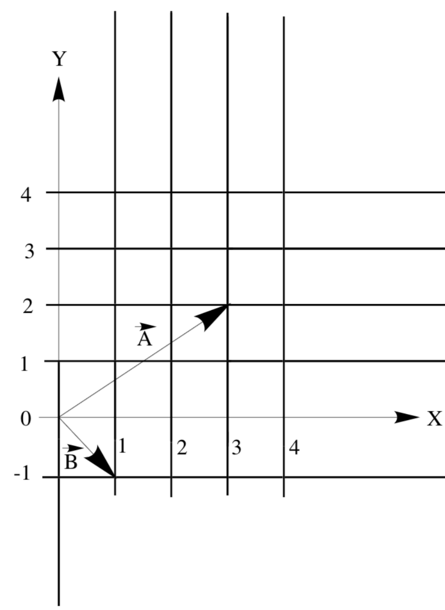

Figure 16:

(b) Calculate the scalar product $\vec{A} \cdot \vec{B}$

$$\vec{A} \cdot \vec{B} = \sum_i A_i B_i = A_x B_x + A_y B_y + A_z B_z = (3)(1) + (2)(-1) + (0)(0) = 1$$

(c) What is the angle between the two vectors

$$\vec{A} \cdot \vec{B} = 1 = AB \cos\theta$$

$$A^2 = \sum_i A_i^2 = (3)(3) + (2)(2) + (0)(0) = 13 \rightarrow A = \sqrt{13}$$

$$B^2 = \sum_i B_i^2 = (1)(1) + (-1)(-1) + (0)(0) = 2 \rightarrow B = \sqrt{2}$$

$$\cos\theta = \frac{\vec{A} \cdot \vec{B}}{AB} = \frac{1}{\sqrt{26}} = 0.20 \rightarrow \theta = 79^\circ$$

Cross or Vector Product ($\times$)

In terms of basis vectors we **define the cross product** of two vectors by the relations:

$$\begin{aligned} \hat{e}_1 \times \hat{e}_2 &= \hat{e}_3 = -\hat{e}_2 \times \hat{e}_1 \\ \hat{e}_2 \times \hat{e}_3 &= \hat{e}_1 = -\hat{e}_3 \times \hat{e}_2 \\ \hat{e}_3 \times \hat{e}_1 &= \hat{e}_2 = -\hat{e}_1 \times \hat{e}_3 \\ \hat{e}_1 \times \hat{e}_1 &= \hat{e}_2 \times \hat{e}_2 = \hat{e}_3 \times \hat{e}_3 = 0 \end{aligned} \tag{2.27}$$

Permutation

In order to make sense of the next definition we must ask the question - what are permutations?

Given a set of number, say (123), then

a **permutation** = a rearrangement of the numbers

We define even and odd permutations in terms of the number of moves, it takes to return a new arrangement or permutation to the original order (123) and where each move can exchange only two adjacent numbers. Therefore,

$$\begin{aligned} 213 = &\text{ odd permutation (requires 1 move } (1 \leftrightarrow 2)) \\ 312 = &\text{ even permutation (requires 2 moves } (3 \leftrightarrow 1), (3 \leftrightarrow 2)) \end{aligned}$$

Epsilon (ε_{ijk}) (Levi-Civita symbol)

We now define a new mathematical object called the **permutation symbol** by the relationship

$$\varepsilon_{ijk} = \begin{cases} 1 & ijk = \text{ even permutation of 123} \\ -1 & ijk = \text{ odd permutation of 123} \\ 0 & \text{ any two indices identical} \end{cases} \tag{2.28}$$

Therefore, we have $\varepsilon_{123} = \varepsilon_{321} = \varepsilon_{231} = 1$, $\varepsilon_{213} = \varepsilon_{132} = \varepsilon_{321} = -1$ and all others $=0$.

Using this new object we can write our defining relations for the cross product as

$$\hat{e}_i \times \hat{e}_j = \sum_{k=1}^{3} \varepsilon_{ijk}\hat{e}_k \tag{2.29}$$

Examples

$$\hat{e}_1 \times \hat{e}_2 = \sum_{k=1}^{3} \varepsilon_{12k}\hat{e}_k = \varepsilon_{121}\hat{e}_1 + \varepsilon_{122}\hat{e}_2 + \varepsilon_{123}\hat{e}_3 = \varepsilon_{123}\hat{e}_3 = \hat{e}_3$$
$$\hat{e}_3 \times \hat{e}_2 = \sum_{k=1}^{3} \varepsilon_{32k}\hat{e}_k = \varepsilon_{321}\hat{e}_1 + \varepsilon_{322}\hat{e}_2 + \varepsilon_{323}\hat{e}_3 = \varepsilon_{321}\hat{e}_1 = -\hat{e}_1$$

The direction of the vector product vector is **perpendicular** to the plane of the two vectors involved (that is, perpendicular to both vectors).

Proof:

$$\hat{e}_i \cdot (\hat{e}_i \times \hat{e}_j) = \hat{e}_i \cdot \sum_k \varepsilon_{ijk}\hat{e}_k = \sum_k \varepsilon_{ijk}\hat{e}_i \cdot \hat{e}_k = \sum_k \varepsilon_{ijk}\delta_{ik} = \varepsilon_{iji} = 0$$

This derivation is a clear demonstration of just how powerful the new mathematical objects are when doing algebra with vectors.

Relations for General Vectors

Now consider two general vectors defined by

$$\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3 = \sum_{i=1}^{3} A_i\hat{e}_i = (A_1, A_2, A_3)$$
$$\vec{B} = B_1\hat{e}_1 + B_2\hat{e}_2 + B_3\hat{e}_3 = \sum_{i=1}^{3} B_i\hat{e}_i = (B_1, B_2, B_3)$$

We then obtain using the defining relations for the cross product

$$\begin{aligned}
\vec{A} \times \vec{B} =&(A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3) \times (B_1\hat{e}_1 + B_2\hat{e}_2 + B_3\hat{e}_3) \\
=&A_1B_2\hat{e}_1 \times \hat{e}_1 + A_2B_1\hat{e}_2 \times \hat{e}_1 + A_3B_1\hat{e}_3 \times \hat{e}_1 \\
&+ A_1B_2\hat{e}_1 \times \hat{e}_2 + A_2B_2\hat{e}_2 \times \hat{e}_2 + A_3B_2\hat{e}_3 \times \hat{e}_2 \\
&+ A_1B_3\hat{e}_1 \times \hat{e}_3 + A_2B_3\hat{e}_2 \times \hat{e}_3 + A_3B_3\hat{e}_3 \times \hat{e}_3 \\
=&A_1B_2\hat{e}_1 \times \hat{e}_2 + A_2B_1\hat{e}_2 \times \hat{e}_1 + A_3B_2\hat{e}_3 \times \hat{e}_2 \\
&+ A_1B_3\hat{e}_1 \times \hat{e}_3 + A_2B_3\hat{e}_2 \times \hat{e}_3 + A_3B_1\hat{e}_3 \times \hat{e}_1 \\
=&A_1B_2\hat{e}_3 - A_2B_1\hat{e}_3 - A_3B_2\hat{e}_1 - A_1B_3\hat{e}_2 + A_2B_3\hat{e}_1 + A_3B_1\hat{e}_2 \\
=&(A_2B_3 - A_3B_2)\hat{e}_1 + (A_3B_1 - A_1B_3)\hat{e}_2 + (A_1B_2 - A_2B_1)\hat{e}_3 \qquad (2.30)
\end{aligned}$$

or using the full power of the new mathematical notation

$$\begin{aligned}
\vec{A} \times \vec{B} &= \sum_i A_i\hat{e}_i \times \sum_j B_j\hat{e}_j = \sum_i \sum_j A_iB_j\hat{e}_i \times \hat{e}_j = \sum_i \sum_j A_iB_j \sum_k \varepsilon_{ijk}\hat{e}_k \\
&= \sum_i \sum_j \sum_k \varepsilon_{ijk}A_iB_k\hat{e}_k = \sum_i \sum_j \sum_k \varepsilon_{jki}A_jB_k\hat{e}_i = \sum_i \sum_j \sum_k \varepsilon_{ijk}A_jB_k\hat{e}_i
\end{aligned} \qquad (2.31)$$

where use has been made of dummy indices and the relation $\varepsilon_{jki} = \varepsilon_{ijk}$

As a check, we expand the last equation to get

$$\begin{aligned}
\vec{A} \times \vec{B} =& \sum_{ijk} \varepsilon_{ijk}A_jB_k\hat{e}_i = \varepsilon_{123}A_2B_3\hat{e}_1 + \varepsilon_{132}A_3B_2\hat{e}_1 + \varepsilon_{213}A_1B_3\hat{e}_2 \\
&+ \varepsilon_{231}A_3B_1\hat{e}_2 + \varepsilon_{321}A_2B_1\hat{e}_3 + \varepsilon_{312}A_1B_2\hat{e}_3 \\
=&A_2B_3\hat{e}_1 - A_3B_2\hat{e}_1 - A_1B_3\hat{e}_2 + A_3B_1\hat{e}_2 - A_2B_1\hat{e}_3 + A_1B_2\hat{e}_3 \\
=&(A_2B_3 - A_3B_2)\hat{e}_1 + (A_3B_1 - A_1B_3)\hat{e}_2 + (A_1B_2 - A_2B_1)\hat{e}_3 \qquad (2.32)
\end{aligned}$$

which agrees with the earlier result. Note that we only wrote down the nonzero terms in the expansion.

Some useful properties of the permutation symbol are:

$$\sum_k \varepsilon_{ijk}\varepsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} \; ; \; \sum_{jk} \varepsilon_{ijk}\varepsilon_{mjk} = 2\delta_{im} \; ; \; \sum_{ijk} \varepsilon_{ijk}\varepsilon_{ijk} = 6 \qquad (2.33)$$

Other Useful Properties

As before, since any two vectors define a plane we can restrict our attention to that plane and write both vectors as 2-dimensional vectors in the plane. Assume that we have chosen the x- and y-axes to lie in this plane. In this case, we have the situation as shown in figure 17:

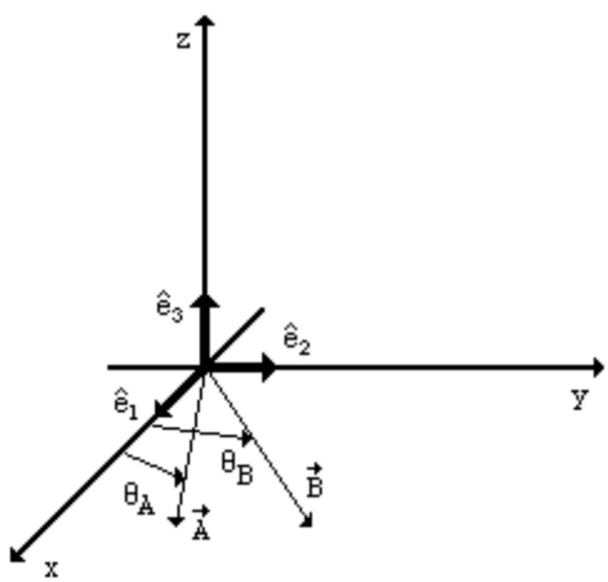

Figure 17:

In this case, we have

$$\begin{aligned}
\vec{A} \times \vec{B} &= (A_1 B_2 - A_2 B_1)\hat{e}_3 = |\vec{A}|\cos\theta_A |\vec{B}|\sin\theta_B \hat{e}_3 - |\vec{A}|\sin\theta_A |\vec{B}|\cos\theta_B \hat{e}_3 \\
&= |\vec{A}||\vec{B}|(\cos\theta_A \sin\theta_B - \sin\theta_A \cos\theta_B \hat{e}_3 \\
&= |\vec{A}||\vec{B}|\sin(\theta_b - \theta_A)\hat{e}_3 = |\vec{A}||\vec{B}|\sin\theta \hat{e}_3 \qquad (2.34)
\end{aligned}$$

or

$$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}|\sin\theta \tag{2.35}$$

This is a very useful relationship since we can now state that

If $\vec{A} \times \vec{B} = 0$, then $\vec{A}$ is parallel to $\vec{B}$ or if $\vec{A}$ is parallel to $\vec{B}$, then $\vec{A} \times \vec{B} = 0$ (2.36)

Right-Hand Rule

For any two vectors $\vec{A}$ and $\vec{B}$, the vector product $\vec{A} \times \vec{B}$ is a vector with magnitude $|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}|\sin\theta$ and direction perpendicular to the plane determined by $\vec{A}$ and $\vec{B}$; the direction is determined by rotating $\vec{A}$ into $\vec{B}$ in the same sense as the defining relations:

$$\hat{e}_1 \times \hat{e}_2 = \hat{e}_3 \ , \ \hat{e}_2 \times \hat{e}_3 = \hat{e}_1 \ , \ hate_3 \times \hat{e}_1 = \hat{e}_2$$

Finally, the cross-product also illustrates a property of an element of surface area that is not obvious, i.e., that a surface area element is not a scalar quantity (one which has magnitude only), but is actually a vector. We will see later on in our discussions that we need to specify the orientation of an area in some physical systems.

Consider the area of a quadrilateral formed by two vectors as shown in figure 18:

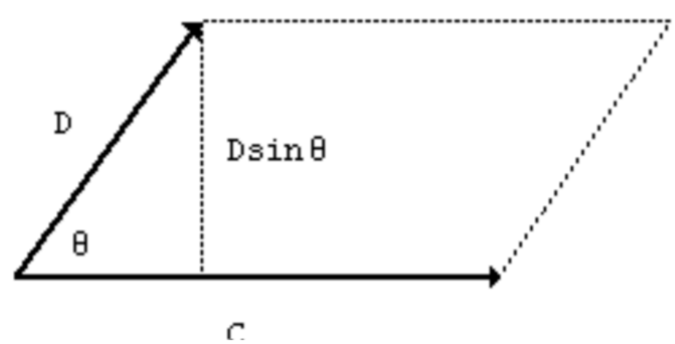

Figure 18:

We have area $= base \times$ height $= CD\sin\theta = |\vec{C} \times \vec{D}|$.

If we assign a direction to an area element as shown in figure 19,

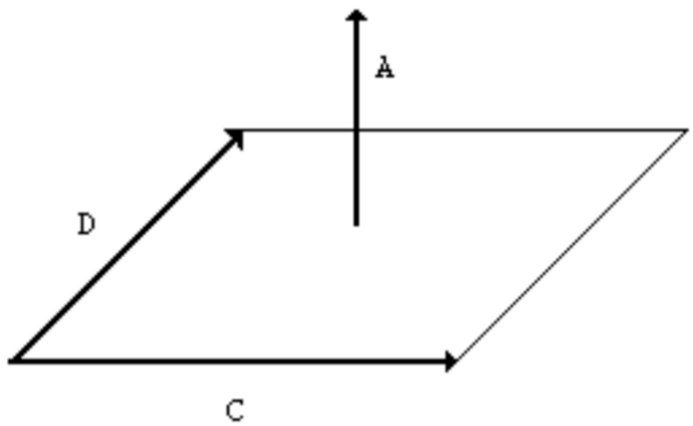

Figure 19:

then we can represent the area element by the vector $\vec{A} = \vec{C} \times \vec{D}$.

The associated direction is **orthogonal** to the area element. We have two possible choices for this vector (±). The usual choice is determined by the right-hand rule(as shown), i.e., rotate $\vec{C}$ into $\vec{D}$ and $\vec{A}$ points in the direction a right-handed screw would move!

Why should we introduce ε_{ijk} ?

(1) It is the easiest way to do complicated vector algebra in 3-dimensions.

(2) For higher dimensions ε_{ijk} it is the only way to go.

(3) It makes for a natural transition to other areas of mathematics that are important in physics in later courses.

Example of use in complex vector identity proof (Illustration of Einstein summation convention):

$$
\begin{aligned}
\vec{A} \times (\vec{B} \times \vec{C}) &= \vec{A} \times (\varepsilon_{ijk} B_j C_k \hat{e}_i) = \varepsilon_{ijk} B_j C_k \vec{A} \times \hat{e}_i = \varepsilon_{ijk} B_j C_k \varepsilon_{mnp} A_n (\hat{e}_i)_p \hat{e}_m \\
&= \varepsilon_{ijk} B_j C_k \varepsilon_{mnp} A_n \delta_{ip} \hat{e}_m = \varepsilon_{ijk} B_j C_k \varepsilon_{mni} A_n \hat{e}_m = \varepsilon_{ijk} \varepsilon_{mni} B_j C_k A_n \hat{e}_m \\
&= (\delta_{mj} \delta_{nk} - \delta_{mk} \delta_{nj}) B_j C_k A_n \hat{e}_m = B_j C_k A_k \hat{e}_j - B_j C_k A_j \hat{e}_k \\
&= (B_j \hat{e}_j)(A_k C_k) - (C_k \hat{e}_k)(A_j B_j) \\
&= \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \qquad (2.37)
\end{aligned}
$$

Examples:

Given the two vectors

$$\vec{A} = \hat{\imath} - \hat{\jmath} + 3\hat{k} \quad , \quad \vec{B} = \hat{\imath} - 5\hat{\jmath} - 2\hat{k}$$

(a) Determine the vector product $\vec{A} \times (\vec{B}$

$$\begin{aligned}\vec{A} \times \vec{B} &= (A_2B_3 - A_3B_2)\hat{e}_1 + (A_3B_1 - A_1B_3)\hat{e}_2 + (A_1B_2 - A_2B_1)\hat{e}_3 \\ &= ((-1)(-2) - (3)(-5))\hat{\imath} + ((3)(1) - (1)(-2))\hat{\jmath} + ((1)(-5) - (-1)(1))\hat{k} \\ &= 17\hat{\imath} + 5\hat{\jmath} - 4\hat{k}\end{aligned}$$

(b) Find the angle between the two vectors

$$\begin{aligned}&|\vec{A} \times (\vec{B}| = AB\sin\theta = \sqrt{330} \\ &A = \sqrt{11}\ ,\ b = \sqrt{30} \\ &\sin\theta = \frac{\sqrt{330}}{\sqrt{11}\sqrt{30}} = 1 \rightarrow \theta = 90^\circ\end{aligned}$$

The vectors are orthogonal! We can check this by calculating the scalar product, which should be zero.

$$\vec{A} \cdot \vec{B} = \sum_i A_iB_i = A_xB_x + A_yB_y + A_zB_z = (1)(1) + (-1)(-5) + (3)(-2) = 0$$

Some physical quantities that we will encounter in this course:

Scalars:

$$m = \text{ mass } \ ,\ E = \text{ energy}$$

Vectors:

$$\vec{r} = \text{position vector } ,d\vec{r} = \text{ displacement vector } ,\vec{v} = \text{ velocity vector}$$

$$\vec{a} = \text{acceleration vector } ,\vec{F} = \text{ Force vector } ,\vec{p} = \text{ momentum vector}$$

$$\vec{\omega} = \text{angular velocity vector } ,\vec{L} = \text{ angular momentum vector } ,\vec{\tau} = \text{ torque vector}$$

Quantities involving scalar or cross products:

$$\text{work } = dW = \vec{F} \cdot d\vec{r} \ , \ \vec{v} = \vec{\omega} \times \vec{r} \ , \ \vec{L} = \vec{r} \times \vec{p} \ , \ \vec{\tau} = \vec{r} \times \vec{F}$$

$$\vec{F} = q\left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right) = \text{ Lorentz force law for charged particle in electromagnetic field}$$

3. Kinematics

Vectors and vector relations are very useful for the description of motion in physical systems. The main task of the subject of mechanics is to determine the **position of an object as a function of time**.

We will derive all of our equations in 2 dimensions. This is sufficient to illustrate all of the vector nature of motion in a physical system. The resulting equations are generalized to 3 dimensions in a straightforward manner and 1 dimensional motion is an easy special case.

3.1. Position Vector

The **position vector** is a vector from the origin of the coordinate system to the object. It is represented by

$$\begin{aligned} \vec{r} = \text{ position vector } &= (x_1, x_2, x_3) = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3 = \sum_k x_k\hat{e}_k \\ &= (x, y, z) = x\hat{x} + y\hat{y} + z\hat{z} \end{aligned} \tag{3.1}$$

where these quantities are shown in figure 20 (in 2 dimensions) for one particular location of the object:

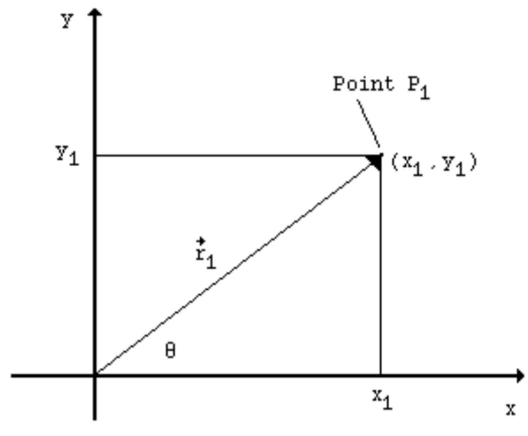

Figure 20:

The position vector is **dependent on choice of origin** (it is actually the **relative** position).

In 3 dimensions, we have the picture shown in figure 21:

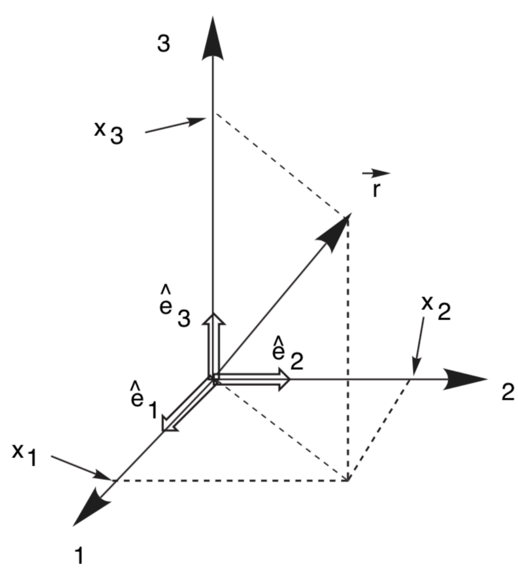

Figure 21:

Path of Motion

As the particle moves, its position vector changes as shown in figure 22.

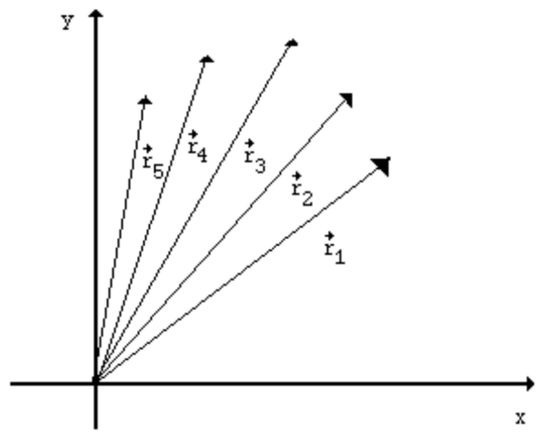

Figure 22:

The curve formed by the tip of the successive position vectors as a function of time is called the path of motion.

In 3 dimensions we have as shown in figure 23

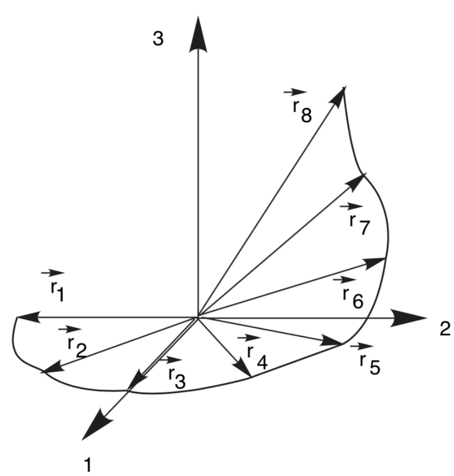

Figure 23:

Displacement Vector

The **change** in the position vector during a time interval is called the **displacement vector** as shown in figure 24.

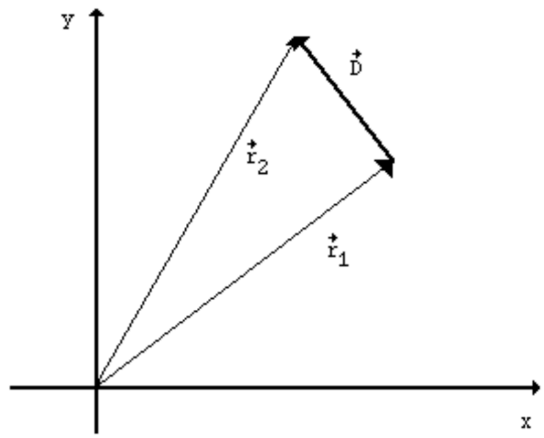

Figure 24:

i.e., $\vec{D} = \vec{r}_2 - \vec{r}_1$ Note that the displacement vector is **independent of choice of origin**.

Now suppose that for general motion between time t and time $t + \Delta t$ we have the path shown in figure 25

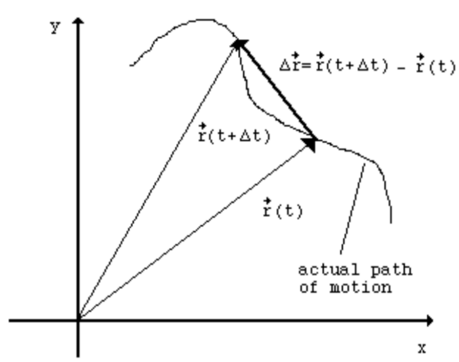

Figure 25:

where

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} \quad , \quad \vec{r}(t+\Delta t) = x(t+\Delta t)\hat{i} + y(t+\Delta t)\hat{j} \tag{3.2}$$

Note that the **Cartesian unit basis vectors do not change with time**. The most important reason for choosing Cartesian coordinates is the fact the basis vectors are independent of time. We will see the difficulty later on when this is not true for a different set of basis vectors when using plane-polar coordinates.

Continuing, we have the displacement vector

$$\begin{aligned}\vec{D} &= \Delta\vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t) \\ &= (x(t+\Delta t) - x(t))\hat{i} + (y(t+\Delta t) - y(t))\hat{j} = \Delta x\hat{i} + \Delta y\hat{j}\end{aligned} \tag{3.3}$$

Equating components, this says that the original **vector equation** is equivalent to two (= number of dimensions) **algebraic** equations

$$\Delta x = x(t+\Delta t) - x(t) \quad , \quad \Delta y = y(t+\Delta t) - y(t) \tag{3.4}$$

3.2. Velocity Vector

The **average velocity** $\vec{v}_{av}$ of the particle as it moves along the path during the time interval t to $t+\Delta t$ is **defined** to be

$$\vec{v}_{av} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

Using the average velocity vector we can **define** the velocity vector(the instantaneous velocity) at time t as the limits of $\vec{v}_{av}$ as $\Delta t \to 0$.

$$\begin{aligned}\vec{v}(t) &= \lim_{\Delta t\to 0}\vec{v}_{av} = \lim_{\Delta t\to 0}\frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \lim_{\Delta t\to 0}\frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} \\ &= \lim_{\Delta t\to 0}\frac{x(t+\Delta t) - x(t)}{\Delta t}\hat{i} + \lim_{\Delta t\to 0}\frac{y(t+\Delta t) - y(t)}{\Delta t}\hat{y} \\ &= \lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t}\hat{i} + \lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t}\hat{y} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{y} \\ &= v_x(t)\hat{i} + v_y(t)\hat{y} = \dot{x}(t)\hat{i} + \dot{y}(t)\hat{j}\end{aligned} \tag{3.5}$$

This implies that

$$\dot{x}(t) = v_x(t) = \frac{dx}{dt} \quad , \quad \dot{y}(t) = v_y(t) = \frac{dy}{dt} \tag{3.6}$$

Remember that this result is simple because the Cartesian unit vectors do not change with time (their directions and magnitudes are constant). Again, this will not be the case later for other types of basis vectors (different coordinate systems).

Before we figure out the direction of the velocity vector, consider the following.

Digression on Properties of the Derivative of a Vector

It is important to realize that the derivative of a vector function is very different than the derivative of a scalar function. This is so because the vector function can change its **direction** as well as change its **magnitude**.

Consider some vector function of t, namely, $\vec{A}(t)$. Then

$$\Delta\vec{A} = \Delta\vec{A}(\text{due to changing magnitude}) + \Delta\vec{A}(\text{due to changing direction}) \tag{3.7}$$

We can always write

$$\Delta\vec{A} = \Delta\vec{A}(\text{parallel to } \vec{A}) + \Delta\vec{A}(\text{orthogonal to } \vec{A}) \tag{3.8}$$

as shown figure 26:

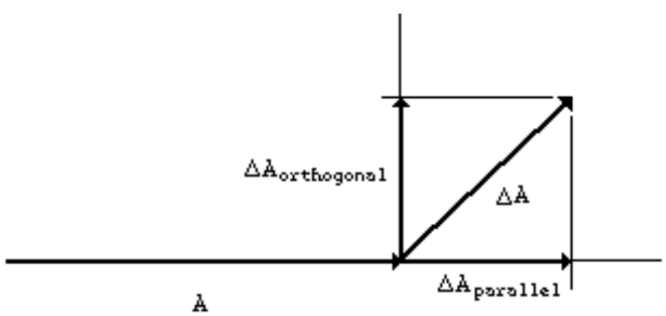

Figure 26:

We now consider two special cases when only one of these two components is non-zero.

(1) The orthogonal component = 0 or the change in the vector is always parallel to the vector.

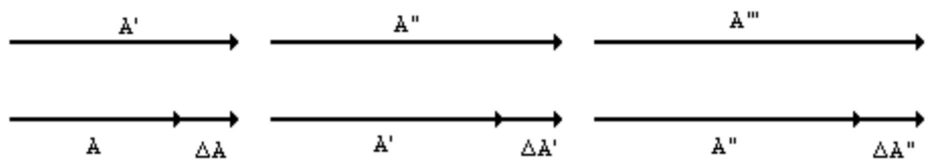

Figure 27:

In this case, only the magnitude (or length) of the vector changes and the direction remains constant as shown figure 27. The vector can only get longer or shorter.

(2) The parallel component = 0 or the change in the vector is always orthogonal to the vector.

In this case, only the direction of the vector changes and the magnitude (or length) remains constant as shown figure 28.

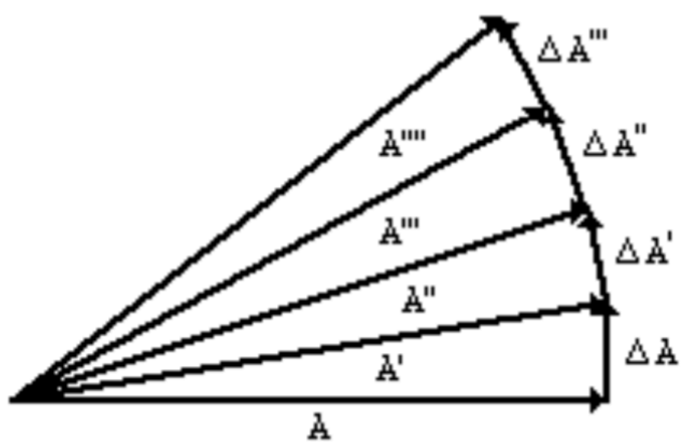

Figure 28:

This can be proved in general using the scalar product. We have

$$\vec{A} \cdot d\vec{A} = 0 = (d\vec{A}) \cdot \vec{A}$$
$$d(A^2) = d(\vec{A} \cdot \vec{A}) = \vec{A} \cdot d\vec{A} + (d\vec{A}) \cdot \vec{A} = 0 \tag{3.9}$$

which says that the **magnitude is constant**. The path of the arrowhead is a circle of radius A.

Relative to Path of Motion

The direction of the velocity vector is the limiting direction of $\Delta\vec{r}$ as $\Delta t \to 0$, since, in this limit we have

$$\Delta\vec{r} = \frac{d\vec{r}}{dt}\Delta t = \vec{v}\Delta t \tag{3.10}$$

Using this result, it can be seen from the figure 29

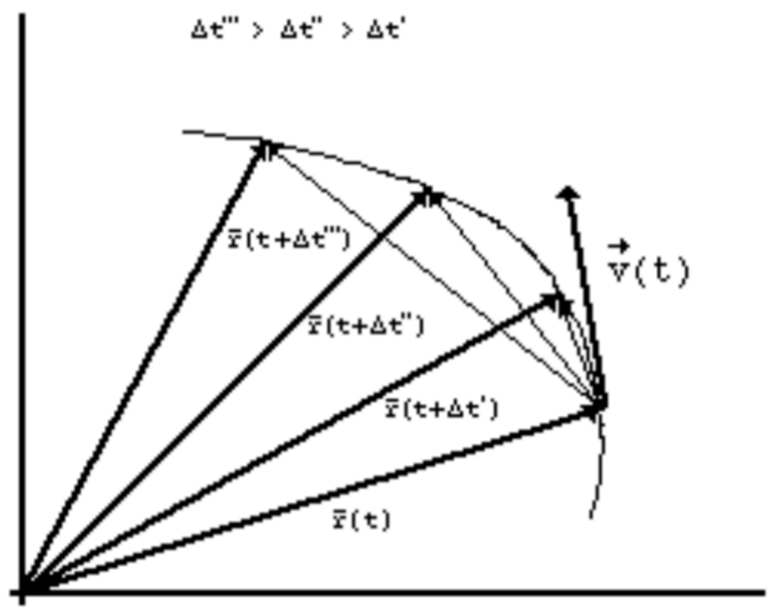

Figure 29:

that the direction of the velocity vector is **tangent to the path of motion at any time**. The formal proof involves tricky geometry and is done at the next level.

Acceleration Vector

In a similar manner we can define the corresponding **acceleration vectors**.

$$\begin{aligned}
\vec{v}(t) &= v_x(t)\hat{i} + v_y(t)\hat{j} \\
\vec{v}(t+\Delta t) &= v_x(t+\Delta t)\hat{i} + v_y(t+\Delta t)\hat{j} \\
\Delta\vec{v} &= \vec{v}(t+\Delta t) - \vec{v}(t) = (v_x(t+\Delta t) - v_x(t))\hat{i} + (v_-(t+\Delta t) - v_y(t))\hat{j} \\
&= \Delta v_x\hat{i} + \Delta v_y\hat{j} \qquad (3.11)
\end{aligned}$$

This vector equation is equivalent to two (= number of dimensions) algebraic equations

$$\begin{aligned}
\Delta v_x &= v_x(t+\Delta t) - v_x(t) \\
\Delta v_y &= v_y(t+\Delta t) - v_y(t) \qquad (3.12)
\end{aligned}$$

The average acceleration $\vec{a}_{av}$ of the particle as it moves along the path during the time interval Δt is defined to be

$$\begin{aligned}
\vec{a}_{av} &= \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t} \\
&= \frac{v_x(t+\Delta t) - v_x(t)}{\Delta t}\hat{i} + \frac{v_y(t+\Delta t) - v_y(t)}{\Delta t}\hat{j} \\
&= \frac{\Delta v_x}{\Delta t}\hat{i} + \frac{\Delta v_y}{\Delta t}\hat{j} = a_{x,av}\hat{i} + a_{y,av}\hat{j} \qquad (3.13)
\end{aligned}$$

In terms of the average acceleration we can define the instantaneous acceleration at a particular instant of time, namely, time t by

$$\begin{aligned}
\vec{a}(t) &= \lim_{\Delta t\to 0}\vec{a}_{av} = \lim_{\Delta t\to 0}\frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \lim_{\Delta t\to 0}\frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t} \\
&= \lim_{\Delta t\to 0}\frac{v_x(t+\Delta t) - v_x(t)}{\Delta t}\hat{i} + \lim_{\Delta t\to 0}\frac{v_y(t+\Delta t) - v_y(t)}{\Delta t}\hat{j} \\
&= \lim_{\Delta t\to 0}\frac{\Delta v_x}{\Delta t}\hat{i} + \lim_{\Delta t\to 0}\frac{\Delta v_y}{\Delta t}\hat{j} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} = a_x(t)\hat{i} + a_y(t)\hat{j} = \ddot{x}\hat{i} + \ddot{y}\hat{j} \\
&\qquad (3.14)
\end{aligned}$$

Relative to Path of Motion

As can be seen from figure 30

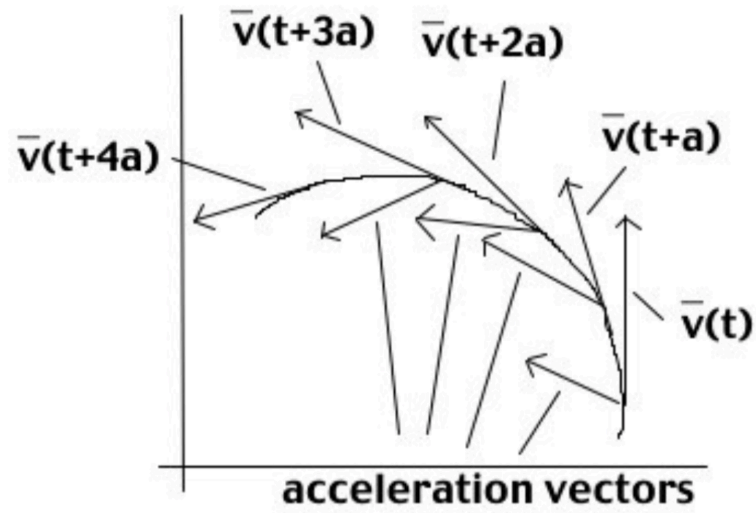

Figure 30:

if the acceleration is not constant in direction, then the acceleration vector always has a component orthogonal to the path of motion directed towards the center of curvature at any time. If the acceleration is constant in direction, then the motion is along a straight line.

Formal Solution of the Equations of Motion

The kinematic equations of motion are:

$$\vec{a} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k} \tag{3.15}$$

$$\vec{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k} \tag{3.16}$$

These correspond to the 6 algebraic equations

$$a_i = \frac{dv_i}{dt} \quad , \quad v_i = \frac{dx_i}{dt} \quad , \quad i = 1, 2, 3 \tag{3.17}$$

Given $\vec{a}(t)$ we can formally solve(using integration) these equations for $\vec{r}(t)$

as follows:

$$dv_i(t) = a_i(t)dt \rightarrow \int_{t_0}^{t} dv_i(dt') = \int_{t_0}^{t} a_i(t')dt' \tag{3.18}$$

$$v_i(t) = v_i(t_0) + \int_{t_0}^{t} a_i(t')dt' \tag{3.19}$$

$$dx_i(t) = v_i(t)dt \rightarrow \int_{t_0}^{t} dx_i(dt') = \int_{t_0}^{t} v_i(t')dt' \tag{3.20}$$

$$x_i(t) = x_i(t_0) + \int_{t_0}^{t} v_i(t')dt' \tag{3.21}$$

Clearly, we need to know the **"initial conditions"** $x_i(t_0)$ and $v_i(t_0)$ to complete the solution.

Special Case - Constant Acceleration

$$\vec{a} = \sum_i a_i \hat{e}_i \ \ , \ \ a_i = \text{ constant} \tag{3.22}$$

We then obtain the results

$$v_i(t) = v_i(t_0) + \int_{t_0}^{t} a_i(t')dt' = v_i(t_0) + a_i(t - t_0) \tag{3.23}$$

$$\begin{aligned} x_i(t) &= x_i(t_0) + \int_{t_0}^{t} v_i(t')dt' = x_i(t_0) + \int_{t_0}^{t} \left(v_i(t_0) + a_i(t' - t_0)\right) dt' \\ &= x_i(t_0) + v_i(t_0)(t - t_0) + a_i \int_{t_0}^{t} (t' - t_0)dt' \\ &= x_i(t_0) + v_i(t_0)(t - t_0) + \frac{1}{2}a_i(t - t_0)^2 \end{aligned} \tag{3.24}$$

Another very useful result can be derived as follows:

$$\begin{aligned} v_i dv_i &= v_i(a_i dt) = a_i(v_i dt) = a_i dx_i \\ \int_{v_{0,i}}^{v_{f,i}} v_i dv_i &= \frac{1}{2}\left(v_{f,i}^2 - v_{0,i}^2\right) = \int_{x_{0,i}}^{x_{f,i}} a_i dx_i = a_i \int_{x_{0,i}}^{x_{f,i}} dx_i = a_i(x_{f,i} - x_{0,i}) \\ v_{f,i}^2 &= v_{0,i}^2 + 2a_i(x_{f,i} - x_{0,i}) \end{aligned} \tag{3.25}$$

These are the key equations of constant acceleration kinematics.

An Example

We consider a particle falling vertically (y-axis) in the earth's gravitational field as in figure 31.

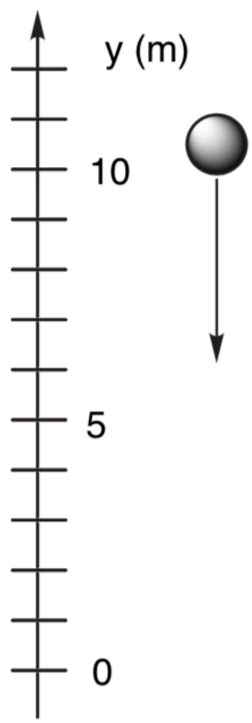

Figure 31:

This is a very easy experiment to do.

If we use a CCD camera and a stroboscope to record the particle positions as a function of time, then we obtain the computer image in figure 32.

Figure 32:

We can then measure the position(distance from the origin) as a function of time $y(t)$ and use the data to discuss the kinematics of the motion of the falling body.

Suppose we have the data shown in the table below:

t	y
0	10.000
0.1	9.951
0.2	9.804
0.3	9.559
0.4	9.216
0.5	8.775
6	8.236
0.7	7.599
0.8	6.864
0.9	6.031
1.0	5.100

A plot of this data looks like (the square symbols):

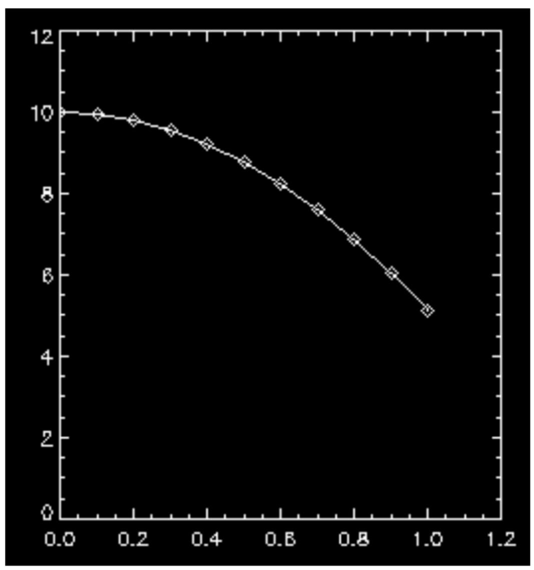

Figure 33:

If we fit a polynomial(curve) to the data, we get a good fit with the parabola formula

$$y(t) = 10.0 - 4.90t^2 \tag{3.26}$$

Therefore, for the data set above we have

$$v_y = \frac{dy}{dt} = -9.80t \quad , \quad a_y = \frac{dv_y}{dt} = -9.80 = \text{ constant} \tag{3.27}$$

So, in one dimensional motion, a constant acceleration produces **quadratic** dependence on t.

This is easy to see by working out the result using the equations we derived.

If, instead of being given the function $y(t)$, which allows us to calculate $v_y(t)$ and $a_y(t)$ by differentiation, we are given $a_y(t)$, we can then calculate $v_y(t)$ and $y(t)$ by integration.

For the experimental data where $a_y(t) = -9.8$ = constant (as in the example above) we have

$$v_y(t) = \int_0^t a_y(t)dt = -9.8 \int_0^t dt = -9.8t \tag{3.28}$$

$$y(t) = 10.0 + \int_0^t v_y(t)dt = 10.0 - 9.8 \int_0^t tdt = 10.0 - 4.90t^2 \tag{3.29}$$

The acceleration of a freely falling body is defined to be g and exact measurements give the value $g = 9.8064$ m/sec^2.

More examples

(1) A water rocket is launched upwards with a velocity of 98 m/s from the top of a building that is 100 m high. Assume that the acceleration due to gravity is 9.8 m/s^2 downwards. See figure 34.

(a) What is the maximum height it reaches above the ground?

(b) What is the time required to reach the maximum height?

(c) What is its velocity when it hits the ground?

(d) What is the total time to reach the ground?

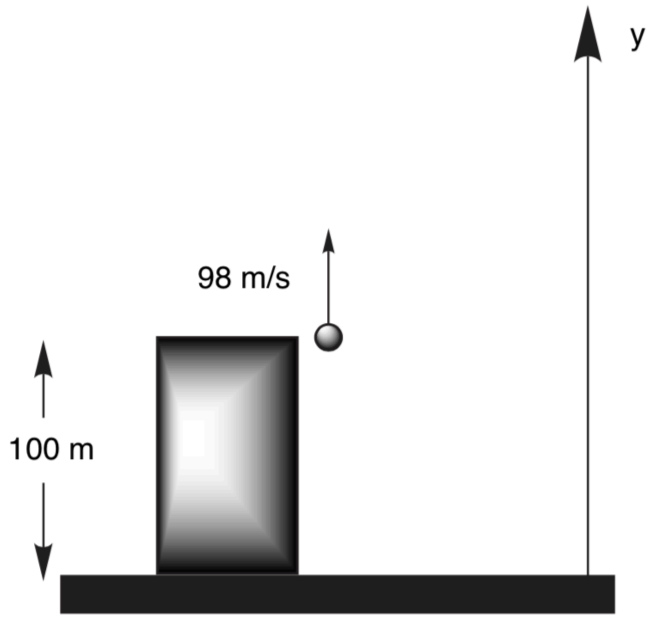

Figure 34:

We choose

positive direction upwards as shown
origin of the y-axis at the ground
$t_0 = 0$
$y(0) = +100$ m
$a_y = -g = -9.8 \text{ m/s}^2$

At time t the kinematic equations of motion are then

$$
\begin{aligned}
\frac{dv_y}{dt} &= a_y(t) = -g = -9.8 \\
&\rightarrow \int_{v_y(0)}^{v_y} dv_y = -9.8 \int_0^t dt \rightarrow v_y(t) = v_y(0) - 9.8t \\
v_y(t) &= 98 - 9.8t \qquad (3.30) \\
\frac{dy}{dt} &= v_y(t) = 98 - 9.8t \\
&\rightarrow \int_{y(0)}^{y} dy = \int_0^t dt \rightarrow y(t) = y(0) + 98i - 4.9t^2 \\
y(t) &= 100 + 98t - 4.9t^2 \qquad (3.31) \\
v_y \frac{dv_y}{dy} &= a_y = -g = -9.8 \\
&\rightarrow \int_{v_y(0)}^{v_y} v_y dv_y = -9.8 \int_{y(0)}^{y} dy \rightarrow \frac{1}{2}(v_y^2(t) - v_y^2(0)) = -9.8(y(t) - y(0)) \\
v_y^2(t) &= v_y^2(0) - 19.6(y(t) - y(0)) \\
v_y^2(t) &= (98)^2 - 19.6(y(t) - 100) \qquad (3.32)
\end{aligned}
$$

At the maximum height $v_y = 0$. Therefore

(a) $v_y^2(t_{max}) = 0 = (98)^2 - 19.6(y_{max} - 100) \rightarrow y_{max} = 590$ m

(b) $v_y(t_{max}) = 0 = 98 - 9.8t_{max} \rightarrow t_{max} = 10$ s

The water rocket hits the ground when $y_{gr} = 0$. Therefore

(c) $v_y^2(t_{gr}) = 0 = (98)^2 - 19.6(0 - 100) \rightarrow v_y(t_{gr}) = v_{y,ground} = -107.41$ m/s

(d) $v_y(t_{gr}) = -107.41 = 98 - 9.8t_{gr} \rightarrow t_{gr} = 20.96$ s

(2) A flower pot falls from a window ledge 36 ft above the pavement.

(a) What is the velocity of the flower pot just before it strikes the ground?

$$
\begin{aligned}
v_{f,y}^2 &= v_{0,y}^2 + 2a_y(y_f - y_0) \\
v_{impact}^2 &= (0)^2 - 2g(0 - y_{ledge}) \\
v_{impact}^2 &= 2gy_{ledge} = (2)(32)(36) = (48)^2 \\
v_{impact} &= 48 \text{ ft/s}
\end{aligned}
$$

(b) The flower pot falls past a window partway down with a speed of 24 ft/s. How far above this window was the ledge?

$$v_{f,y}^2 = v_{0,y}^2 + 2a_y(y_f - y + 0)$$
$$v_{window}^2 = (0)^2 - 2g(y_{window} - y_{ledge})$$
$$(y_{ledge} - y_{window}) = \text{ height above window } = \frac{v_{window}^2}{2g} = \frac{(24)^2}{2(32)} = 9 \text{ ft}$$

Two-Dimensional Motion

Case #1 - No air resistance Suppose that we have a projectile that is launched with a velocity v_0 in a direction making an angle θ with the horizontal as shown below.

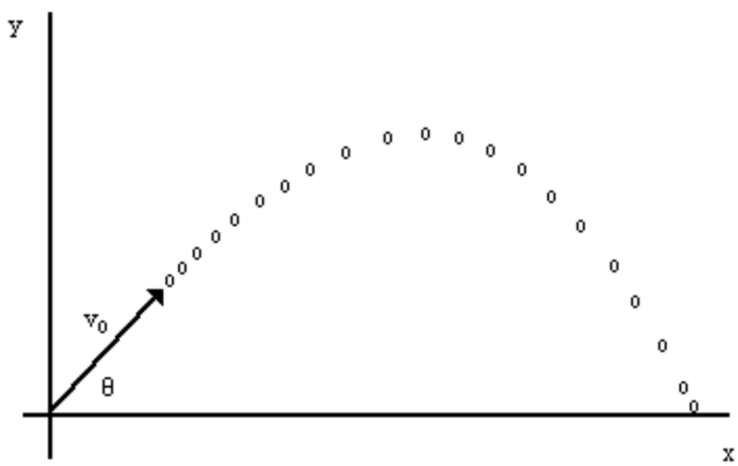

Figure 35:

The acceleration in the case of motion near the surface of the earth is given by

$$\vec{a} = -g\hat{j} \quad \text{(constant acceleration)} \tag{3.33}$$

later when we consider motion farther from the earth we will need to use the full version of Newton's law of gravitation.

We start off with the initial conditions

$$x_0 = x(0) = y_0 = y(0) = t_0 = 0 \;\; ; \;\; v_{x0} = v_0 \cos\theta \; , \; v_{y0} = v_0 \sin\theta \quad \rightarrow \text{components}$$

The kinematic equations of motion are

$$\begin{aligned} v_x(t) &= v_{x0} = v_0 \cos\theta = \text{ constant} \quad (a_x = 0) \\ v_y(t) &= v_{y0} - gt = v_0 \sin\theta - gt \qquad (3.34) \\ x(t) &= v_{x0}t = (v_0 \cos\theta)t \\ y(t) &= v_{y0}t - \frac{1}{2}gt^2 = (v_0 \sin\theta)t - \frac{1}{2}gt^2 \qquad (3.35) \end{aligned}$$

These are easily solved for all relevant physical quantities as follows.

At maximum height or altitude

$$y(t_{max}) = y_{max} \quad , \quad v_y(t_{max}) = 0$$

or

$$0 = v_0 \sin\theta - gt_{max} \to t_{max} = \frac{v_0 \sin\theta}{g}$$

We then have

$$y_{max} = y(t_{max}) = (v_0 \sin\theta)t_{max} - \frac{1}{2}gt_{max}^2 = \frac{v_0^2 \sin^2\theta}{2g}$$

The equation of the path of motion is given by

$$\begin{aligned} y &= (v_0 \sin\theta)t - \frac{1}{2}gt^2 \quad , \quad x = (v_0 \cos\theta)t \to t = \frac{x}{v_0 \cos\theta} \\ y &= x \tan\theta - \frac{g}{2v_0^2 \sin^2\theta}x^2 \to \text{ a parabola} \qquad (3.36) \end{aligned}$$

Finally, the range R or maximum x-distance is given by

$$R = x(2t_{max}) = (v_0 \cos\theta)2\frac{v_0 \sin\theta}{g} = \frac{v_0^2}{g} \sin 2\theta$$

since the time (t_{max}) to fall from the maximum height is the same as the time (t_{max}) to reach maximum height. This can be seen as follows: starting from y_{max} with $v_y = 0$ at $t = 0$ we have the equation for the y-motion

$$y(t) = y_{max} - \frac{1}{2}gt^2 \to y(t_{fall}) = \frac{v_0^2 \sin^2\theta}{2g} - -\frac{1}{2}gt_{fall}^2 = 0 \to t_{fall} = \frac{v_0 \sin\theta}{g} = t_{max}$$

The parabola is symmetric around $x = R/2$ and the angle that the velocity vector makes with the horizontal at impact is the same as that at launch.

As we shall show later in the course, the path of motion(when we do not assume that the acceleration due to gravity is a constant) is actually a part of an ellipse which is well approximated by a parabola **near** the surface of the earth.

Example - A Monkey and a Gun

A monkey hunter sits on the ground armed with a tranquilizer dart gun. She aims at a monkey hanging from a tree. Startled by the noise of the gun, the monkey lets go of the branch at the same instant that the dart leaves the gun.

Explain why the dart will strike the monkey as he falls to the ground. As shown in the figure 36

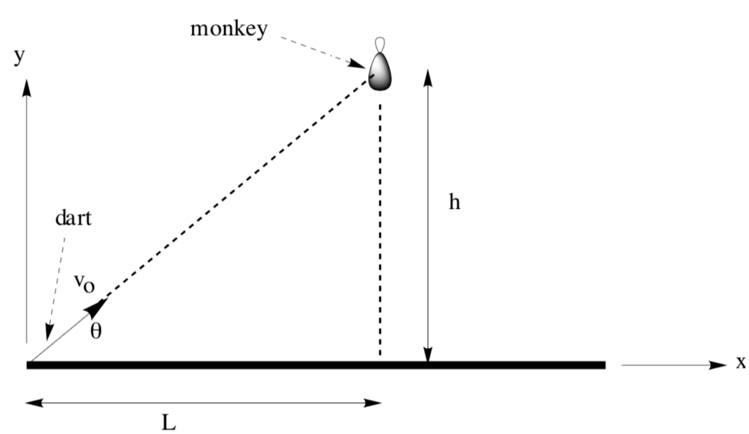

Figure 36:

the monkey position is given by

$$\vec{r}_m = L\hat{i} + \left(h - \frac{1}{2}gt^2\right)\hat{j}$$

and the dart position is given by

$$\vec{r}_d = (v_0 \cos\theta)t\hat{i} + \left((v_0 \sin\theta)t - \frac{1}{2}gt^2\right)\hat{j}$$

Let us calculate the time when the dart reaches $x_d = L$. Since the speed in the x?direction is constant we have

$$(v_0 \cos\theta) t_L = l \to t_L = \frac{L}{v_0 \cos\theta}$$

We now calculate $y_d(t_L)$, $y_m(t_L)$ using $\tan\theta = h/L$.

$$y_d(t_L) = (v_0 \sin\theta) t_L - \frac{1}{2} g t_L^2 = v_0 \frac{L}{v_0 \cos\theta} \sin\theta - \frac{1}{2} g \left(\frac{L}{v_0 \cos\theta} \right)^2 = h - \left(\frac{L}{v_0 \cos\theta} \right)^2$$

$$y_m(t_L) = h - \frac{1}{2} g t_L^2 = h - \frac{1}{2} g \left(\frac{L}{v_0 \cos\theta} \right)^2 = y_d(t_L)$$

We get a tranquilized monkey!

For Class Discussion

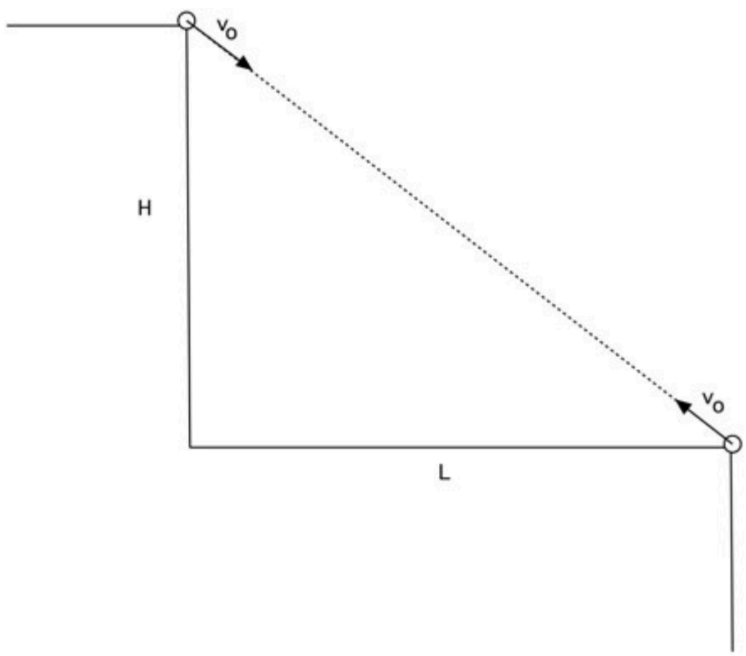

Figure 37:

Two identical balls are aimed at each other as in figure 37. The balls are thrown simultaneously with the same speed.

Discuss which of the following statements about the subsequent motion is correct.

(a) Ball thrown from top hits the ground first.

(b) The two balls collide in midair.

(c) They reach the ground at the same time.

Case #2 - Air resistance proportional to velocity

In this case, we have the acceleration $\vec{a} = -kv_x\hat{\imath} - (g + kv_y)\hat{\jmath}$, which now includes an air resistance term. We start off with the same initial conditions $x_0 = x(0) = y_0 = y(0) = t_0 = 0$; $v_{x0} = v_0\cos\theta$, $v_{y0} = v_0\sin\theta$. The kinematic equations of motion are

$$\frac{dv_x}{dt} = a_x = -kv_x \quad , \quad \frac{dv_y}{dt} = a_y = -g - kv_y$$

These are more complicated to solve. Formally these equations are called **differential equations** (during the semester we will learn techniques that enable us to solve several types of differential equations). In this case, they can be solved using standard integration methods as follows

$$\begin{aligned}
\frac{dv_x}{dt} = -kv_x \to \frac{dv_x}{v_x} = -kdt \to \int_{v_x(0)}^{v_x} \frac{dv_x}{v_x} &= -k\int_{t_0}^{t} dt \\
\ln(v_x) - \ln(v_x(0)) &= -k(t - t_0) \\
\ln\frac{v_x}{v_0\cos\theta} = -kt \to \frac{v_x}{v_0\cos\theta} &= e^{-kt} \\
v_x &= v_0\cos\theta e^{-kt} \qquad (3.37)
\end{aligned}$$

which says that the velocity in the x-direction exponentially decreases to zero. Then we have

$$\begin{aligned}
\frac{dv_y}{dt} &= a_y = -g - kv_y \\
\frac{dv_y}{v_t + v_y} &= -kdt \quad , \quad v_t = \frac{g}{k} \\
\int_{v_y(0)}^{v_y} \frac{dv_y}{v_t + v_y} &= -k\int_{t_0}^{t} dt \to \ln\frac{v_t + v_y(t)}{v_t + v_0\sin\theta} = -kt \\
\frac{v_t + v_y(t)}{v_t + v_0\sin\theta} &= e^{-kt} \to v_y(t) = -v_t + (v_t + v_0\sin\theta)e^{-kt} \qquad (3.38)
\end{aligned}$$

which says that the velocity in the y-direction exponentially decreases to a constant velocity $v_t = \frac{g}{k}$ (downwards) called the **terminal velocity**.

We can now use the integral $\int e^{-kt}dt = -\frac{1}{k}e^{-kt}+C$ to integrate these equations once more and determine $x(t)$ and $y(t)$. We have

$$\begin{aligned}
v_x(t) &= (v_0 \cos\theta e^{-kt} = \frac{dx}{dt} \rightarrow dx = (v_0 \cos\theta e^{-kt} dt \\
\int_{x(0)}^{x} dx &= v_0 \cos\theta \int_{t_0}^{t} e^{-kt} dt \rightarrow x - x(0) = -\frac{v_0}{k}\cos\theta(e^{-kt} - e^{-kt_0}) \\
x(t) &= \frac{v_0}{k}\cos\theta(1 - e^{-kt}) \qquad (3.39)
\end{aligned}$$

which says that the maximum range in the x-direction is $\frac{v_0}{k}\cos\theta$.

$$\begin{aligned}
v_y(t) &= -v_t + (v_t + v_0 \sin\theta)e^{-kt} = \frac{dy}{dt} \\
dy &= -v_t dt + (v_t + v_0 \sin\theta)e^{-kt} dt \\
\int_{y(0)}^{y} dy &= -v_t \int_{t_0}^{t} dt + (v_t + v_0 \sin\theta) \int_{t_0}^{t} e^{-kt} dt \\
y(t) - y(0) &= -v_t(t - t_0) - \frac{1}{k}(v_t + v_0 \sin\theta)(e^{-kt} - e^{-kt_0}) \\
y(t) &= -v_t t + \frac{1}{k}(v_t + v_0 \sin\theta)(1 - e^{-kt}) \qquad (3.40)
\end{aligned}$$

which says that $y(t)$ decreases **linearly** with time after a long time.

Physical properties of the solutions

The maximum possible value of $x(t \rightarrow \infty)$ is $\frac{v_0 \cos\theta}{k}$. In addition as $t \rightarrow \infty$, $v_x \rightarrow 0$ so that the velocity as it returns to the ground is approaching the vertical(no x-component). A plot of these results helps us visualize what is happening. The plot in figure 38 is a set of curves representing the path of motion for projectiles with different air-resistance values(k), where $k = 0$ corresponds to the zero air-resistance case (parabolic motion).

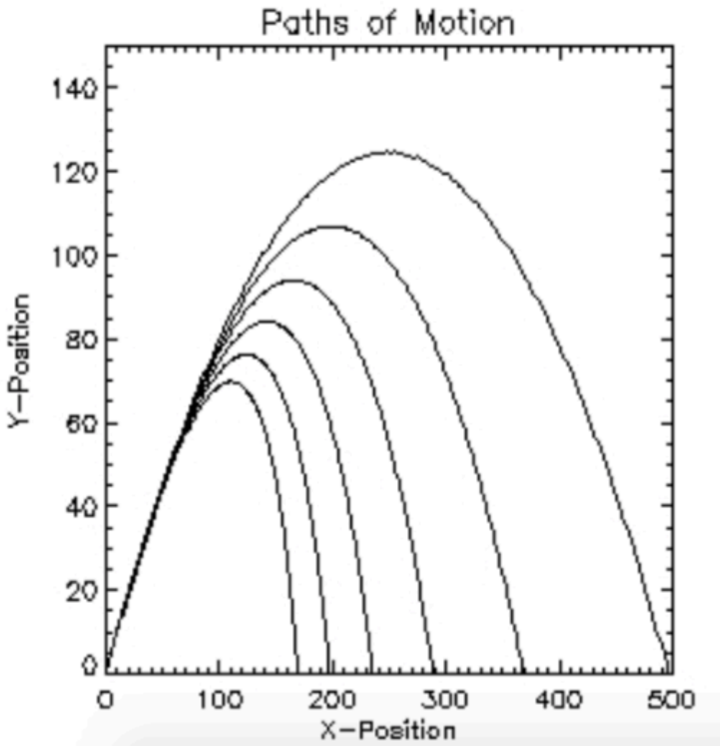

Figure 38:

The paths shown are for $k = 0.0, 0.05, 0.10, 0.15, 0.20, 0.25$.

So using our kinematic relations and our ability to use calculus we can now solve some non-trivial problems.

Plane Polar Coordinates ("Toto, we are not in Kansas anymore!"

An alternate set of coordinates in 2 dimensions that is more appropriate for some problems (such as our later discussion of central forces and planetary motion) is **plane polar coordinates** defined in the diagram in figure 39.

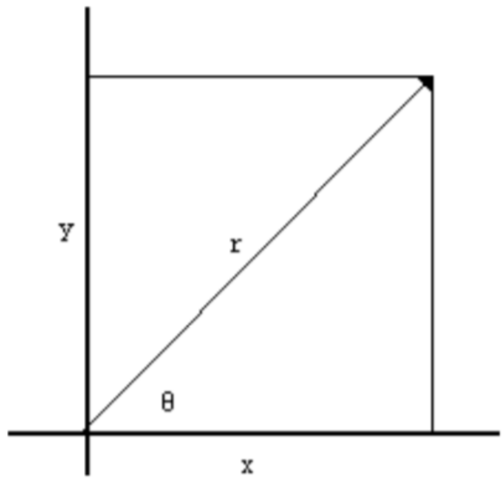

Figure 39:

where

$$r = \sqrt{x^2 + y^2} \ , \ \tan\theta = \frac{y}{x} \quad \text{or} \quad x = r\cos\theta \ \text{ and } \ y = r\sin\theta \tag{3.41}$$

Uniform Circular Motion in 2 Dimensions

We begin our discussion with a simple two dimensional motion, namely, **uniform circular motion**. In uniform circular motion we have the following restrictions:

$$r = \text{ constant (the direction of } \vec{r} \text{ is not constant)} \tag{3.42}$$

and

$$\frac{d\theta}{dt} = \dot{\theta} = \text{ constant } = \omega \rightarrow \theta = \omega t \tag{3.43}$$

The time T that it takes the particle to complete one complete revolution or rotate around the origin through an angle of $\Delta\theta = 2\pi$ is called the **period**. We have

$$\Delta\theta = 2\pi = \omega\Delta t = \omega T \rightarrow T = \frac{2\pi}{\omega} \tag{3.44}$$

Therefore, we can write for the rotating position vector

$$\vec{r} = r(\cos\omega t\hat{e}_x + \sin\omega t\hat{e}_y) = r\left(\cos\frac{2\pi t}{T}\hat{e}_x + \sin\frac{2\pi t}{T}\hat{e}_y\right) \tag{3.45}$$

This corresponds to constant length r and the position vector rotating with constant angular speed ω.

The velocity vector is then given by

$$\vec{v} = \frac{d\vec{r}}{dt} = -r\omega \sin \omega t \hat{e}_x + r\omega \cos \omega t \hat{e}_y \tag{3.46}$$

We note that

$$\vec{r} \cdot \vec{r} = r^2 = \text{ constant} \tag{3.47}$$

$$\vec{v} \cdot \vec{v} = (r\omega)^2 = v^2 = \text{ constant } \rightarrow v = r\omega \tag{3.48}$$

and

$$\vec{r} \cdot \vec{v} = 0 \tag{3.49}$$

which says that the velocity vector is **always perpendicular** to the position vector. It looks like figure 40:

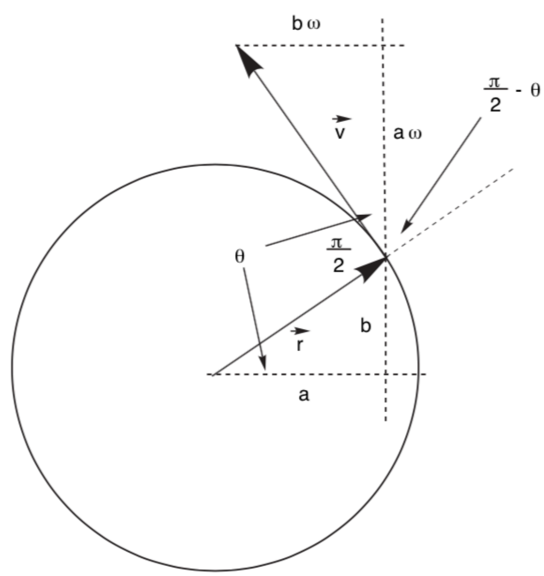

Figure 40:

In fact, for any circular motion (r = constant), whether uniform or not, and the position vector is orthogonal to the velocity vector since

$$\vec{r} \cdot \vec{v} = \vec{r} \cdot \frac{d\vec{r}}{dt} = \frac{1}{2}\frac{d}{dt}(\vec{r} \cdot \vec{r}) = \frac{d}{dt}(r^2) = 0 \tag{3.50}$$

We calculate the acceleration vector in the same manner. How do we know that the acceleration is nonzero? We have

$$\begin{aligned}\vec{a} &= \frac{d\vec{v}}{dt} = \frac{d}{dt}(-r\omega\sin\omega t\hat{e}_x + r\omega\cos\omega t\hat{e}_y) \\ &= -r\omega^2(\cos\omega t\hat{e}_x + \sin\omega t\hat{e}_y) \\ &= -r\omega^2\hat{r} \rightarrow \text{ magnitude } = a = r\omega^2 = \frac{v^2}{r} \qquad (3.51)\end{aligned}$$

Thus, in uniform circular motion, the acceleration vector always points radially inward as shown in figure 41. It is called **centripetal acceleration**.

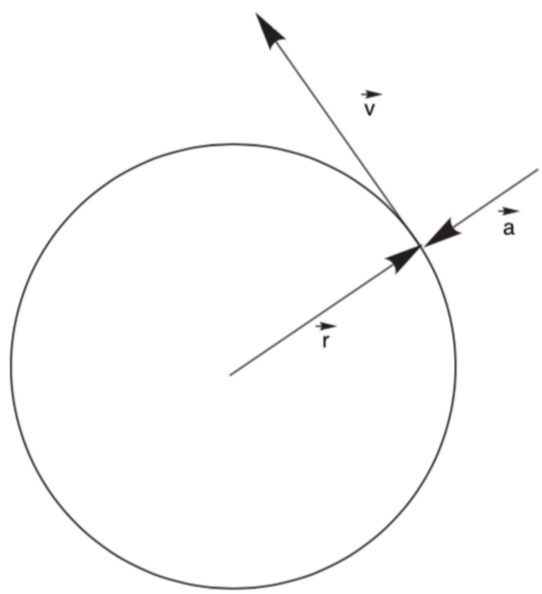

Figure 41:

One again we get a constant magnitude velocity vector because the velocity vector and the acceleration vector are always orthogonal.

General Motion

Let us now consider polar coordinates for **general motion**.

This is difficult stuff and we will regard this as a **first pass** only.

We will return to the subject several times, hopefully increasing your understanding each time.

We will need to use polar coordinates extensively outside of the circular motion case near the end of the class when we discuss motion in a gravitational field, the solar system and interplanetary rockets.

Position, Velocity and Acceleration in General

Let us define two new unit vectors $\hat{r}$ (= $\hat{e}_r$) and $\hat{\theta}$ (= $\hat{e}_\theta$) as shown in the figure 42.

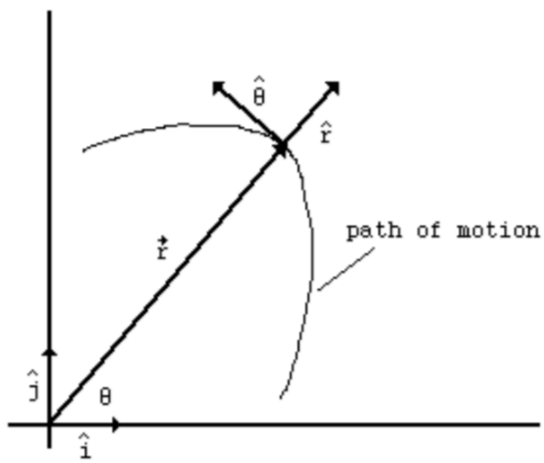

Figure 42:

This pair of unit vectors will form the basis set for plane-polar coordinates.

We have

$$\vec{r} = r\hat{r} \;\;,\;\; \hat{r} = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath} \;\;,\;\; \hat{\theta} = -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath} \tag{3.52}$$

Can you draw the diagrams showing these components?

Note that the three unit vectors $\hat{r}, \hat{\theta}, \hat{k}$ form a basis for 3-dimensional space since:

$$\hat{r}\cdot\hat{\theta} = \hat{\theta}\cdot\hat{r} = \hat{r}\cdot\hat{k} = \hat{k}\cdot\hat{r} = \hat{k}\cdot\hat{\theta} = \hat{\theta}\cdot\hat{k} = 0$$
$$\hat{r}\cdot\hat{r} = \hat{\theta}\cdot\hat{\theta} = \hat{k}\cdot\hat{k} = 1$$
$$\hat{r}\times\hat{r} = \hat{\theta}\times\hat{\theta} = \hat{k}\times\hat{k} = 0$$
$$\hat{r}\times\hat{\theta} = -\hat{\theta}\times\hat{r} = \hat{k} \;\;,\;\; -\hat{r}\times\hat{k} = -\hat{k}\times\hat{r} = \hat{\theta} \;\;,\;\; -\hat{k}\times\hat{\theta} = -\hat{\theta}\times\hat{k} = \hat{r}$$

This pair will form the basis(set of vectors that we can use to construct all other vectors) for plane-polar coordinates. We know we can write any arbitrary vector in terms of the Cartesian unit vectors

$$\vec{A} = A_x\hat{e}_x + A_y\hat{e}_y \tag{3.53}$$

where

$$A_x = \hat{e}_x\cdot\vec{A} \quad , \quad A_y = \hat{e}_y\cdot\vec{A} \tag{3.54}$$

A similar construction works for the pair $\hat{r}$ and $\hat{\theta}$. Since

$$\hat{r}\cdot\hat{r} = 1 = \hat{\theta}\cdot\hat{\theta} \text{ and } \hat{r}\cdot\hat{\theta} = 0 \tag{3.55}$$

we have

$$\vec{A} = A_r\hat{e}_r + A_\theta\hat{e}_\theta \tag{3.56}$$

where

$$A_r = \hat{e}_r\cdot\vec{A} \quad , \quad A_\theta = \hat{e}_\theta\cdot\vec{A} \tag{3.57}$$

Generalizing these notions we can say that the three unit vectors $\hat{e}_r$, $\hat{e}_\theta$ and $\hat{e}_k$ form a basis for 3-dimensional space since:

$$\hat{e}_i\cdot\hat{e}_j = \delta_{ij} \quad i,j = r,\theta,z \tag{3.58}$$

$$\hat{e}_i\times\hat{e}_j = \sum_k \varepsilon_{ijk}\hat{e}_k \tag{3.59}$$

These are called **cylindrical polar** coordinates.

Mathematically, all of these sets of vectors are equivalent and all present the same difficulties when used in proofs. In physics they are still mathematically equivalent, but there is a big operational difference between these new unit vectors and Cartesian unit vectors in that the new unit vectors are **not necessarily constant in time** (their **directions change** as the particle moves around).

You can easily see this by drawing a picture of a particle moving along a curve and attaching unit vectors at several points.

In some problems however, they greatly simplify the derivations and lead to an enhanced understanding of the physical system under investigation and that is why we study them.

We can now work out the velocity and acceleration vectors by differentiation. Follow this closely.... it is straightforward but very tedious.

$$\vec{r} = r\hat{r} \ , \ \hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j} \ , \ \hat{\theta} = -\sin\theta\hat{i} + \cos\theta\hat{j} \tag{3.60}$$

Differentiating and using the chain rule, we get

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d(r\hat{r})}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt} \tag{3.61}$$

$$\begin{aligned}\frac{d\hat{r}}{dt} &= \frac{d(\cos\theta\hat{i} + \sin\theta\hat{j})}{dt} = \frac{d\cos\theta}{dt}\hat{i} + \frac{d\sin\theta}{dt}\hat{j} \\ &= \frac{d\theta}{dt}\frac{d\cos\theta}{dt}\hat{i} + \frac{d\theta}{dt}\frac{d\sin\theta}{d\theta}\hat{j} \\ &= \dot{\theta}(-\sin\theta\hat{i} + \cos\theta\hat{j}) = \dot{\theta}\hat{\theta} \end{aligned} \tag{3.62}$$

so that

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \tag{3.63}$$

Does it make sense? A theoretical physicist always checks new relations using special cases to make sure they make sense. Consider the case of uniform circular motion. We then have

$$\dot{r} = 0 \quad \text{and} \quad \dot{\theta} = \omega = \text{ constant}$$

$$\rightarrow \vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} = r\omega\hat{\theta}$$

as we found before.

Continuing with the **supernova** of all calculations

$$
\begin{aligned}
\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})}{dt} &= \frac{d\dot{r}}{dt}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \frac{dr}{dt}\dot{\theta}\hat{\theta} + r\frac{d\theta}{dt}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt} \\
&= \ddot{r}\hat{r} + \dot{r}(\dot{\theta}\hat{\theta}) + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt} \\
&= \ddot{r}\hat{r} + 2\dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}
\end{aligned}
$$

Now

$$
\begin{aligned}
\frac{d\hat{\theta}}{dt} = \frac{d(-\sin\theta\hat{i} + \cos\theta\hat{j})}{dt} &= -\frac{d\sin\theta}{dt}\hat{i} + \frac{d\cos\theta}{dt}\hat{j} \\
&= -\frac{d\theta}{dt}\frac{d\sin\theta}{d\theta}\hat{i} + \frac{d\theta}{dt}\frac{d\cos\theta}{d\theta}\hat{j} \\
&= \dot{\theta}(-\cos\theta\hat{i} - \sin\theta\hat{j}) = -\dot{\theta}\hat{r} \qquad (3.64)
\end{aligned}
$$

so that

$$
\vec{a} = \ddot{r}\hat{r} + 2\dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} \qquad (3.65)
$$

Does it make sense? Again consider the case of uniform circular motion. We then have

$$
\dot{r} = 0 = \ddot{r} \quad \text{and} \quad \dot{\theta} = \omega \ , \ \ddot{\theta} = 0
$$

$$
\rightarrow \vec{a} = \frac{d\vec{v}}{dt} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} = -r\omega^2\hat{r} = -\frac{v^2}{r}\hat{r}
$$

as we found before.

Let us return to circular motion and review our earlier discussion and see if we can learn anything new using these general relations and maybe generalize the earlier results. Theoretical physicists often do this.

Circular Motion - using plane-polar coordinates

From our earlier derivation we have

$$
\vec{r} = r\hat{r} \ , \ \hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j} \ , \ \hat{\theta} = -\sin\theta\hat{i} + \cos\theta\hat{j} \qquad (3.66)
$$

$$
\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \ , \ \vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} \qquad (3.67)
$$

$$
\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta} \quad , \quad \frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r} \qquad (3.68)
$$

For uniform circular motion we have

$$\dot{r} = 0 \to r = R = \text{ constant } , \ \ddot{\theta} = 0 \to \dot{\theta} = \omega = \text{ constant} \tag{3.69}$$

which says that

$$\vec{v} = R\omega\hat{\theta} \text{ (tangent to the circular path of motion)} \tag{3.70}$$

$$\vec{a} = -R\omega^2\hat{r} \text{ (directed towards the center of the circle)} \tag{3.71}$$

Let us assume that this circular path is in the $x-y$ plane and define a new vector

$$\vec{\omega} = \dot{\theta}\hat{k} = \omega\hat{k} = \text{ angular velocity vector (constant direction)} \tag{3.72}$$

The picture of this motion with associated vectors is shown in figure 43.

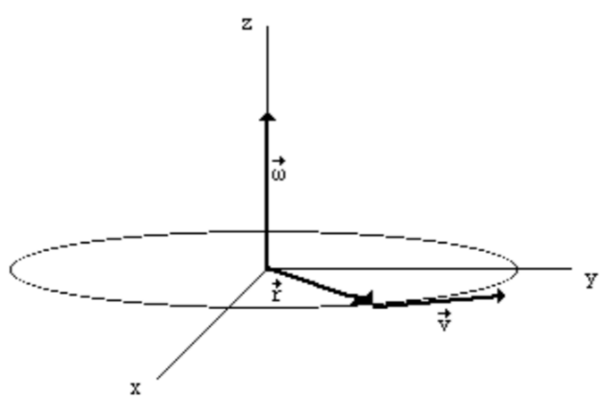

Figure 43:

We can then write

$$\vec{v} = v\hat{\theta} \to v = R\omega \tag{3.73}$$

The three vectors $(\vec{r}, \vec{v}, \vec{\omega})$ and their relationships **suggest** the existence of a single equation relating these quantities given by

$$\vec{v} = \vec{\omega} \times \vec{r} \tag{3.74}$$

This is the kind of thing a theoretical physicist does ....

Clearly, it works in this special case since all three vectors are mutually orthogonal. Try it!

We will see later that this equation is, in fact, generally true. We will return to this equation later in the course as we try to understand a quantity called **angular momentum**.

The acceleration is then given by

$$\vec{a} = -R\omega^2\hat{r} = \vec{\omega} \times \vec{v} = \vec{\omega} \times (\vec{\omega} \times \vec{r}) \tag{3.75}$$

Again, this vector relationship clearly works in this special case. Try it!

We will find out later in the course that it is also a general result.

A picture with a more general choice of origin, which illustrates all the relationships, is shown in figure 44.

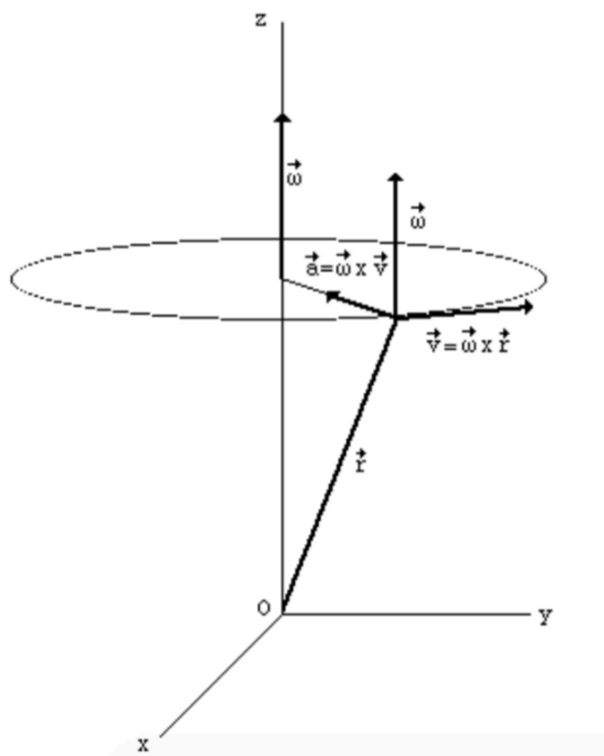

Figure 44:

As we have seen, in the special case where r = constant(circular motion), the angular velocity vector $\vec{\omega}$ has a constant direction.

If the circle lies in the $x-y$ plane, then $\vec{\omega} = \dot{\theta}\hat{k} = \omega\hat{k}$. If we define the **angular acceleration** vector by

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} \quad , \quad \text{magnitude } = \alpha \tag{3.76}$$

then since the direction of $\vec{\omega}$ is constant we can write

$$\alpha = \frac{d\omega}{dt} = \frac{d\dot{\theta}}{dt} = \ddot{\theta} = \frac{d^2\theta}{dt^2} \tag{3.77}$$

Now, suppose that α is a constant. We can then integrate the equation above to determine $\omega(t)$ and $\theta(t)$ as functions of time.

$$d\omega = \alpha dt \rightarrow \omega(t) = \omega t_0 + \alpha(t - t_0) \tag{3.78}$$

$$d\theta = \omega(t)dt \rightarrow \theta t = \theta(t_0) + \omega(t_0)(t - t_0) + \frac{1}{2}\alpha(t - t_0)^2 \tag{3.79}$$

Another very useful result can be derived as follows:

$$\omega d\omega = \alpha\omega dt = \alpha d\theta \rightarrow \omega^2(t) = \omega^2(t_0) + 2\alpha(\theta - \theta(0)) \tag{3.80}$$

If $\alpha = 0$, then we have **uniform circular motion** as described above.

Example

The angular velocity of a wheel increases uniformly from 20 sec^{-1} to 30 sec^{-1} in 5 sec. Calculate the angular acceleration and the total angle through which it has rotated.

$$\alpha = \text{ constant } = \frac{\Delta\omega}{\Delta t} = \frac{\omega(\Delta t) - \omega(0)}{\Delta t} = \frac{30 \text{ sec}^{-1} - 20 \text{ sec}^{-1}}{5 \text{ sec}} = 2 \text{ sec}^{-2}$$

$$\alpha = 2 \text{ sec}^{-2} = \frac{d\omega}{dt} \rightarrow \omega(t) - \omega(0) = \int_0^t \alpha dt = \alpha t$$

$$\omega(t) = \omega(0) + \alpha t = \frac{d\theta}{dt} \rightarrow \theta(t) - \theta(0) = \int_0^t \omega(0)dt + \int_0^t \alpha dt = \omega(0)t + \frac{1}{2}\alpha t^2$$

Now choosing $t = \Delta t = 5$ sec we get

$$\omega(\Delta t) = \omega(0) + \alpha\Delta t = 20 \text{ sec}^{-1} + 2 \text{ sec}^{-2} \cdot 5 \text{ sec} = 30 \text{ sec}^{-1}$$

$$\Delta\theta = \theta(\Delta t) = \theta(0) = \omega(0)\Delta t + \frac{1}{2}\alpha(\Delta t)^2 = 20 \text{ sec}^{-1} \cdot 5 \text{ sec} + \frac{1}{2} 2 \text{ sec}^{-2} \cdot (5 \text{ sec})^2 = 125 \text{ rad}$$

Finally, we look at an example which shows that non-Cartesian coordinates can be very tricky and non-intuitive.

Example - Radial Motion without Acceleration

Suppose that a particle moves with $\dot{\theta} = \omega$ = constant and $r(t) = r_0 e^{\beta t}$ We then have for the acceleration in plane-polar coordinates

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} = (\beta^2 r_0 e^{\beta t} - r_0 e^{\beta t}\omega^2)\hat{r} + (2\beta r_0 e^{\beta t}\dot{\theta})\hat{\theta}$$

$$\vec{a} = (\beta^2 - \omega^2) r_0 e^{\beta t}\hat{r} + (2\beta\dot{\theta}) r_0 e^{\beta t}\hat{\theta}$$

It is clear that in the special case(s) where $\beta = \pm\omega$ the radial component of the acceleration $a_r = 0$.

Is this surprising? Our assumption is that $r(t) = r_0 e^{\beta t}$, $\dot{r} \neq 0$, $\ddot{r} \neq 0$, but this **does not imply** a non-zero radial component of the acceleration a_r.

We must avoid thinking of all coordinate system as Cartesian where $a_x = 0 \rightarrow \ddot{x} = 0$. This correspondence is not true in this case because the basis vectors are also changing with time and contributing extra terms to the radial acceleration.

An important thing to remember is

> **Everything is not Cartesian coordinates. Be careful how you interpret some system behaviors in non-Cartesian coordinates.**

Example

A particle is moving in a plane according such that $r = 9t^2 + 6t$ and $\theta = 3t^2 + 2t$, where r is measured in meters, θ is measured in radians and t is in seconds. Determine the velocity and acceleration vectors as functions of time. We have

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \quad \text{and} \quad \vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

$$\dot{r} = 18t + 6 \;\; , \;\; \ddot{r} = 18$$

$$\dot{\theta} = 6t + 2 \;\; , \;\; \ddot{\theta} = 6$$

$$\vec{v} = (18t + 6)\hat{r} + (9t^2 + 6t)(6t + 2)\hat{\theta} = (18t + 6)\hat{r} + (54t^3 + 54t^2 + 12t)\hat{\theta}$$

$$\vec{a} = (18 - (9t^2 + 6t)(6t + 2)^6)\hat{r} + (2(18t + 6)(6t + 2) + 6(9t^2 + 6t))\hat{\theta}$$
$$= -(324t^4 + 532t^3 + 144t^2 + 12t - 18)\hat{r} + (270t^2 + 180t = 24)\hat{\theta}$$

Plots of the radial acceleration versus time and the x-position versus the y-position are shown in figures 45 and 46.

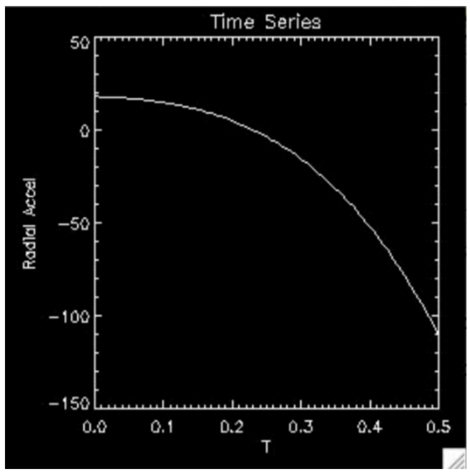

Figure 45:

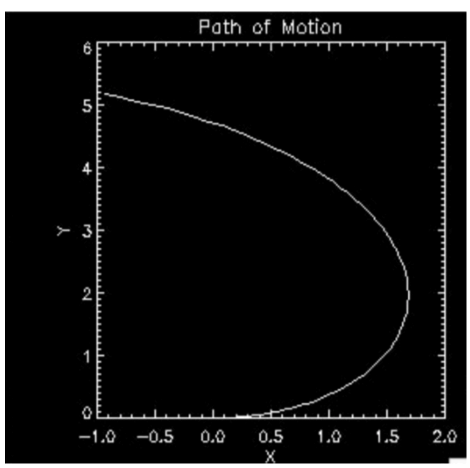

Figure 46:

The graphs clearly indicate that the radial acceleration is zero at some time even though the radius is always increasing! The curve is a spiral.

Some Extra Mathematics - For later use.

Taylor Series

Functions can be expanded in power series.

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then we get by differentiation $f(0) = a_0$, $f'(0) = a_1$, $\frac{1}{2!}f''(0) = a_2$ and so on.... or in general $\frac{1}{n!}f^{(n)}(0) = a_n$. Therefore

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) x^n \tag{3.81}$$

which is called a **Maclaurin series** for $f(x)$ or a **Taylor series** for $f(x)$ **about the origin**.

A Taylor series, in general, means a power series in powers of $(x - a)$ where a = some constant. The derivation of the coefficients is identical to the last derivation except that we use $x = a$ instead of $x = 0$. Let

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \tag{3.82}$$

Then, we get by differentiation $f(a) = a_0$, $f'(a) = a_1$, $\frac{1}{2!}f''(a) = a_2$ and so on.... or in general $\frac{1}{n!}f^{(n)}(a) = a_n$. Therefore

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n \tag{3.83}$$

which is a **Taylor series** for $f(x)$ about the point $x = a$.

Binomial Series

Now consider the following function $f(x) = (1 + x)^n$. If we expand this as a Taylor series we get

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) x^n \tag{3.84}$$

where

$$f(0) = 1 \ , \ f'(0) = n \ , \ \frac{1}{2!}f''(0) = n(n-1) \text{ and so on} \tag{3.85}$$

so that

$$f(x) = (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots = \sum_{m=0}^{\infty} \frac{n!}{m!(n-m)!}x^m = \sum_{m=0} \binom{n}{m} x^m \tag{3.86}$$

which is the **Binomial series**. For n = integer, the series terminates and we have an n^{th} order polynomial.

Examples:

Taylor Series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{3.87}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{3.88}$$

$$e^{\alpha x} = 1 + \frac{\alpha x}{1!} + \frac{(\alpha x)^2}{2!} + \frac{(\alpha x)^3}{3!} + \cdots \tag{3.89}$$

Binomial Series

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots \Rightarrow \frac{1}{(1 \pm x)^n} \approx 1 \mp nx \quad x \ll 1 \tag{3.90}$$

Complex Exponential

$$\begin{aligned} e^{i\alpha x} &= 1 + \frac{i\alpha x}{1!} + \frac{(i\alpha x)^2}{2!} + \frac{(i\alpha x)^3}{3!} + \frac{(i\alpha x)^4}{4!} + \cdots \\ &= 1 + \frac{i\alpha x}{1!} - \frac{(\alpha x)^2}{2!} - i\frac{(\alpha x)^3}{3!} + \frac{(\alpha x)^4}{4!} + \cdots \\ &= \left(1 - \frac{(\alpha x)^2}{2!} + \frac{(\alpha x)^4}{4!} - \cdots\right) + i\left(\frac{\alpha x}{1!} - \frac{(\alpha x)^3}{3!} + \cdots\right) \\ &= \cos \alpha x + i \sin \alpha x \end{aligned} \tag{3.91}$$

$$\cos \alpha x = \frac{e^{+i\alpha x} + e^{-i\alpha x}}{2} \quad , \quad \sin \alpha x = \frac{e^{+i\alpha x} - e^{-i\alpha x}}{2i} \tag{3.92}$$

4. Newton's Laws

In our earlier discussions, we learned how to determine the position vector as a function of time $\vec{r}(t)$ if we know initial conditions and the **acceleration**

function.

$$\vec{r}(t) = \vec{r}(t_0) + \vec{v}(t_0)(t - t_0) + \int_{t_0}^{t} \vec{a}(t)dt \tag{4.1}$$

We now turn to the task of determining the acceleration function.

All new theories are a blend of definitions and experimental results.

Let us first look at experiments for some help in understanding what is going on.

We have this very low friction track to carry out some experiments. We will define the statement that **"an interaction is present"** to mean that we observe the velocity of the cart changing in any manner or that the cart is accelerating.

We make sure that the cart on the track is not **"interacting"** with anything by adjusting it until the cart remains at rest when we let it go... then "nothing" is "affecting" the motion of the cart on the track or "interacting with" the cart on the track - by definition.

Now place a cart on the level track - at rest(velocity = 0):

- the cart remains at rest(because that is how we set things up)

Give the cart a small push

- the cart moves along the track with essentially constant velocity
- the cart will actually slow down and stop after some time has passed, but we will come back to that problem later in our discussion

This is not a surprising result. Motion only has meaning with respect to the particular coordinate system we are using. For example, we are implicitly assuming a coordinate system attached to the track (at rest with respect to the track). Remember, however, we are free to choose any coordinate system we please. Suppose as I pushed the cart, you chose to run along side of it ... that is, you decided to choose a coordinate system that is moving uniformly with respect to the track. In particular, if we choose a coordinate system moving with exactly the same velocity as the cart, then it seems to be at rest, and we already decided that if we set the cart at rest, then it would remain that way (as long as nothing is interacting with it and we already said that was the case) ... so we are not surprised if it continues to move with constant velocity in the other frame attached to the track.

We define any coordinate system which is moving uniformly with respect to the track as an **inertial frame**. We therefore must have defined the track as an inertial frame (the original one)Note that we have not as yet defined an"inertial frame".

In any such frame the cart that was at rest in the original inertial frame (the track), moves uniformly.

However, not all coordinate systems are inertial. In a coordinate system accelerating with respect to the track, the cart that was at rest does not have constant velocity.

If we define an **"isolated"** body as one that is experiencing no interaction, hence is not accelerating, then it always possible to find a coordinate system such that the isolated body is moving uniformly or is even at rest.

This seemingly **circular set of statements** is basically **Newton's 1st law**.

Newton's 1st law:

- **inertial frames** exist for **isolated bodies** (must be defined carefully)
- a body at **rest** or **moving uniformly** remains at rest or moving uniformly unless something **interacts** with it (must be defined carefully)

If we do not seem to be getting anywhere, that is the nature of the first law. It is true, but its meaning is not clear until we have the 2nd law.

Alternatively, I might say that the set of all frames where a body is observed to be at rest or moving uniformly are all **equivalent**. If one of those frames is inertial, then all the equivalent frames are also inertial.

I still have not defined how to find one inertial frame.

Newton's 2nd Law

What happens when the cart is not isolated ..... when it interacts with its surroundings ..... when it accelerates.

Let us make the cart accelerate in a controlled manner .... that is, let us do a physics experiment, well, sort of (remember, I am a theoretical physicist).

Experiment: reporting results

- string attached to the cart + objects attached to the string (hanging over edge of table)
- 1 object attached → a_1 (constant)
- 2 identical objects attached → $2a_1$ (constant)

So, **more attached objects** → **more** interaction → larger accelerations.

- if add objects to cart → acceleration decreases

So, more **stuff** → **less** effect from interaction → smaller accelerations

- define amount of stuff = **mass**

We now use such an experiment to define the quantity **mass** in an **operational manner**.

- we arbitrarily pick some body and define its mass to be 1 unit = 1 kg (Old standard is kept in Paris - a platinum-iridium bar)
- let some object interact with another object on a string and measure its acceleration a_1; call its mass m_1
- let a different object interact with the same object on the string and measure it acceleration a_2, then we define its mass to be

$$m_2 = m_1 \frac{a_1}{a_2} \tag{4.2}$$

This is called an **operational** definition of the mass ratio. If one of the objects is the unit mass, then this is a determination of the absolute value of the mass (ratio with 1).

We can use this procedure to define the mass of all bodies.

This **"operational"** definition would not have any significance unless it is independent of the experiment that we used, i.e., that the ratio of the masses is independent of how we produce the accelerations.

It is!

The mass ratio is independent of the source of the acceleration and appears to be an intrinsic property of a body.

An operational definition of this sort is useful if it has wide applicability. The concept of mass defined in this way is an example. It holds for everyday objects, atoms, nuclei, elementary particles, planets, stars, galaxies and so on.

Now we **define** a quantity called **force**.

We define the **"interaction"** or the operation of the object acting through the string on the cart as the object (or the string) applying a **force** to the cart. This is a description of **how to produce a force** it is **NOT** an answer to the question - **what is a force?**. That will come later.

Experimental results:

- apply force → acceleration in a given direction (given by the string!)
- double the force (add 2nd identical object and assume their effect is cumulative) → 2× acceleration in same direction (if added mass small compared to cart)
- apply force in different direction → same acceleration but direction changes to new direction
- apply identical forces in opposite direction → acceleration = 0 or effect of forces cancel somehow

All of these results can be **"explained or accounted for"** if we make the following assumptions:

- **force** is a **directional** quantity .... it has magnitude and direction – it should be a **vector quantity**
    - remember that acceleration is already known to be a vector quantity

- define a **unit amount** of force as that force that produces a unit of acceleration ($a = 1$) when applied to a unit of mass ($m = 1$)
- the experiments then say (or can be interpreted as saying) that
  - F units of force accelerate the unit mass by F units of acceleration
  - F units of force accelerate m units of mass by F/m units of acceleration

Thus, from experiment (and various definitions) we have

$$F = ma \quad \textbf{in any single direction} \tag{4.3}$$

If we do experiments in more than one dimension we find

$$F_x = ma_x \ , \ F_y = ma_y \ , \ F_z = ma_z \quad , \quad \text{all using the same mass} \tag{4.4}$$

and

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k} = ma_x\hat{i} + ma_y\hat{j} + ma_z\hat{k} \tag{4.5}$$

Units of force:

- mass = kilograms → kg
- acceleration = m/s^2
- force =kg m/s^2 = Newton

Experiment also shows that if two different forces act on the same body, that is, if

$$\vec{F}_1 = m\vec{a}_1 \quad \text{and} \quad \vec{F}_2 = m\vec{a}_2 \tag{4.6}$$

then the total force acting on the body is given by

$$\vec{F} = \vec{F}_1 + \vec{F}_2 = m(\vec{a}_1 + \vec{a}_2) = m\vec{a} \tag{4.7}$$

or the **acceleration produced by several forces** acting on a body is equal to the **vector sum of the accelerations** produced by each of the forces acting separately, which confirms that the force is a vector. Clearly, then, the total force is equal to the vector sum of all the forces.

This fact will be crucial to our ability to solve problems involving

many forces.

The result is called the **Principle of Superposition**. It will be found to hold in many different areas of physics, namely, electromagnetism, quantum physics, and so on.

We summarize all of these experimental results by writing

total(net) or resultant force on a body = $\vec{F} = \sum_i \vec{F}_i$

$\vec{a}$ = **total(net) or resultant acceleration** due to $\vec{F}$

If $\vec{a}_i$ = acceleration due to $\vec{F}_i$ alone, then

$$\vec{F} = \sum_i \vec{F}_i = \sum_i m\vec{a}_i = m\sum_i \vec{a}_i = m\vec{a} \tag{4.8}$$

which is **Newton's 2nd law**.

All of mechanics is derived from this single law.

The existence of a **"force"** is the same as saying two bodies have **"interacted"**. The **"interaction"** is responsible for the **"force"**.

An **"isolated"** body is one subjected to **no interactions** or **no forces** and thus experiences **no acceleration** and hence has **constant velocity** (which is just **Newton's 1st law**).

All known forces or interactions decrease with distance and hence it should be possible to actually create an isolated object by getting it far away from everything else.

Later, we will return to these ideas and try to define what is **"really"** happening during an interaction and hence what is **"really"** meant by the concept of a force and whether or not it is **"really"** necessary.

Newton's 3rd Law

We have seen that a force is necessarily the result of an interaction between two systems. This is made explicit by the 3rd law.

The 3rd law states that **forces ALWAYS appears in pairs**, that is

If a system B exerts a force $\vec{F}_{AB}$ on a system A, there must be a force $\vec{F}_{BA}$ acting on a system B, due to system A, such that

$$\vec{F}_{AB} = -\vec{F}_{BA} \tag{4.9}$$

Summary of Newton's Laws

Newton's 1st law

Inertial frames exist for isolated bodies.

A body at rest remains at rest unless it interacts with another body.

The velocity of a body remains constant unless it interacts with another body.

MODERN VIEW : In the absence of interactions, an object moves with constant velocity.

Newton's 2nd law

Resultant or Net Force = mass x resultant acceleration (a vector equation) (mass must be defined for each body)

$$\vec{F} = m\vec{a}$$

Newton's 3rd law

MODERN VIEW : When systems A and B interact, the force that the interaction exerts on A is equal in magnitude and opposite in direction to the force it exerts on B.

So, If you push a book across a table, the force that you feel is NOT the force that makes the book move. It is the force that the book exerts on you. According to Newton's 3rd law these two forces are equal and opposite.

We will return to these laws (definitions) after we define linear momentum and find that after introducing conservation laws, the 3rd law is the most fundamental of the three Newton's laws.

Applications of Newton's Laws

There are a number of distinct steps that we must be careful to carry out when we try to apply Newton's laws to any physical system.

We will now outline this procedure and then apply it to a range of problems in order to learn the method.

This procedure is not the only way to approach these problems. It is just the most straightforward and easiest to apply at the level of this course. Later, I will show you another procedure that is used at a more advanced level.

In many of our examples it will not be clear how to extract the proper information that will allow us to apply our well-defined procedure.

This is simply an example of the well-known fact that no procedure, no matter how well it is stated, can substitute for intelligent analytical thinking on your part.

You will have to think about and understand the physical systems under investigation well enough so that you can extract the information you will need to apply the procedure.

The procedure just organizes your thinking processes and gives you a step-by-step process for determining the information you need to solve the problem at hand.

Procedure:

(1) **Divide the physical system (collection of physical objects) into smaller systems, each of which can be treated as a point mass.**

This is the standard **reductionist** approach to solving problems.

It has been a successful approach in science for centuries. It relies on the assumption that the smaller a system is, the more fundamental it must be, and therefore, the easier to understand (it is assumed).

In addition, it assumes that once we understand the small fundamental systems, we can reconstruct the behavior or the larger system that they are part of by combining the behaviors of the smaller, fundamental systems in some way.

As I said, this has worked very well for centuries. It **DOES NOT** mean that it will always work well. It probably will fail in chaotic systems(as we shall see later), in the study of elementary particles and gravity, in complex biological systems and so on.

(2) **Draw a force diagram for each mass as follows:**

   (a) represent the body by a point and label it

   (b) draw a force vector on the mass for each force **acting** on it

   - **do not** draw force vectors for forces exerted by the body on other parts of the system
   - according to Newton's laws **ONLY** forces acting on a body can influence its motion

Example: We consider 2 blocks at rest on a table as shown in figure 47.

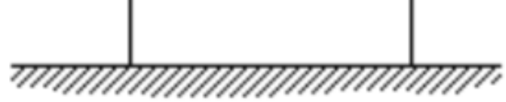

Figure 47:

Replace with points masses.....and add force vectors......see figure 48.

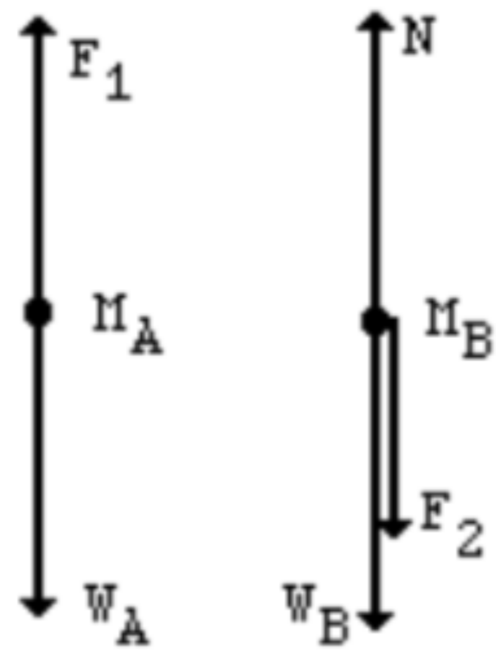

Figure 48:

For body A we have

F_1 = the force exerted on block A by block B

W_A = the force on body A due to gravity (= weight of A)

For body B we have

F_2 = the force exerted on block B by block A

W_B = the force on body B due to gravity (= weight of B)

N = the force exerted on body B by the table (called the **normal force**)

Note that all **"contact"** forces, that is, F_1, F_2, N act in a direction perpendicular to the interface(surface) between the bodies.

There are no other physical interactions that would produce a force on body A or body B.

When you say there is a force acting on an object you should always be able to identify which other object external to the first object is causing the force. These are REAL forces.

These diagrams are called **FREE-BODY DIAGRAMS**.

Do not confuse forces with accelerations! We only draw **REAL** forces on free-body diagrams.

Repeating this important point, in inertial systems, you should be able to answer the question for any force that might be included on the free-body diagram:

What system(object) exerts this force on the body ?

(3) **Introduce a coordinate system.**- See figure 49

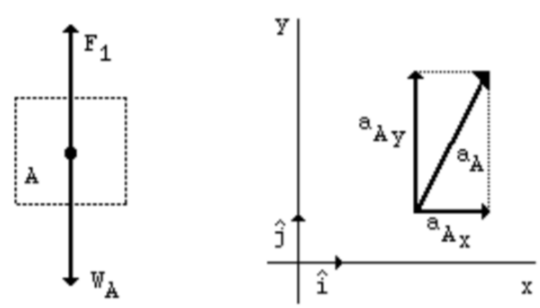

Figure 49:

This must be an inertial coordinate system (fixed to an inertial frame) for Newton's laws to be valid.

Using the force diagrams write the equations for the force (acceleration) components for each body.

These equations are of the form for the i^{th} component

$$
\begin{aligned}
F_i = F_{1i} + F_{2i} + F_{3i} + \cdots &= \sum_{k=1}^{n} F_{ki} \\
&= ma_{1i} + ma_{2i} + ma_{3i} + \cdots = m\sum_{k=1}^{n} a_{ki} = ma_i
\end{aligned}
\tag{4.10}
$$

where $i = x, y$, and z (or 1,2, and 3). **The algebraic signs of each terms must be consistent with the directions assumed in the force diagrams and with the choice of coordinate axis directions.**

In the example above for block A, we assume that the acceleration has a direction as shown (this choice will be arbitrary as long as internal consistency is maintained in the system). Then, Newton's 2nd law gives

$$\vec{F}_1 + \vec{W}_A = m_A\vec{a}_A$$
$$F_1\hat{j} - W_A\hat{j} = m_A a_{Ax}\hat{i} + m_A a_{Ay}\hat{j}$$

which says that

$$F_1 - W_A = m_A a_{Ay}$$
$$0 = m_A a_{Ax}$$

In a similar manner for block B we get

$$\vec{F}_2 + \vec{W}_B + \vec{N} = m_B\vec{a}_B$$
$$-F_2\hat{j} - W_A\hat{j} + N\hat{j} = m_B a_{Bx}\hat{i} + m_B a_{By}\hat{j}$$

which says that

$$-F_2 - W_A + N = m_B a_{By}$$
$$0 = m_B a_{Bx}$$

(4) **If two bodies in the same system interact, the forces between them must be equal and opposite by Newton's 3rd law.**

These equations should be written down explicitly. In this systems we have such a case:

$$\vec{F}_1 = -\vec{F}_2 \rightarrow F_1 = F_2$$

Newton's 3rd law will always relate forces on two different bodies (never on the same body).

(5) **In many systems, some or all of the bodies involved are constrained to move along certain paths** (like a ball bearing that is rolled down a track with a loop).

Other examples are

a pendulum bob which moves on a circle

a block sliding on a plane must move in the plane

Each constraint is expressed by a kinematical equation → a **constraint** equation. We must write down all constraint equations **explicitly**.

We also have implicit constraints like the statement that the blocks are at rest on the table top. The constraint equations in this case are then

$$\vec{a}_A = \vec{a}_B = 0$$

(6) **Keep track of which variables are known and which are unknown.**

The force equations and the constraint equations should provide the **same number** of equations as there are unknowns.

This is the best way of finding out that you either overlooked an equation (too few equations) or that you are saying something that is incorrect (too many equations).

For the blocks on the table we now have the full set of equations:

$F_1 - W_A = m_A a_{Ay}$ from Newton's 2nd law

$0 = m_A a_{Ax}$ from Newton's 2nd law

$-F_2 - W_A + N = m_B a_{By}$ from Newton's 2nd law

$0 = m_B a_{Bx}$ from Newton's 2nd law

$F_1 = F_2$ from Newton's 3rd law

$\vec{a}_A = \vec{A}_B$ from constraint

We can then carry out the mathematical solution of the equations to get

$$F_1 = F_2 = W_A$$
$$N = W_A + W_B$$

which make physical sense!

Having just crushed a peanut with a sledgehammer in the above simple system, let us now do some other examples. There is no other way to learn how to carry out this procedure except by doing a lot of practice problems. Along the way we will discuss all of the everyday forces we see around us so that we can use them in the problems.

Newton's 2nd Law —- More Examples

#1 - Simple Block - see figure 50.

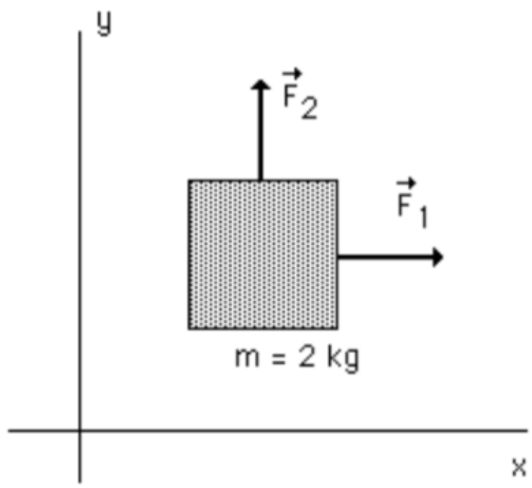

Figure 50:

If $\vec{F}_1 = 2\hat{i}$ and $\vec{F}_2 = 1\hat{j}$ calculate the resultant acceleration of the block.

Solution:

$$\vec{F} = \vec{F}_1 + \vec{F}_2 = 2\hat{i} + 1\hat{j} = m\vec{a} = ma_x\hat{i} + ma_y\hat{j}$$

$$a_x = \frac{F_x}{m} = \frac{\vec{F}\cdot\hat{i}}{m} = \frac{F_1}{m} = 1\ \text{m/s}^2$$

$$a_y = \frac{F_y}{m} = \frac{\vec{F}\cdot\hat{i}\hat{j}}{m} = \frac{F_2}{m} = 0.5\ \mathrm{m/s}^2$$

$$\vec{a} = 1\hat{i} + 0.5\hat{j}$$

#**2 - Blocks on a Plane** - see figure 51.

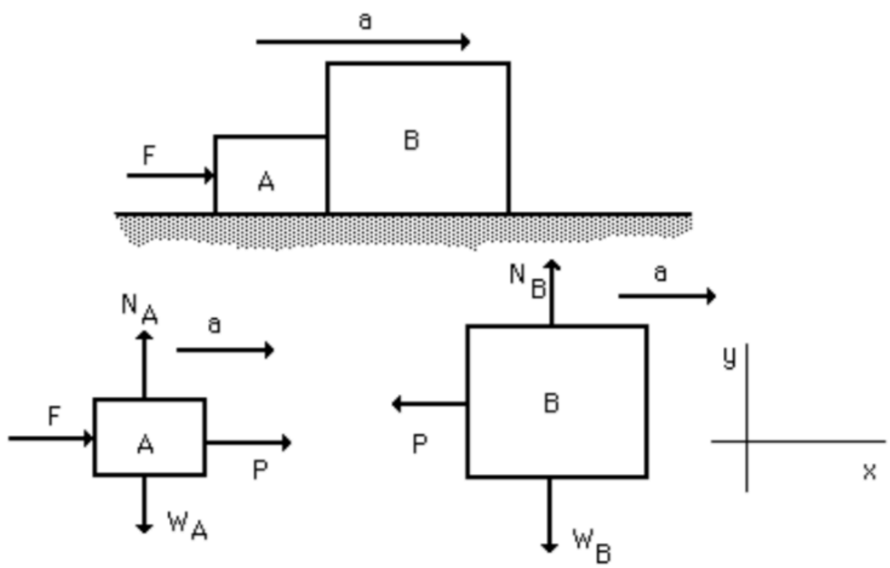

Figure 51:

y-direction

$$N_A - W_A = 0 \to N_A = W_A$$

$$N_B - W_B = 0 \to N_B = W_B$$

x-direction

$$F + P = m_A a$$

$$-P = m_B a \to a = \frac{F}{m_A + m_B}$$

which makes sense since the internal force P should not contribute to the net acceleration. Note that I purposely chose P in the wrong direction and it does not matter. Solving for P we have

$$P = -m_B a = -\frac{F m_B}{m_A + m_B}$$

The negative sign indicates that we have chosen the incorrect direction for P.

#3 What is the effect of pulling on a string?

Case 1: The Massless String

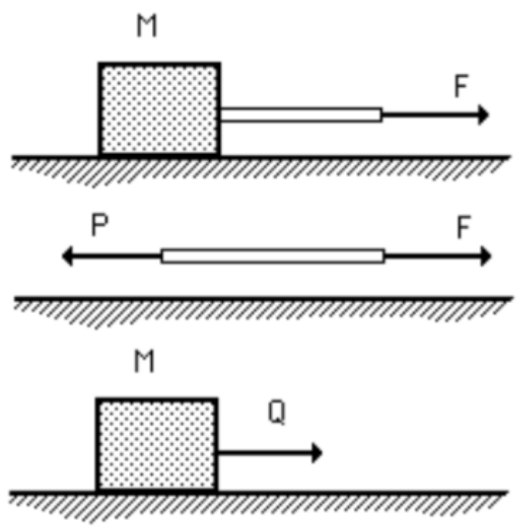

Figure 52:

We have a block of mass M being pulled along a horizontal table force F as in figure 52. The force F actually pulls on a massless rope which in turn then pulls on the block. If we isolate all of the bodies as shown, then the 3rd law says that

$$\vec{P} = -\vec{Q} \to P = Q$$

The 2nd law says

$$F - P = m_{rope} a_{rope} = 0 \to F = P = Q$$

$$Ma = Q = F \to a = \frac{F}{M}$$

That is, because the rope is massless, the net force on it must vanish. This means that the force(tension) anywhere inside the rope is = F and that

$$\vec{Q} = \vec{F}$$

A **massless** rope simply **transfers the force** to whatever object it attached. If a **massless** rope **changes direction** (a pulley) then all that happens is the **tension changes direction without changing magnitude**.

Case 2: Rope with mass

The same arguments give

$$\vec{P} = -\vec{Q} \rightarrow P = Q$$

and

$$F - P = m_{rope}a_{rope}$$
$$Ma = Q = P = F - m_{rope}a_{rope}$$
$$a = \frac{F}{M + m_{rope}}$$

which again makes sense - we have simply increased the total mass of the system. The tension in the string is not constant!

We will consider other cases involving massive ropes as we go along.

#**4 - Simple Pulley** - see figure 53.

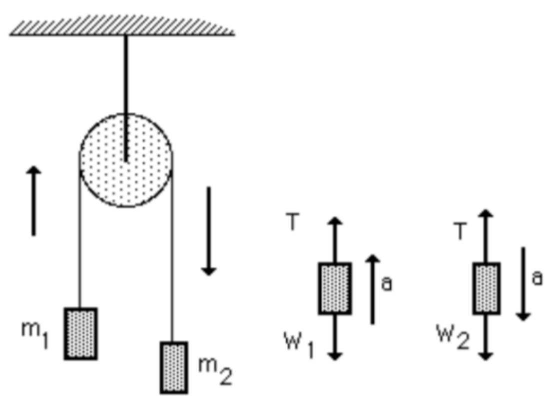

Figure 53:

$$T - W = m_1 a$$

$$W_2 - T = m_2 a$$

$$W_2 - W_1 = (m_1 + m_2)a$$

$$a = \frac{W_2 - W - 1}{m_1 + m_2}$$

$$T = \frac{m_1 W_1 + m_2 W_2}{m_1 + m_2}$$

What happens if $W_1 > W_2$?

The acceleration turns out to be negative. This just means that we assumed the wrong direction. Notice that the tension T is not affected.

#5 **- Pulleys and Blocks Together** - see figure 54.

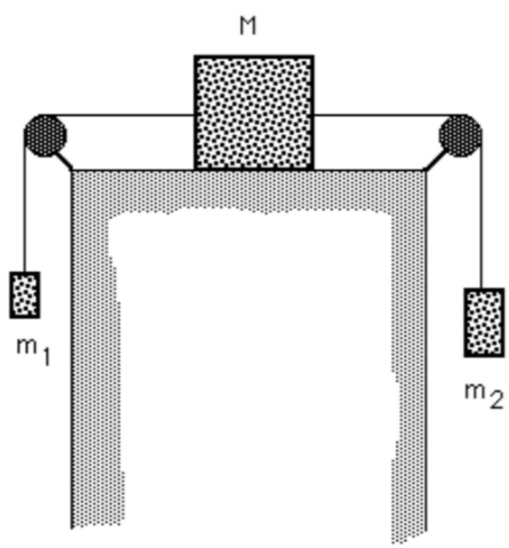

Figure 54:

This implies the free-body diagrams in figure 55.

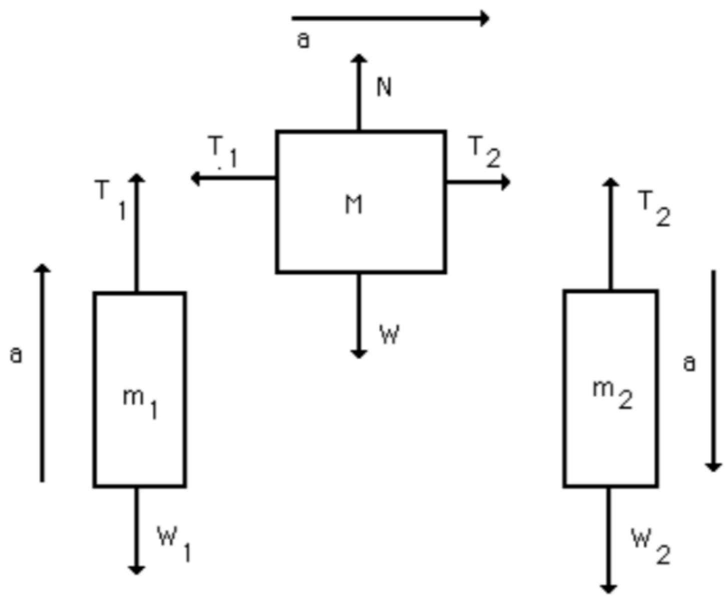

Figure 55:

All the bodies must have the same acceleration (we assume that the strings do not stretch).

We can choose an arbitrary direction for the acceleration (since in general we do not know the direction until after we do the calculation). We just must make all choices internally consistent.

If we make the wrong choice for the direction the answer will come out negative, indicating the vector actually points in the opposite direction to what we assumed.

From Newton's 2nd law we now get

$$T_1 - W_1 = m_1 a \ , \ T_2 - T_1 = Ma$$

$$W_2 - T_2 = m_2 a \ , \ N - W = 0 \to N = W$$

or substituting for T_2

$$W_2 - m_2 a - T_1 = Ma$$

then substituting for T_1

$$W_2 - W_1 = (M + m_1 + m_2)a$$

$$a = \frac{W_2 - W_1}{M + m_1 + m_2}$$

Return to Circular Motion - now using plane-polar coordinates

Earlier we found, in general,

$$\vec{r} = r\hat{r} \ , \ \hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j} \ , \ \hat{\theta} = -\sin\theta\hat{i} + \cos\theta\hat{j}$$

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

$$\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta} \quad , \quad \frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r}$$

For uniform circular motion we then have

$$\dot{r} = 0 \to r = R = \text{ constant} \quad , \quad \ddot{\theta} = 0 \to \dot{\theta} = \omega = \text{ constant}$$

which says that

$$\vec{v} = R\omega\hat{\theta} \text{ (tangent to the circular path of motion)}$$

$$\vec{a} = -R\omega^2\hat{r} \text{ (directed towards the center of the circle)}$$

Circular Orbits

If an object is in circular orbit around the earth(or any other body) then we must have

$$\vec{a} = -\frac{v^2}{r}\hat{r} = -r\dot{\theta}^2\hat{r} = -r\omega^2\hat{r}$$

with directions as shown in figure 56:

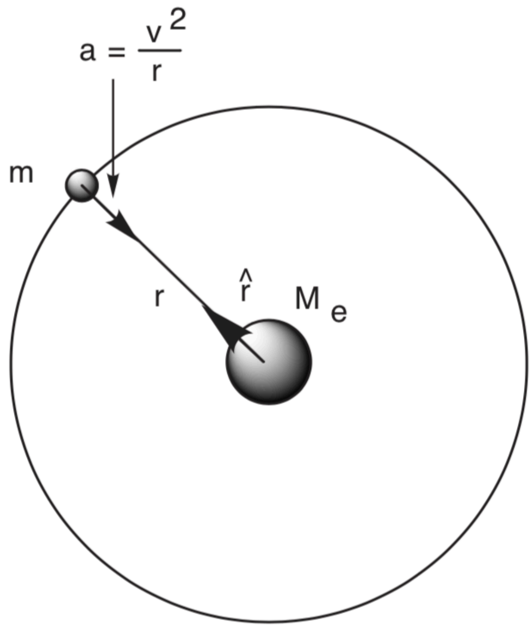

Figure 56:

The **only force** acting on the mass m in orbit is the gravitational attraction of the earth

$$\vec{F} = -G\frac{mM_e}{r^2}\hat{r} \tag{4.11}$$

Therefore, Newton's second law gives

$$\vec{F} = -G\frac{mM_e}{r^2}\hat{r} = m\vec{a} = -m\frac{v^2}{r}\hat{r} \tag{4.12}$$

$$G\frac{mM_e}{r^2} = \frac{v^2}{r} \tag{4.13}$$

$$v = \sqrt{\frac{GM_e}{r}} \tag{4.14}$$

The period of the motion in orbit is given by

$$T = \frac{2\pi r}{v} = 2\pi\sqrt{\frac{r^3}{GM_e}} \rightarrow \text{ Kepler's 3rd Law} \tag{4.15}$$

Suppose I wanted to be in orbit and remain stationary over a single spot on the earth (geosynchronous satellite). The period of the motion would have to be 1 day.

$$T = 2\pi\sqrt{\frac{r^3}{GM_e}} = 24 \times 60 \times 60 \text{ sec } = 86400 \text{ sec } \to \sqrt{\frac{r^3}{GM_e}} = 1.376 \times 10^4$$

$$\frac{r^3}{GM_e} = 1.893{\times}10^8 \to r^3 = 1.893{\times}10^8 \text{ sec}^2(6.67{\times}10^{-11} \text{ N-m}^2/\text{kg}^2)(5.98{\times}10^{24} \text{ kg})$$

$$r = 4.227 \times 10^7 \text{ m } = 26268 \text{ mi } - 3959 \text{ mi } = 22309 \text{ mi}$$

$$\text{distance above surface } = 22309 \text{ mi } - 3959 \text{ mi } = 22309 \text{ mi}$$

More Uniform Circular Motion

A **conical pendulum** looks like figure 57.

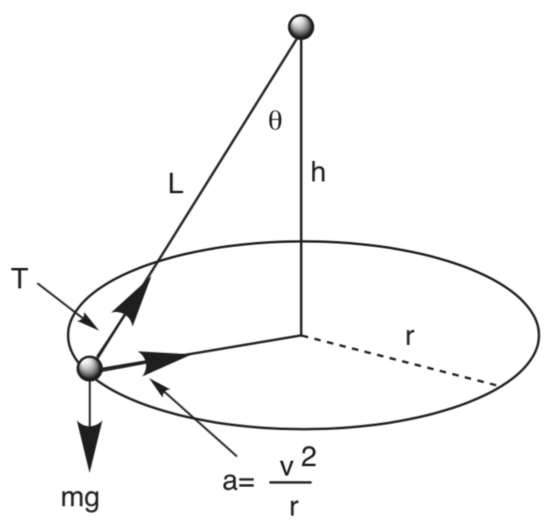

Figure 57:

The only forces acting on the mass m are gravity and the tension. Since we have circular motion(say in the $x-y$ plane) we have the following solution

$$\vec{T} + \vec{W} = m\vec{a} \to \text{ Newton' s 2nd law vector equation}$$

$$T_z\hat{e}_z - T_r\hat{r} - mg\hat{e}_z = -mr\omega^2\hat{r} \to \text{ inserting unit vectors}$$

$$T\cos\theta\hat{e}_z - T\sin\theta\hat{r} - mg\hat{e}_z = -mr\omega^2\hat{r} \rightarrow \text{ inserting components}$$

$$T\cos\theta = mg \quad \text{and} \quad T\sin\theta = mr\omega^2 \rightarrow \text{ equating components}$$

$$T = \frac{mr\omega^2}{\sin\theta} = mL\omega^2$$

$$\cos\theta = \frac{mg}{T} = \frac{g}{L\omega^2}$$

The Vertical Circle

If we are on a ferris wheel(constrained motion) that moves in a vertical circle with constant speed we have the situations shown in figure 58:

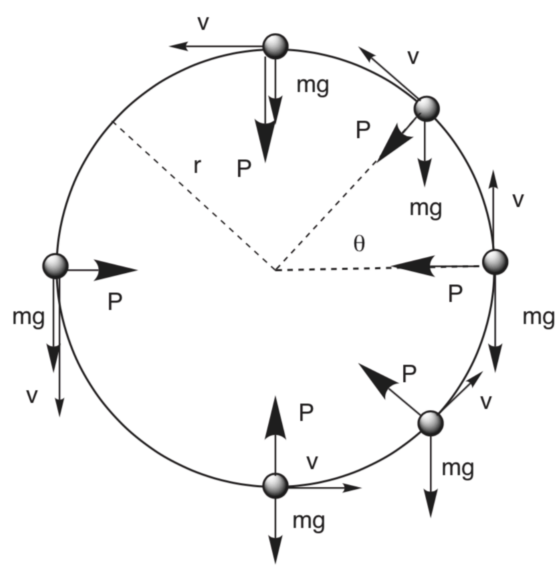

Figure 58:

At each point the acceleration of the mass m is directed towards the center of the circle(constrained motion) and has a magnitude $a = \frac{v^2}{r}$, where v is the velocity at that point. The force due to the ferris wheel on the mass m can be considered as two components. One is tangent(in the $\hat{\theta}$ direction) to the path of motion (the circle) and the other is radial(perpendicular to the path of motion) in the $\hat{r}$ direction. Let us called the radial component P. The only other force on the mass m is it's weight, which is always vertically downwards. The radial and tangential components of the weight change as the mass goes around the circle. At a point with angle θ (as shown) we have

$$W_r = -mg\sin\theta \quad \text{and} \quad W_\theta = -mg\cos\theta$$

In this discussion, we are only interested in the radial equation of motion(as we will see later, the tangential equation determines the speed changes). We have

$$-P - mg\sin\theta = -m\frac{v^2}{r} \rightarrow P = m\left(\frac{v^2}{r} - g\sin\theta\right)$$

Therefore

$$\theta = 0 \rightarrow P = m\left(\frac{v^2}{r}\right) \quad , \quad \theta = 90° \rightarrow P = m\left(\frac{v^2}{r} - g\right)$$

$$\theta = 180° \rightarrow P = m\left(\frac{v^2}{r}\right) \quad , \quad \theta = 270° \rightarrow P = m\left(\frac{v^2}{r} + g\right)$$

What is happening?

At the sides, the weight is not helping us stay in a circular orbit, that is, since $P > 0$, We must rely on being strapped into the ferris wheel seat.

At the bottom, the ferris wheel always pushes upwards on us to keep us going in a circle.

At the top, the ferris wheel force can go either way depending on the speed as we pass the top.

If $\frac{v^2}{r} > g$, then P is downwards. What does that mean?

If $\frac{v^2}{r} < g$, then P is upwards. What does that mean?

If $\frac{v^2}{r} = g$, then $P = 0$. What does that mean?

Contact Forces

These are forces which are transmitted between bodies by short-range atomic or molecular interactions(electromagnetic forces). Examples are tension in strings and ropes, sliding friction and viscosity in a fluid.

Force is transmitted through a body at a finite speed (order of 4000 m/s). It is a direct interatomic electromagnetic force. If a string is massless, then the tension is constant throughout the string. If the string has mass then the tension varies through the string in a manner that guarantees that all parts of the string have the same acceleration.

Pulleys with no mass simply redirect a tension force in a string. If the pulley has a mass, this is a more difficult case. We must study the rotational motion of a solid body before we can deal with this case. We will do that later.

Normal forces are the equal and opposite forces that exist on the interface between two objects. It is a direct interatomic electromagnetic force.

Frictional Forces

Friction cannot be described by a simple theory. It arises when one surface slides over the surface of a second body. The details of the frictional force are very complicated. The general things we can say are:

(1) The frictional force always opposes the motion; it is opposite to the instantaneous velocity

(2) The frictional force is different if the body is at rest than if it is moving

(3) If a body is not moving due to a force acting on it because of friction then the frictional force is equal in magnitude to whatever force has been applied and opposite in direction

   (a) There exists a maximum such static frictional force

$$f \leq \mu_{static} N \ , \ \ N = \text{ normal force at interface} \tag{4.16}$$

(4) For surfaces moving over each other

$$f = \mu_{kinetic} N \ , \ \ N = \text{ normal force at interface} \tag{4.17}$$

(5) For regular surfaces – not very smooth – the friction force does not depend on the area in contact because actually very little of the area is in contact.

   The frictional force is due to direct atomic forces; if we can get more atoms close to each other than the frictional force should increase with the contact area (proportional to number of atoms interacting).

If we brought two perfectly flat(perfectly smooth) surfaces together the frictional force would be very large (not zero) and the surfaces would actually stick together.

Viscosity

A body moving through a liquid or gas is retarded by a force due to a property called **viscosity**. In many cases, this force has a simple velocity dependence – it is proportional to the velocity. If the velocity gets too large this simple relationship breaks down because of turbulence effects. So we can write

$$\vec{F}_v = -c\vec{v} \ \ , \ \ C = \text{ constant dependent on fluid and geometry} \tag{4.18}$$

Inertial Systems and Fictitious Forces

Suppose that we have two frames of reference as shown below. Suppose that one of the frames is an inertial frame so that Newton's laws are valid and the second frame (see their origins) is separated from the first by a vector $\vec{R}(t)$ as shown in figure 59.

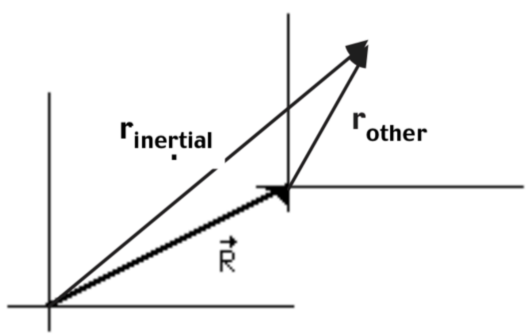

Figure 59:

We then have the relations (for any particle being observed by both frames)

$$\vec{r}_{other} = \vec{r}_{inertial} - \vec{R} \tag{4.19}$$

This gives

$$\ddot{\vec{r}}_{other} = \ddot{\vec{r}}_{inertial} - \ddot{\vec{R}} \tag{4.20}$$

which implies that

$$\vec{F}_{apparent} = \vec{F}_{other} = \vec{F}_{inertial} - m\ddot{\vec{R}}(t) = \vec{F}_{true} - m\ddot{\vec{R}}(t) \tag{4.21}$$

If $\ddot{\vec{R}}(t) = 0$, then $\vec{F}_{apparent} = \vec{F}_{true}$ or the second coordinate system is **also** inertial.

If we carry out measurement in a non-inertial frame (earth for instance) what can we do to get the correct equations of motion?

We will use the equation

$$\vec{F}_{apparent} = \vec{F}_{true} - m\ddot{\vec{R}}(t) \tag{4.22}$$

and think of the last term as an additional force → a **fictitious** force, because **NO REAL INTERACTION IS INVOLVED**; it is there only because we chose to use a non-inertial frame where Newton's law does not hold.

We can write

$$\vec{F}_{apparent} = \vec{F}_{true} + \vec{F}_{fictitious} \tag{4.23}$$

where

$$\vec{F}_{fictitious} = -m\ddot{\vec{R}}(t) \tag{4.24}$$

Fictitious forces are useful in solving many problems but they must be used with great care. **Centrifugal force** is such a fictitious force. It only exists in a rotating reference frame!

Equilibrium

The earlier example with the blocks on the table represents a system in equilibrium, which is the simplest application of Newton's laws.

Let us define equilibrium in general.

> **Equilibrium corresponds to the absence of any acceleration**

Newton's second law then says that we must have

$$\vec{F} = \sum_i \vec{F}_i = 0 \tag{4.25}$$

or the system can be acted upon by forces but their **vector sum must add to zero**.

We will see shortly that equilibrium requires a second condition to prevent rotational motion.

Newton's 2nd Law —- Everyday Forces and More Examples

The theoretical physicist spends a lot of time trying to understand experimental results. She will then try to synthesize them into some theory. This usually involves guessing the form the interaction between objects in the system under investigation.

In the study of mechanics, this involves coming up with force or acceleration functions to represent the forces that affect the motion of objects in the macroscopic world.

There are only three fundamental interactions at this moment

> Color charge → quark-gluon interactions → strong interactions.
>
> Electric charge→electroweak interactions of leptons, vector bosons like the photon, and quarks.
>
> Mass or energy → gravitational interactions between all bodies.

The so-called Standard Model of Elementary Particles combines the first two into a single interaction and the ultimate hope of Loop Quantum Gravity is to combine all into one fundamental interaction governing all things.

A manifestation of the electroweak interaction are electric and magnetic fields.

Another is the weak interaction.

Strong and weak interactions are extremely short range (less than 1 fermi $= 10^{-15}$ m) and we can neglect them in this course.

Electromagnetic and gravitational interactions are essentially infinite range. If both are present electromagnetic interactions will usually dominate. If bodies are neutral (no net electric charge) then gravity dominates as in planetary motion.

What kind of force is exerted by common objects?

Force Due to Gravity

The study of mechanics really started with investigations about gravity. Newton figured out the force law for gravitational interactions and was able to derive Kepler's laws of motion for planets, which were obtained from experiment.

Newton's Law of Gravitation

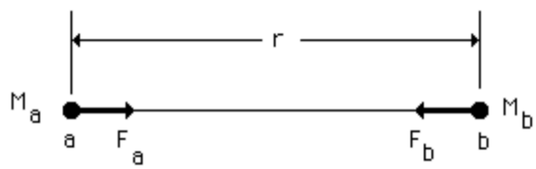

Figure 60:

Figure 60 above represents two massive bodies interacting via the gravitational force. We have

$$\vec{F}_b = \text{ force on } b \text{ due to } a = -\vec{F}_a = \text{ force on } a \text{ due to } b$$

$$\vec{F}_b = -\frac{GM_aM_b}{r^2}\hat{r}_{ab} \quad , \quad \hat{r}_{ab} = \frac{\vec{r}_{ab}}{r} \tag{4.26}$$

or

$$\vec{F}_a = \frac{GM_aM_b}{r^2}\hat{r}_{ab} = -\frac{GM_aM_b}{r^2}\hat{r}_{ba} \quad , \quad \hat{r}_{ab} = -\hat{r}_{ba} \tag{4.27}$$

where the vectors are defined as shown in figure 61

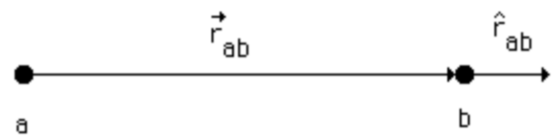

Figure 61:

That is, the force is proportional to the product of the masses and inversely proportional to the distance between them. The constant of proportionality is $G = 6.67 \times 10^{-11}$ N-m^2/kg^2.

The gravitational force has a special property in relation to Newton's 2nd law.

$$\vec{F}_b = -\frac{GM_aM_b}{r^2}\hat{r}_{ab} = M_b\vec{a}_b \rightarrow \vec{a}_b - \frac{GM_a}{r^2}\hat{r}_{ab} \tag{4.28}$$

or the acceleration due to gravitational interaction is **INDEPENDENT of the MASS being accelerated !!**

Gravity Near the Earth

For now we will assume that the **force due to gravity** on the earth can be represented by the simple expression

$$\vec{W} = \vec{F}_g = m\vec{g} = -mg\hat{e}_2 = -mg\hat{j} = -mg\hat{e}_y \tag{4.29}$$

or that everywhere in the laboratory (not too large a volume) the force of gravity points vertically downwards(a constant direction) and has a constant magnitude mg where

$$g = \text{ acceleration due to gravity } = 9.80 \text{ m/s}^2 \tag{4.30}$$

How good an approximation is this?

The exact gravitational force law is

$$\begin{aligned}\vec{F}_g &= -G\frac{mM_{earth}}{r^2}\hat{r} = -G\frac{mM_{earth}}{(R_{earth}+h)^2} \\ &= -m\frac{GM_{earth}}{R_{earth}^2}\frac{1}{\left(1+\frac{h}{R_{earth}}\right)^2}\hat{r} = -mg\frac{1}{\left(1+\frac{h}{R_{earth}}\right)^2}\hat{r}\end{aligned} \tag{4.31}$$

where R_{earth} = radius of the earth, M_{earth} = mass of the earth, r = distance from center of the earth to the object, m = mass of object and h = height of object above the surface.

Clearly, the force is actually along a radial line $\hat{r}_{ab} = \hat{r}$ (we will also need a correction for the rotating earth later on) and varies with height above the surface.

In a laboratory let us say that

$$h \leq 1000 \text{ m } \rightarrow \frac{h}{R_{earth}} \leq \frac{10^3}{6.4 \times 10^7} = 1.5 \times 10^{-5}$$

$$\frac{1}{\left(1 + \frac{h}{R_{earth}}\right)^2} \approx 1 - \frac{2h}{R_{earth}} = 1 - 3 \times 10^{-5} \rightarrow 1$$

and as shown in the figure 62

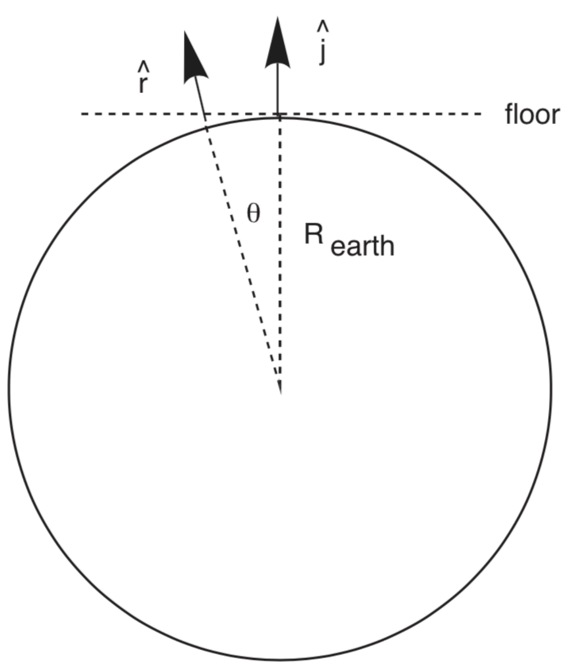

Figure 62:

$$\hat{r}\cdot\hat{j} = \cos\theta = \sqrt{1-\sin^2\theta} \approx 1 - \frac{1}{2}\sin^2\theta = 1 - \frac{1}{2}\left(\frac{\text{width of lab}}{R_{earth}}\right)^2$$

$$\cos\theta \approx 1 - \frac{1}{2}\left(\frac{1000}{R_{earth}}\right)^2 \approx 1 - \frac{1}{2}(2.25\times 10^{-5})^2 \to 1 \to \hat{r} \approx \hat{j}$$

Therefore, we have $\vec{F}_g = -mg\hat{j}$ to a **very good approximation**.

Weight

The **"weight of a body"** = the reading of a scale that it rests on.

Consider the following situation.

Imagine a turtle on a scale in an elevator and the elevator is accelerating **upwards** with acceleration a. It looks like figure 63.

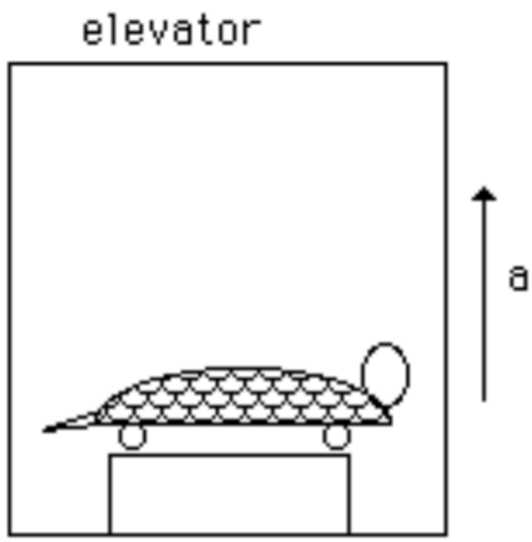

Figure 63:

The force diagram is as shown in figure 64

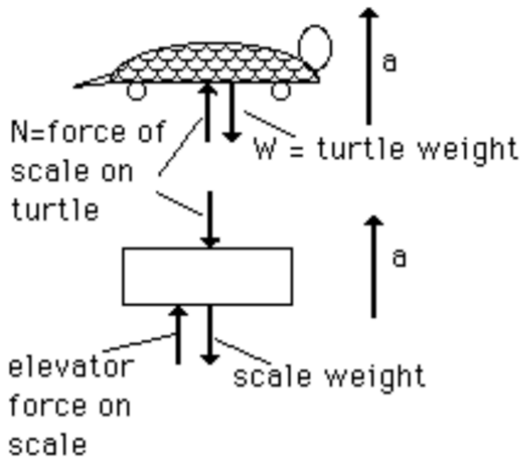

Figure 64:

where N = force scale exerts on turtle and thus = **"scale reading"** = **apparent weight**. Setting up and solving Newton's 2nd law we notice that the weight is equal to

$$N - W = N - F_{gravity} = ma \rightarrow N = m(a + g) = \text{ scale reading } = \text{ apparent weight}$$

If the elevator was not accelerating, then the apparent weight is mg = force due to gravity. We define this to be the **"normal"** weight of the object on the surface of the earth.

We can now understand what **"weightless"** means. If our acceleration is $-g$ (= g downwards) then the scale reading = 0 and we are **weightless**. So, if the elevator cable broke, the turtle would be weightless until the elevator crashed into the bottom! More about this when we go into orbit later.

Springs

The force exerted by a stretched spring (lying along x-direction) is given by

$$F_x = -kx \quad , \quad k = \text{ constant } = \text{ spring constant} \tag{4.32}$$

and x is measured from the equilibrium position of the spring as shown in figure 65 below.

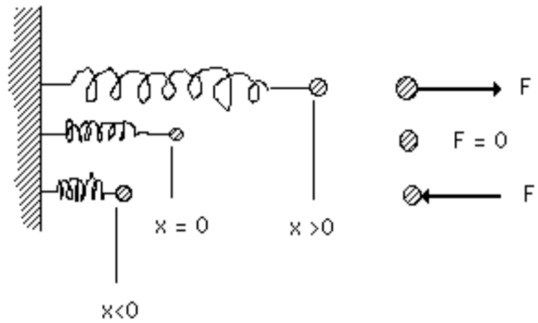

Figure 65:

Ropes and Rods

Non-rigid objects like ropes can only pull, while objects like rods can both push and pull. The pull of a rope is directed along its length. Newton's 3rd law says that the object pulls back on the rope with the same force in the opposite direction, putting the rope under **tension**, which is then transmitted throughout the rope.

What happens to the tension as it is transmitted through the rope?

We will assume for a while that ropes do not have any mass. Consider a piece of rope with a tension of T on one end. Assume the rope is split in two and the tension changes as shown in figure 66.

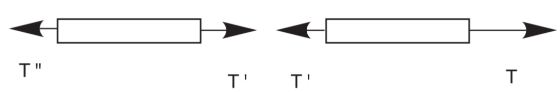

Figure 66:

Look at the piece on the right and apply Newton's 2nd law. We get

$$T - T' = m_{rope} a_{rope} = 0 \rightarrow T = T' \tag{4.33}$$

because the **mass is zero**. This says that the tension is transmitted unchanged through a massless rope or $T = T' = T'' = \cdots$ and so on.

Example

Consider an object which is held up (in equilibrium) by a person using two ropes in the configuration shown in figure 67:

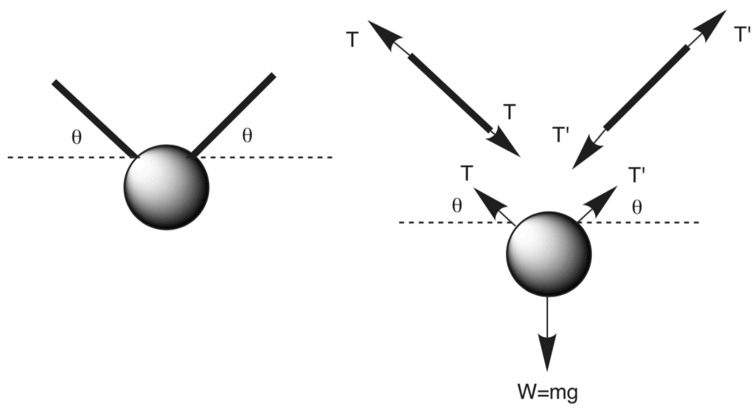

Figure 67:

The vector sum of the forces must equal zero. This implies that

$$\sum_i \vec{F}_i = 0 = \sum_i F_{ix}\hat{e}_x + \sum_i F_{iy}\hat{e}_y$$

$$\sum_i F_{ix} = 0$$

$$\sum_i F_{iy} = 0$$

We have

$$T'\cos\theta - T\cos\theta = 0 \rightarrow T' = T \text{ as expected in this symmetric configuration}$$

$$T'\sin\theta + T\cos\theta - mg = 0 \rightarrow T = \frac{mg}{2\sin\theta}$$

What happens as the person holding the weight makes the angle smaller and smaller?

Wedges - Fixed and Moving

Just a fixed wedge

Let the wedge in figure 68 be fixed. All measurements are from an inertial frame.

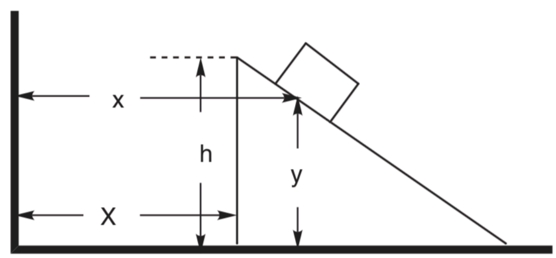

Figure 68:

Assume the block has acceleration a **down the plane**. See figure 69.

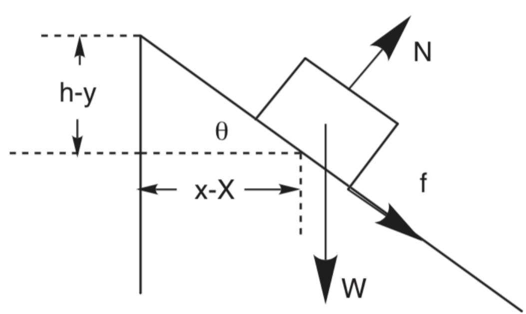

Figure 69:

The free-body diagram says that

$$mg\sin\theta = ma \rightarrow a = g\sin\theta$$

$$N = mg\cos\theta$$

If friction was present, then

$$mg\sin\theta - \mu N = ma$$

$$N = mg\cos\theta$$
$$a = g(\sin\theta - \mu\cos\theta)$$
$$mg\sin\theta + \mu N = ma$$
$$N = mg\cos\theta$$

and(if block is going up plane)

$$a = g(\sin\theta + \mu\cos\theta)$$

A rather straightforward result.

What happens if the wedge can move?

In this case, we need to see how constraints affect the solution of problems.

What is the **relationship(constraint)** between the accelerations of the block and wedge?

Since the wedge is in contact with the table we have the trivial constraint that the **vertical** acceleration of the wedge = 0.

The less obvious constraint has to do with the **horizontal** direction.

Looking at figure 69 we have

$$x - X = (h - y)\cot\theta \to \dot{x} - \dot{X} = -\dot{y}\cot\theta \to \ddot{x} - \ddot{X} = -\ddot{y}\cot\theta$$

which is the **equation of constraint between the various accelerations**.

Important things to note: we defined all coordinates in an **inertial frame** ... fixed to the floor. If we had attached the reference frame to the wedge(**a non-inertial frame**) then it would have been a more difficult problem, especially to interpret(fictitious forces would arise). We will discuss this case in a later chapter.

Unimportant parameters such as the height of the wedge (h) vanish from the equations when we take time derivatives but they are absolutely necessary to set up the coordinates.

Note that constraint equations are independent of applied forces, i.e., if we added friction, the constraint equation would not change.

An Example - A 45° wedge is pushed along a table with constant acceleration A. A block of mass m slides without friction on the wedge. Find its acceleration a.

Newton's 2nd law gives ($x - y$ directions) for the block of mass m:

$$N\cos\theta = m\ddot{x} \quad , \quad N\sin\theta - mg = m\ddot{y}$$

where we have used $W = mg$ and the figures above. The constraint equation gives

$$x - X = (h - y)\cot\theta \rightarrow \ddot{x} - A = -\ddot{y}\cot\theta$$

Now $\cos\theta = \sin\theta = \sqrt{2}/2$, so we get

$$\frac{N}{\sqrt{2}} = m\ddot{x} \quad , \quad \frac{N}{\sqrt{2}} - mg = m\ddot{y}$$

$$\ddot{x} - A = \ddot{y}$$

or

$$\ddot{x} = \frac{A+g}{2} \quad , \quad \ddot{y} = \frac{A-g}{2}$$

Pulleys and More Constraint Problems

Two masses are connected by a string which passes over a pulley that is accelerating upwards at A as shown in figure 70. How are the accelerations of the two bodies related? Assume that the motion is only vertical.

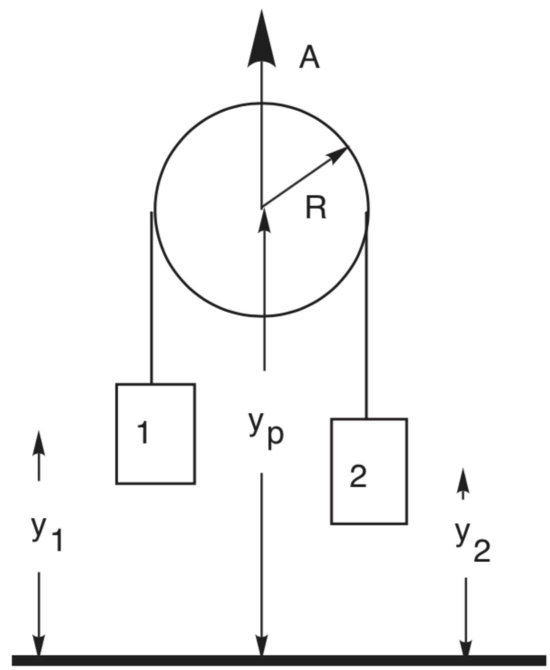

Figure 70:

We use coordinates as shown (measured from inertial frame - the floor). The constraint is that the length of the string L is constant.

y_p is measured to the center of the pulley; $R =$ radius of the pulley

$$L = \pi R + (y_p - y_1) + (y_p - y_2) \rightarrow \text{ constraint equation}$$

Differentiating the constraint equation and using $A = \ddot{y}_p$ we get

$$0 = 2\ddot{y}_p - \ddot{y}_1 - \ddot{y}_2 \quad , \quad A = \frac{1}{2}(\ddot{y}_1 + \ddot{y}_2)$$

The equations of motion are

$$T - m_1 g = m_1 \ddot{y}_1 \quad , \quad T - m_2 g = m_2 \ddot{y}_2$$

and we get

$$(m_1 - m_2)g = m_1 \ddot{y}_1 + m_2 \ddot{y}_2$$

$$\frac{(m_1 - m_2)g + 2m_2 A}{m_1 + m_2} = \ddot{y}_1 \quad , \quad \frac{2m_1 A - (m_1 - m_2)g}{m_1 + m_2} = \ddot{y}_2$$

Does it agree with earlier result when $A = 0$?

Now look at the pulley system in figure 71.

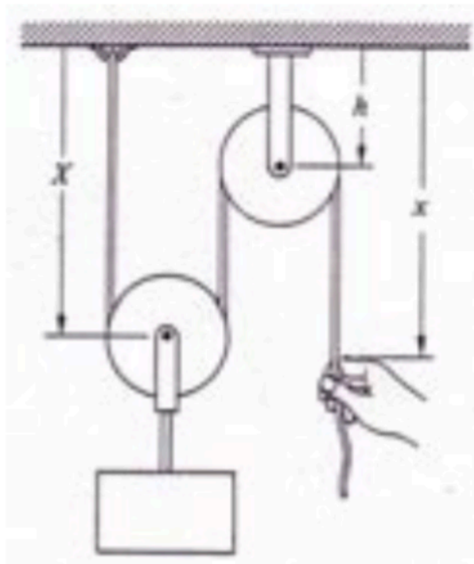

Figure 71:

How does the acceleration of the rope in the hand compare with the acceleration of the block that is, being lifted? Use the coordinates(again measure from an inertial frame - the ceiling) as shown.

In this case we have the constraint equation

$$L = X + \pi R + (X - h) + \pi R + (X - h) \rightarrow \ddot{X} = -\frac{1}{2}\ddot{x}$$

Another example - The Double Pulley

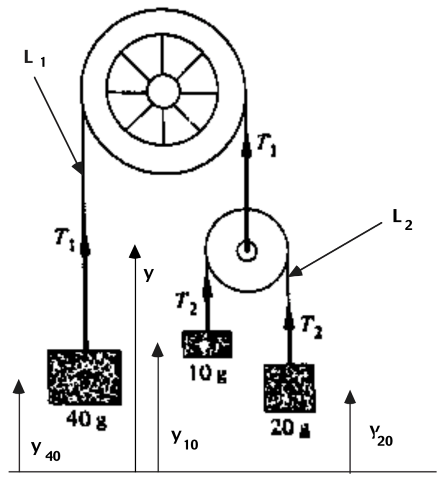

Figure 72:

We define the positions and lengths as shown in figure 72.

We then have the equations of motion

$$m_{40}g - T_1 = m_{40}a_{40}$$

$$T_1 - 2T_2 = 0$$

$$T_2 - m_{10}g = m_{10}a_{10}$$

$$T_2 - m_{20}g = m_{20}a_{20}$$

and the constraint equation

$$a_{40} = -\frac{1}{2}(a_{10} + a_{20})$$

The constraint equation follows from this argument:

$$y_{40} + L_1 + y = \text{ constant}$$

$$2y = L_2 + y_{20} + y_{10}$$

Differentiating we get

$$\ddot{y}_{40} + \ddot{y} = 0$$

$$2\ddot{y} = \ddot{y}_{20} + \ddot{y}_{10}$$

$$\ddot{y}_{40} = -\frac{1}{2}(\ddot{y}_{10} + \ddot{y}_{20})$$

Masses and Pulleys all over the place - See figure 73.

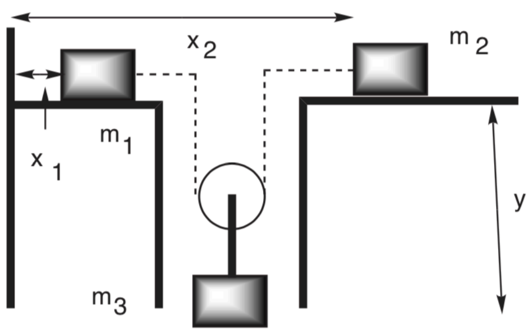

Figure 73:

Note that my choice of coordinates uses an inertial reference frame.

We assume that the coefficient of friction between 1 and 2 and the table is μ.

The force diagrams look like those in figure 74.

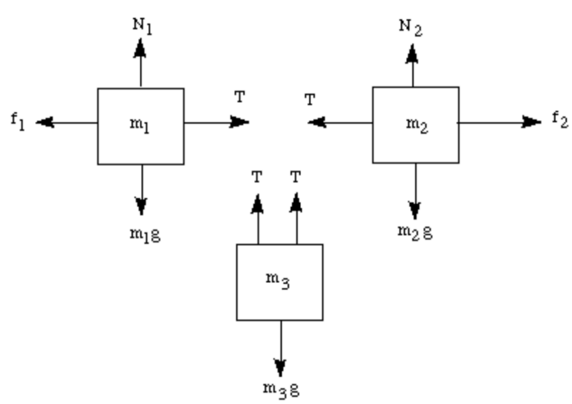

Figure 74:

and we have

$$x_2 - x_1 + 2y = \text{ length of the string } = \text{ constant } \rightarrow \text{ constraint}$$

$$\ddot{x}_2 - \ddot{x}_1 + 2\ddot{y} = 0$$

$$T - f_1 = M_1\ddot{x}_1 \quad , \quad f_1 = \mu N_1 = \mu M_1 g$$

$$f_2 - T = M_2\ddot{x}_2 \quad , \quad f_2 = \mu N_2 = \mu M_2 g$$

$$M_3 g - 2T = M_3\ddot{y}$$

$$T = \frac{2(\mu+1)g}{\frac{1}{M_1} + \frac{1}{M_2} + \frac{4}{M-3}}$$

The gravitational force of a sphere (like the earth)

The following result can be proven. It requires a lot of calculus and is done at the end of this section for the two-dimensional case. We will only need the result. **Newton invented calculus to do this derivation!!**

The gravitational force obeys the **superposition principle** ..... the force due to a collection of particles is equal to the vector sum of all the individual forces. So if we think of extended bodies as small particles all super-glued together we can find the force due to an extended body by vector addition (difficult but doable).

The first result is for a thin shell of mass as shown in figure 75:

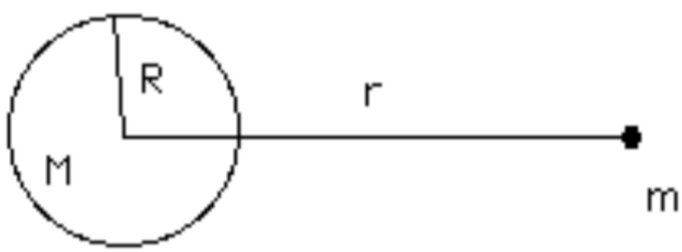

Figure 75:

Newton used calculus to show that

$$\vec{F} = \begin{cases} -G\frac{Mm}{r^2}\hat{r} & r > R \\ 0 & r < R \end{cases} \tag{4.34}$$

i.e., the force due to the shell of mass M (on a mass m outside the shell) is the same as if all the mass M were concentrated at the center as in figure 76.

Figure 76:

This result can be made plausible by a simple argument.

Consider the mass m inside the shell as shown in figure 77

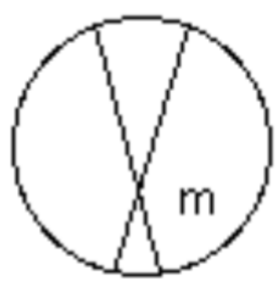

Figure 77:

and consider the two regions of mass on the shell inside the lines on opposite sides of the mass. The amount of mass is proportional to the area on the shell which is proportional to the distance-squared from the mass. However the gravitational force due to this mass decreases as the distance-squared. Since the forces due to each piece are in opposite directions, it seems possible that they could cancel (they do exactly because of the joint distance-squared dependence) and so the force anywhere inside a shell due to the shell is $= 0$.

Now a uniform(solid) sphere can be thought of as being made up of spherical shells and it follows for particles outside of it, a sphere behaves gravitationally as if its mass were all concentrated at the center of the sphere. The result also holds if the density is not constant(non-uniform) as long as it only varies with radius (spherically symmetric).

Therefore, for a body like the earth - see figure 78

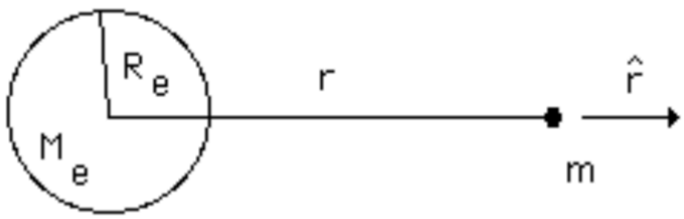

Figure 78:

to a good approximation, for masses outside the earth we have

$$\vec{F} = -G\frac{M_e m}{r^2}\hat{r} \quad r \geq R_e \tag{4.35}$$

Inside the earth, only the mass at points inside the shell at the position of the particle contributes to the force. The shells outside that position give zero force. This means that inside the earth, say at a position $r < R_e$, the force on a particle of mass m is given by

$$\vec{F}(r) = -G\frac{M_e(r)m}{r^2}\hat{r} \quad r < R_e$$

$$M_e(r) = \text{ volume } \cdot \text{ density } = \frac{4\pi}{3}r^3\rho = \frac{4\pi}{3}r^3\frac{M_e}{\frac{4\pi}{3}R_e^3} = \frac{r^3}{R_e^3}M_e$$

Therefore

$$\vec{F}(r) = -G\frac{\frac{r^3}{R_e^3}M_e m}{r^2}\hat{r} = -G\frac{M_e m}{R_e^3}r\hat{r} \quad r < R_e \tag{4.36}$$

It is a **LINEAR** force inside the sphere!!!

The Gravitational Field

We define the **gravitational field** at a point in space a as the force per unit mass due to a mass at b (really the acceleration). It is given by

$$\vec{G}_a(\vec{r}_{ab}) = \frac{\vec{F}_{ab}}{M_b} = -\frac{GM_a}{r^2}\hat{r}_{ab} \tag{4.37}$$

At this point , it is just a **mathematical construct** that **associates** a vector with **each point** in space. At the moment it looks like a mathematical rearrangement that is really not needed, does not introduce anything new, and thus cannot have any new physical meaning.

Nothing could be further from the truth.

We will not do much with fields in this course, but next course one studies is electricity and magnetism and in that subject the Electric and Magnetic fields will dominate all physical discussions and it will become clear that classical fields of this type are **"real"** physical things.

Gravitational Proof for a Ring in 2 Dimensions

Consider the ring of mass shown in figure 79.

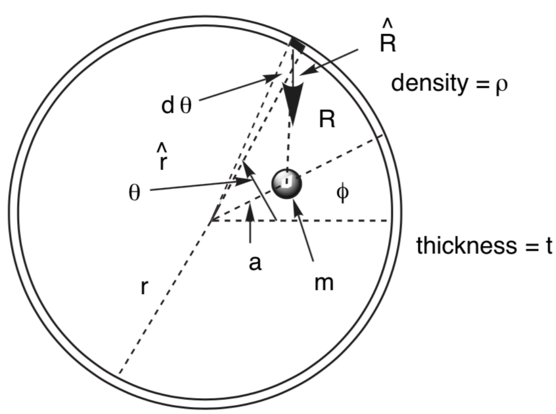

Figure 79:

The small element of mass is $dM = rt\rho d\theta$. The total mass of the ring is $M = 2\pi rt\rho$. The gravitational force on the mass m due to the small mass element is

$$d\vec{F}_g = -G\frac{mdM}{R^2}\hat{R} \tag{4.38}$$

We have

$$R^2 = r^2 + a^2 - 2ra\cos(\theta - \phi) \tag{4.39}$$

$$\vec{r} = r\hat{r} = r\cos\theta\hat{e}_x + r\sin\theta\hat{e}_y \tag{4.40}$$

$$\vec{a} = a\cos\theta\hat{e}_x + a\sin\theta\hat{e}_y \tag{4.41}$$

$$\vec{R} = R\hat{R} = \vec{a} - \vec{r} = (a\cos\theta - r\cos\theta)\hat{e}_x + (a\sin\theta - r\sin\theta)\hat{e}_y \tag{4.42}$$

Therefore,

$$d\vec{F}_g = -Gmrt\rho d\theta\frac{(a\cos\theta - r\cos\theta)\hat{e}_x + (a\sin\theta - r\sin\theta)\hat{e}_y}{(r^2 + a^2 - 2ra\cos(\theta - \phi))^{3/2}}$$

and

$$\vec{F}_g = \int d\vec{F}_g = -Gmrt\rho\int_0^{2\pi}\frac{(a\cos\theta - r\cos\theta)\hat{e}_x + (a\sin\theta - r\sin\theta)\hat{e}_y}{(r^2 + a^2 - 2ra\cos(\theta - \phi))^{3/2}}d\theta \tag{4.43}$$

If we do the integral we get

$$\vec{F}_g = -G\frac{mM}{a^2}\hat{a} \tag{4.44}$$

as if the entire mass was located at the center of the circle!!

Electrostatic Forces

Similar to the gravitational force between two masses, we can define the electric force between two electric charges as

$$\vec{F}_b = -k\frac{q_a q_b}{r^2}\hat{r}_{ab} = q_b\vec{E}_a \tag{4.45}$$

$$\vec{E}_a = -k\frac{q_a}{r^2}\hat{r}_{ab} = \text{ electric field vector} \tag{4.46}$$

Force on a Charged Particle in an Electromagnetic Field

An electromagnetic field is characterized by two vector fields

$$\vec{E} = \text{ electric field vector} \quad , \quad \vec{B} = \text{ magnetic field vector}$$

A charged particle (q) experiences a force given by the Lorentz force law

$$\vec{F} = q\left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right) \tag{4.47}$$

Spring and a Block

Consider the situation shown in figure 80.

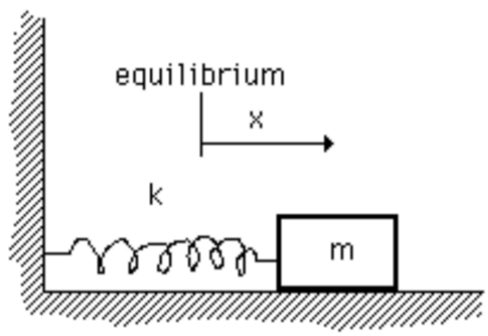

Figure 80:

The equation of motion is

$$m\ddot{x} = -kx \rightarrow \ddot{x} + \omega^2 x = 0 \quad , \quad \omega^2 = \frac{k}{m} \tag{4.48}$$

This is an example of a 2nd-order differential equation. Later in the course we will learn how to solve such equations.

We can guess the solutions in this case because we can experimentally observe the motion.

Since it is periodic in time, we will assume sinusoidal solutions of the form

$$x(t) = A \sin \omega t + B \cos \omega t \tag{4.49}$$

Substitution into the original equation shows that it is a solution for arbitrary A and B values, i.e.,

$$x(t) = A\sin\omega t + B\cos\omega t$$

$$\dot{x}(t) = A\omega\cos\omega t - B\omega\sin\omega t$$

$$\ddot{x}(t) = -A\omega^2\sin\omega t - B\omega^2\cos\omega t = -\omega^2 x(t)$$

Therefore

$$m\ddot{x} = -m\omega^2 x = -kx \rightarrow \omega^2 = \frac{k}{m} \tag{4.50}$$

The period of the periodic motion is the repetition time, that is, the time it takes for $x(t)$ to return to the same value, which corresponds to the time T it take for the argument of the sine or cosine to change by 2π or

$$\omega T = 2\pi \rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \tag{4.51}$$

The frequency of the motion is

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{4.52}$$

This motion is called **Simple Harmonic Motion (SHM)**.

Hole Through the Earth

The gravitational force on a body located at a distance R from the center of a uniform spherical mass is due solely to the mass lying at distance $r \leq R$, measured from the center of the sphere. The mass exerts a force as if it were a point mass at the origin. Let us use this result to figure out what happens if we drill a hole through the earth and drop a mass into it. We first consider a hole through the center of the earth and then an off-center hole. See figure 81.

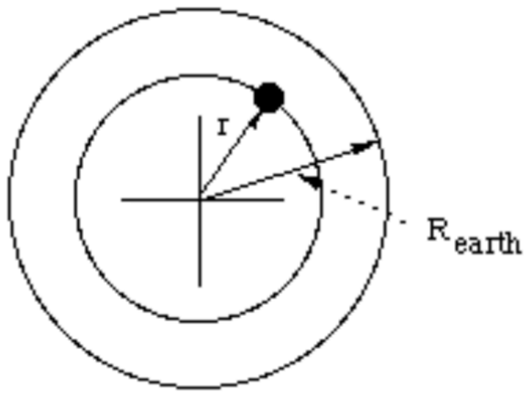

Figure 81:

The gravitational force on m at radius r is

$$F = G\frac{Mm}{r^2} \quad \text{with} \quad M = M_{earth}\left(\frac{r}{R_{earth}}\right)^3 \tag{4.53}$$

or force on m is

$$F = G\frac{m}{r^2}M_{earth}\left(\frac{r}{R_{earth}}\right)^3 = \frac{GmM_{earth}}{R^3_{earth}}r = \frac{mg}{R_{earth}}r \tag{4.54}$$

where $g = \frac{GM_{earth}}{R^2_{earth}}$ is the acceleration due to gravity at the surface of the earth.

This is a one-dimensional problem (along the line through the center of the earth). The equation of motion is

$$-F = ma_r = m\ddot{r} = -\frac{mg}{R_{earth}}r \to \ddot{r} + \frac{g}{R_{earth}}r = 0 \tag{4.55}$$

which corresponds to simple harmonic motion with a period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R_{earth}}{g}} \approx 84 \text{ minutes} \tag{4.56}$$

Now figure 82 shows a circular orbit at the surface of the Earth.

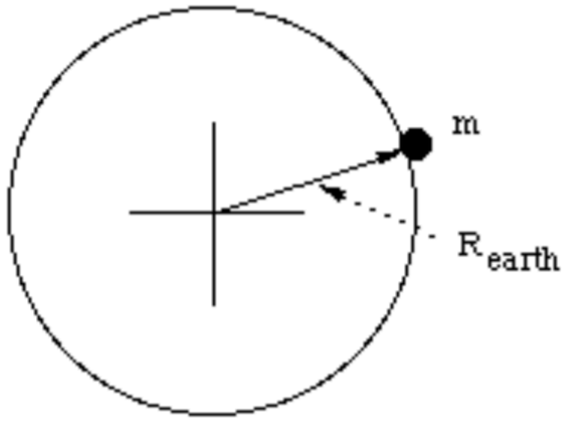

Figure 82:

For this circular motion(orbit) at the surface of the earth we have the equation of motion ($r \approx R_{earth}$ in our earlier equations)

$$-F = ma_r = m\ddot{r} = -\frac{mg}{R_{earth}}R_{earth} = -mg = -m\frac{v^2}{R_{earth}}$$

$$\rightarrow v^2 = gR_{earth} \rightarrow \omega = \frac{2\pi}{T} = \frac{v}{R_{earth}} = \sqrt{\frac{g}{R_{earth}}}$$

$$\rightarrow T = 2\pi\sqrt{\frac{R_{earth}}{g}}$$

The same result as going through the center! Why?

Off-center hole....

As can be seen in figure 83

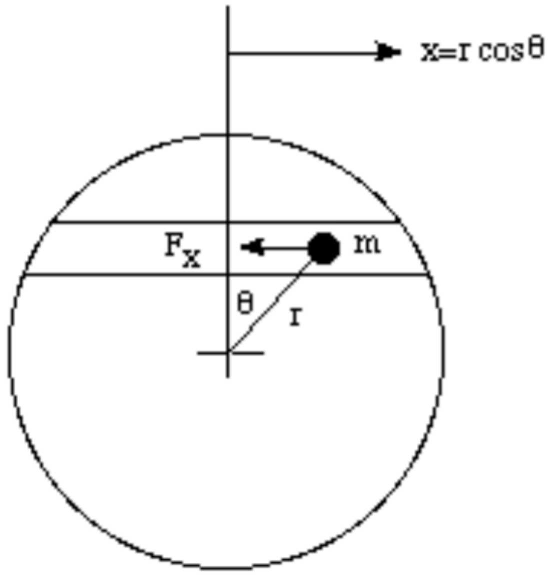

Figure 83:

this is also a one dimensional problem along the hole. The equation of motion in the x-direction is (using earlier results)

$$F_x = F_r \cos\theta = \left(\frac{mg}{R_{earth}} r\right) \cos\theta = \frac{mg}{R_{earth}} (r \cos\theta = \frac{mg}{R_{earth}} x$$

$$\rightarrow m\ddot{x} = -F_x = -\frac{mg}{R_{earth}} x \rightarrow \ddot{x} + \frac{g}{R_{earth}} x = 0$$

which is again simple harmonic motion with the **SAME** period as when the hole was through the center! Can we use this result in any way?

Back to the Mundane – Blocks on strings

Now let us consider the dynamics of rotational motion.

A particle undergoing circular motion must have a radial acceleration.

Suppose that mass m whirls with constant speed vv at the end of a string of length R as in figure 84(top).

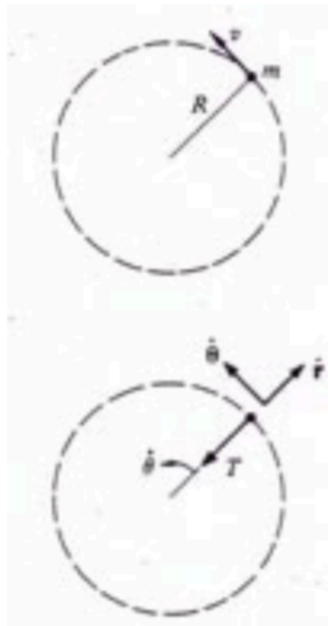

Figure 84:

What is the force on m (assume no gravity or friction)?

The only force on m is the string force or tension T, which acts toward the center as shown in figure 84(bottom).

We use polar coordinates.

The radial acceleration is $a_r = \ddot{r}?r\dot{\theta}^2$ where $\dot{\theta}$ is the angular velocity and a_r is positive if it points outward.

The tension $\vec{T}$ is directed toward the origin, $\vec{T} = -T\hat{r}$ and thus the **radial equation of motion** is

$$-T = ma_r = m(\ddot{r} - r\dot{\theta}^2)$$

For circular motion, we have

$$\ddot{r} = \ddot{R} = 0 \ , \ \text{and} \ , \ \dot{\theta} = \frac{v}{R}$$

Thus

$$a_r = -\frac{v^2}{R} \ , \ T = m\frac{v^2}{R}$$

Note that T is directed toward the origin ... there is NO outward force on m.

The force you feel does not act on the block, it acts on you.

It is equal in magnitude and opposite in direction to the force with which you pull the block.

Now let us do an example with both radial and tangential accelerations.

Again we whirl a mass m on a string, except that now it is in a vertical plane in the gravitational field of the earth.

The forces are as shown in figure 85.

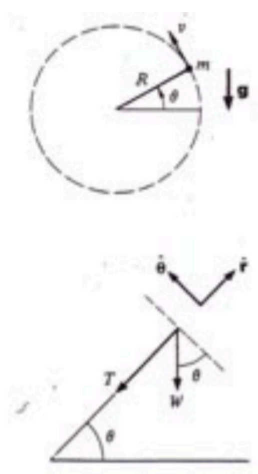

Figure 85:

The geometry is shown in the figure and v = the instantaneous speed. The radial equation is

$$-T - W\sin\theta = ma_r = m(\ddot{r} - r\dot{\theta}^2)$$

The tangential equation is

$$-W\cos\theta = ma_\theta = m(2\dot{r}\dot{\theta} + r\ddot{\theta})$$

Since $r = R$ = constant, we have

$$a_r = -R\dot{\theta}^2 = -\frac{v^2}{R}$$

and

$$T = m\frac{v^2}{R} - W\sin\theta$$

Now a string can **PULL** but not **PUSH** on a mass, so that T **cannot be negative** (point the opposite way). This requires that

$$T \geq 0 \rightarrow m\frac{v^2}{R} \geq W\sin\theta$$

The maximum value of $W\sin\theta$ occurs when the mass is vertically up. In this case

$$m\frac{v^2}{R} \geq W$$

If this condition is not satisfied, the mass does not follow the circular path but starts to fall and $\ddot{r}$ is no longer $= 0$.

During the circular motion, the tangential acceleration is given by

$$a_\theta = R\ddot{\theta} = -\frac{W\cos\theta}{m}$$

The mass does not move with constant speed, that is, it accelerates tangentially.

On the downswing its speed increases and on the upswing it decreases.

Putting It All Together Now

Now let us consider a system in which **everything is happening** ... rotational motion, translational motion and constraints.

A horizontal frictionless table has a small hole in its center. Block A on the table is connected to block B hanging beneath the table by a string (no mass) which passes through the hole as shown in figure 86.

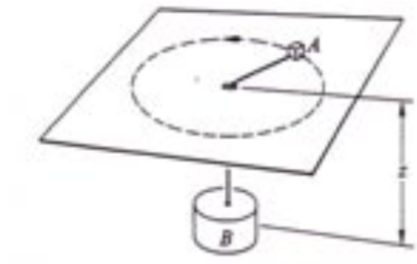

Figure 86:

Initially B is held stationary and A rotates at constant radius r_0 with steady angular velocity ω_0.

If B is released at $t = 0$, what is the acceleration immediately afterward?

The force diagrams for A and B are shown in figure 87.

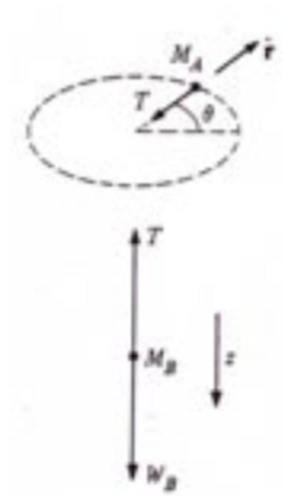

Figure 87:

The vertical forces acting on A are in balance and we can ignore them. The only horizontal force on A is the string force T. The forces on B are the string force T and the weight W_B.

We use polar coordinates(they are natural here) for A and a single linear coordinate for B as shown in figure 87. The equations of motion are

$$-T = M_A(\ddot{r} - r\dot{\theta}^2)$$

$$0 = M_A(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

$$W_B - T = M_B\ddot{z}$$

Since the string length $L = r + z$ is constant we have

$$\ddot{r} = -\ddot{z}$$

as a **constraint equation**. What does the negative sign mean? We get

$$r = r_0 \ , \ \dot{\theta} = \omega_0 \ , \ \ddot{z}(0) = \frac{W_B - M_A r_0 \omega_0^2}{M_A + M_B}$$

Thus, $\ddot{z}(0)$ can be $>, =,$ or < 0 - depending on the numerator; if ω_0 is large enough block B will start to rise after release!

Solving a Differential Equation

A block of mass m slides on a frictionless table. It is constrained to move inside a ring of radius L which is fixed to the table. At $t = 0$ the block is moving along inside the ring (in the tangential direction) as shown in figure 88 with velocity v_0.

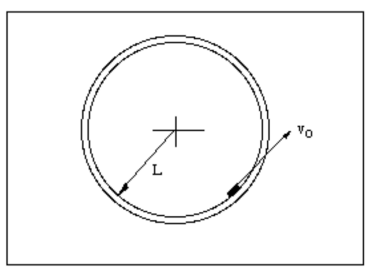

Figure 88:

The coefficient of friction between the block and the ring is μ.

The force diagram is shown in figure 89:

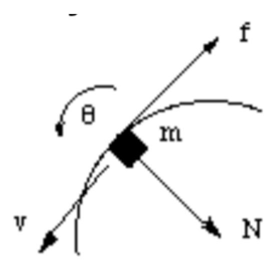

Figure 89:

where N is the reaction force due to the ring.

The general acceleration is given by:

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

In the special case above we have $r = \text{ constant } = L \rightarrow \dot{r} = \ddot{r} = 0$ so that

$$\vec{a} = -L\dot{\theta}^2\hat{r} + L\ddot{\theta}\hat{\theta} = -\frac{v^2}{L}\hat{r} + L\ddot{\theta}\hat{\theta}$$

The equations of motion in the radial and tangential directions are then:

$$N = \frac{mv^2}{L} = \text{ normal force on the block}$$

$$-f = -\mu N = mL\ddot{\theta} = mL\frac{d\dot{\theta}}{dt} = mL\frac{d\left(\frac{v}{L}\right)}{dt} = m\frac{dv}{dt}$$

Combining these equations we have the differential equation

$$\frac{dv}{dt} = -\frac{\mu}{L}v^2$$

which we can solve by separation of variables and integration ...

$$\frac{dv}{v^2} = -\frac{\mu}{L}dt \rightarrow \int_{v_0}^{v}\frac{dv}{v^2} = -\frac{\mu}{L}\int_0^t dt$$

$$\rightarrow \frac{1}{v_0} - \frac{1}{v} = -\frac{\mu}{L}t \rightarrow v = \frac{v_0}{1 + \frac{\mu v_0 t}{L}} = L\frac{d\theta}{dt}$$

$$\rightarrow \int_{\theta_0}^{\theta} d\theta = \frac{1}{L}\int_0^t \frac{v_0}{1 + \frac{\mu v_0 t}{L}}dt \rightarrow \theta = \theta_0 + \frac{1}{\mu}\ln\left(1 + \frac{\mu v_0 t}{L}\right)$$

The Conical Pendulum (we started this discussion earlier)

A mass M hangs by a massless rod of length L which rotates at constant angular velocity ω as shown in figure 90.

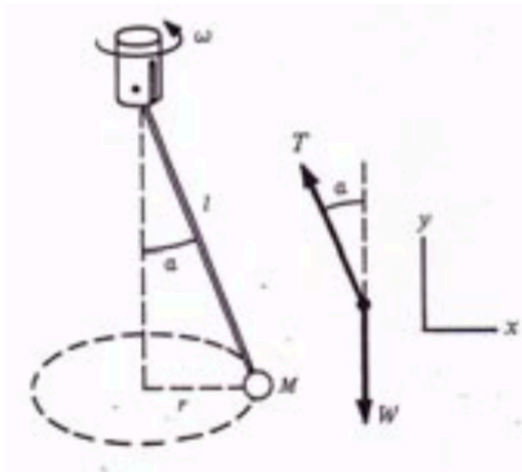

Figure 90:

The mass moves with constant speed in a circular path of constant radius. What is the angle the rod makes with the vertical?

The force diagram is shown in figure 90. T is the rod force and W is the weight of the mass. **No other forces are present**. There is no vertical acceleration. There is no tangential acceleration. There is only a radial acceleration $a_r = -\omega^2 r$ in the plane of the circular motion. This means that the mass is rotating around the vertical with fixed angle. The equations of motion are then

$$T\cos\alpha - W = 0$$

$$-T\sin\alpha = -Mr\omega^2$$

$$r = L\sin\alpha$$

$$(T - ML\omega^2)\sin\alpha = 0 \to T = ML\omega^2$$

$$\cos\alpha = \frac{W}{ML\omega^2} = \frac{g}{L\omega^2}$$

This **seems** like the solution to the problem.

Let us look, however, at some limits ....

At high speed we get

$$\omega \to \infty \ , \ \cos\alpha \to 0 \ , \ \alpha \to \frac{\pi}{2} \to \text{ horizontal}$$

but at low speed the result predicts **nonsense**, that is,

$$\omega \to 0 \ , \ \cos\alpha > 1$$

as in diagram figure 91.

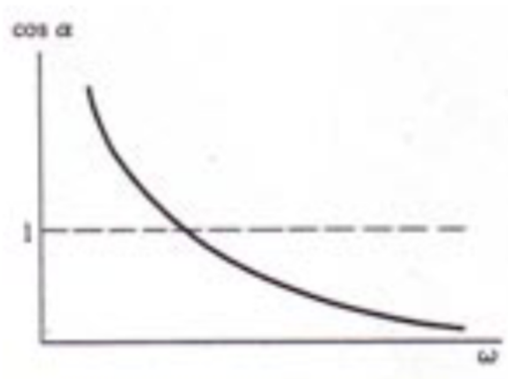

Figure 91:

In fact the solution predicts

$$\cos\alpha > 1 \text{ if } \omega < \sqrt{\frac{g}{L}}$$

When $\omega = \sqrt{\frac{g}{L}}$, $\cos\alpha = 1$ and $\sin\alpha = 0$ and the mass hangs **vertically**.

What happened to that solution? Let us look at the equations.

For this solution, we actually divided by 0, which is not allowed.

That is why we **missed** this solution.

A general picture of the entire solution is shown in figure 92.

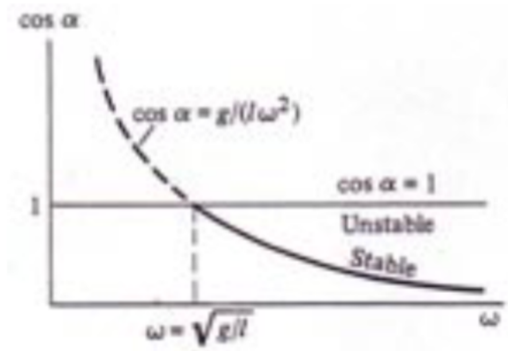

Figure 92:

Physically for $\omega < \sqrt{\frac{g}{L}}$, the only acceptable solution is $\alpha = 0$, $\cos\alpha = 1$.

For $\omega > \sqrt{\frac{g}{L}}$ there are **two acceptable solutions**

(1) $\cos\alpha = 1$

(2) $\cos\alpha = \frac{g}{L\omega^2}$

Solution #1 corresponds to the mass rotating rapidly but hanging vertically. Solution #2 corresponds to the mass flying around at an angle to the vertical.

For $\omega > \sqrt{\frac{g}{L}}$ solution #1 is unstable – if the system is in that state and is even slightly perturbed, it will jump outward to solution #2.

Someday you might learn that the reason the system behaves this way is the same reason why electromagnetic and weak elementary particle forces are two aspects of the same thing and why quarks cannot be observed and so on. WOW! It is called **spontaneous symmetry breaking**.

The Dangling Rope

Consider the section of the rope of length x, as shown in figure 93..

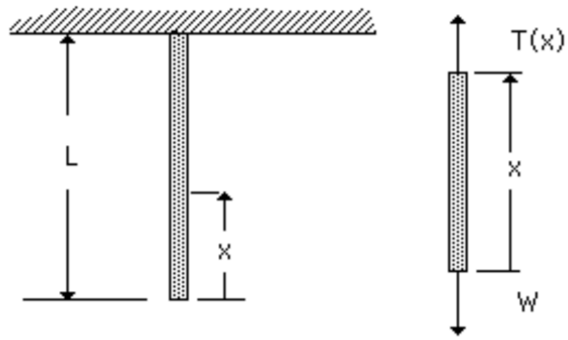

Figure 93:

Let $T(x)$ = the tension at x, which pulls upwards. The weight = $W = \frac{Mgx}{L}$ pulls downwards. The equation of motion (the rope is at rest) gives

$$T(x) = \frac{Mgx}{L}$$

Therefore at bottom of rope the tension is Mg and it is = 0 at the top!

The Whirling Rope

Consider a small section of a whirling rope as shown in figure 94.

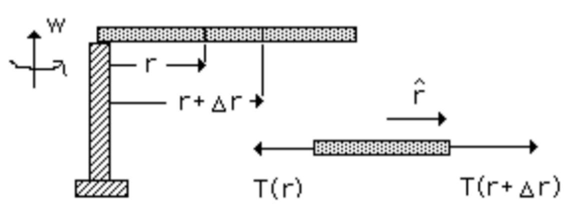

Figure 94:

Let the size = Δr and the mass = $\Delta m = \frac{M\Delta r}{L}$

Since we have circular motion, we have a radial acceleration. This means the forces pulling the two ends of the segment of rope cannot be equal. The tension must vary in the rope.

The equation of motion is

$$T(r+\Delta r) - T(t) = -(\Delta m) r\omega^2 = -\frac{Mr\omega^2}{L}\Delta r$$

Dividing by Δr and taking the limit as $\Delta r \to 0$, we get

$$\lim_{\Delta r \to 0} \frac{T(r+\Delta r) - T(t)}{\Delta r} = \frac{dT}{dr} = -\frac{Mr\omega^2}{L}$$

Now integrating we have

$$\int_{T_0}^{T(r)} dT = -\frac{Mr\omega^2}{L}\int_0^r r dr \to T(r) = T_0 - \frac{M\omega^2}{2L} r^2$$

where T_0 = tension at $r = 0$.

Since the tension = 0 at the free end of the rope, we have $T(L) = 0$. This gives

$$0 = T_0 - \frac{M\omega^2}{2L} L^2 \to T_0 = \frac{M\omega^2 L}{2}$$

The final result is then

$$T(r) = \frac{M\omega^2 L}{2} - \frac{M\omega^2}{2L} r^2 = \frac{M\omega^2 L}{2}\left(1 - \left(\frac{r}{L}\right)^2\right)$$

Now a rigid rod, on the other hand, can both push and pull. If we stretch the rod then it is under tension and it will pull on another object. If we compress a rod then it is under compression and it will push on another object.

A rigid rod can have transverse force also while a rope can only have a tension along its length. This fact forces us to rethink the definition of general equilibrium. Consider the rod shown in figure 95:

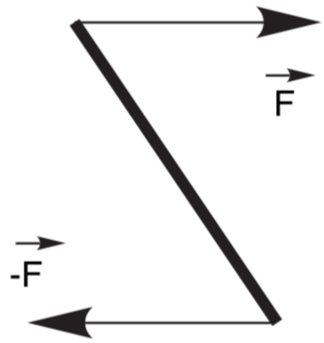

Figure 95:

In this case

$$\sum_i \vec{F}_i = \vec{F} - \vec{F} = 0$$

and, thus, there will be no **translational** motion of the rod (no linear acceleration). However, the rod will **rotate**, so it **cannot** be in equilibrium. We will fix up our equilibrium conditions shortly to deal with this case.

A More Complicated Example

Assume the system shown in figure 96 is in equilibrium.

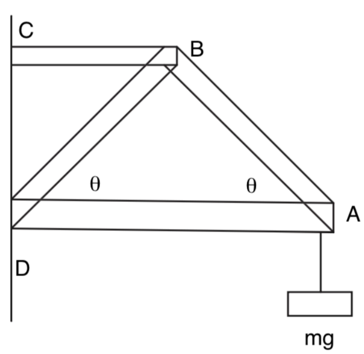

Figure 96:

We concentrate on the 4 pins as shown in figure 97:

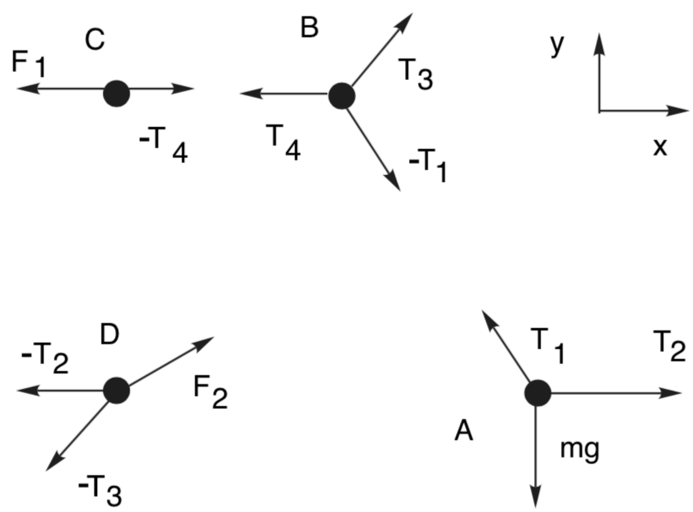

Figure 97:

where we have removed the rods and the wall from the problem and replaced them by the forces they exert on the pins.

The wall exerts the F_1 and F_2 forces and the internal forces of the ends of each rod are equal and opposite.

We can assume a wrong direction for the force as long as we maintain consistency. **It will simply come out as a negative number in the end if the assumed direction was wrong!!**

The vector sum of the forces at each pin must equal zero. We write

$$\vec{F}_1 = F_{1x}\hat{e}_x + F_{1y}\hat{e}_y$$

$$\vec{F}_2 = F_{2x}\hat{e}_x + F_{2y}\hat{e}_y$$

$$\vec{T}_1 = -T_1\cos\theta\hat{e}_x + T_1\sin\theta\hat{e}_y$$

$$\vec{T}_2 = T_2\hat{e}_x$$

$$\vec{T}_3 = T_3\cos\theta\hat{e}_x + T_3\sin\theta\hat{e}_y$$

$$\vec{T}_4 = -T_4\hat{e}_x$$

At A:

$$\vec{T}_1 + \vec{T}_2 - mg\hat{e}_y = 0 = -T_1\cos\theta\hat{e}_x + T_1\sin\theta\hat{e}_y + T_2\hat{e}_x - mg\hat{e}_y$$

$$0 = (T_2 - T_1\cos\theta)\hat{e}_x + (T_1\sin\theta - mg)\hat{e}_y$$

$$T_2 = T_1\cos\theta \text{ and } T_1 = \frac{mg}{\sin\theta} \rightarrow T_2 = \frac{mg\cos\theta}{\sin\theta} = mg\cot\theta$$

At B:

$$-\vec{T}_1 + \vec{T}_3 + \vec{T}_4 = 0 = T_1\cos\theta\hat{e}_x - T_1\sin\theta\hat{e}_y + T_3\cos\theta\hat{e}_x + T_3\sin\theta\hat{e}_y - T_4\hat{e}_x$$

$$0 = (T_1\cos\theta + T_3\cos\theta - T_4)\hat{e}_x + (T_3\sin\theta - T_1\sin\theta)\hat{e}_y$$

$$T_3\sin\theta = T_1\sin\theta \text{ and } T_1\cos\theta + T_3\cos\theta = T_4$$

$$T_3 = T_1 = \frac{mg}{\sin\theta} \text{ and } T_4 = 2mg\cot\theta$$

At C:

$$\vec{F}_1 - \vec{T}_4 = 0 = F_{1x}\hat{e}_x + F_{1y}\hat{e}_y + T_4\hat{e}_x$$

$$(F_{1x} + T_4)\hat{e}_x + F_{1y}\hat{e}_y = 0$$

$$F_{1x} = -T_4 = -2mg\cot\theta$$

$$F_{1y} = 0$$

At D:

$$\vec{F}_2 - \vec{T}_2 - \vec{T}_3 = 0 = F_{2x}\hat{e}_x + F_{2y}\hat{e}_y - T_2\hat{e}_x - T_3\cos\theta\hat{e}_x - T_3\sin\theta\hat{e}_y$$

$$0 = (F_{2x} - T_2 - T_3\cos\theta)\hat{e}_x + (F_{2y} - T_3\sin\theta)\hat{e}_y$$

$$F_{2x} = T_2 + T_3\cos\theta = 2mg\cot\theta$$

$$F_{2y} = T_3\sin\theta = mg$$

A tedious procedure, but if you are careful in applying it, it will always work.

We now deal with the problem of rotational equilibrium. If equal and opposite forces are applied to a body at different point , we get translational equilibrium(net force is zero), but we do not get rotational equilibrium.

Let us do a set of experiments. Consider the seesaw in figure 98.

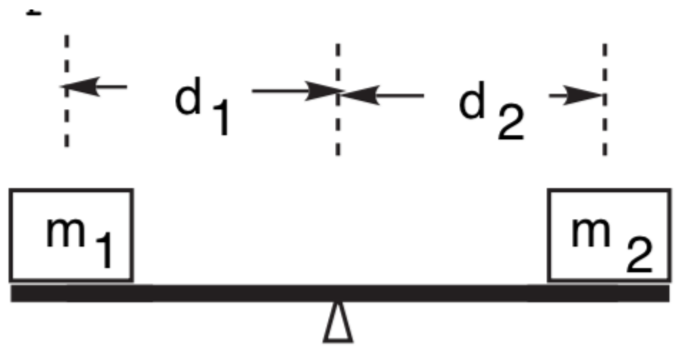

Figure 98:

If $d_1 = d_2$ and $m_1 = m_2$ then we get no rotational motion.

If $d_1 > d_2$ and $m_1 = m_2$ then we get counterclockwise rotational motion.

If $d_1 = d_2$ and $m_1 > m_2$ then we get counterclockwise rotational motion.

If $d_1 < d_2$ and $m_1 = m_2$ then we get clockwise rotational motion.

If $d_1 = d_2$ and $m_1 < m_2$ then we get clockwise rotational motion.

So the tendency to rotate has something to do with amount of mass(force due to gravity in this case) and distance from the axis of rotation.

Now consider the experiments shown in figure 99:

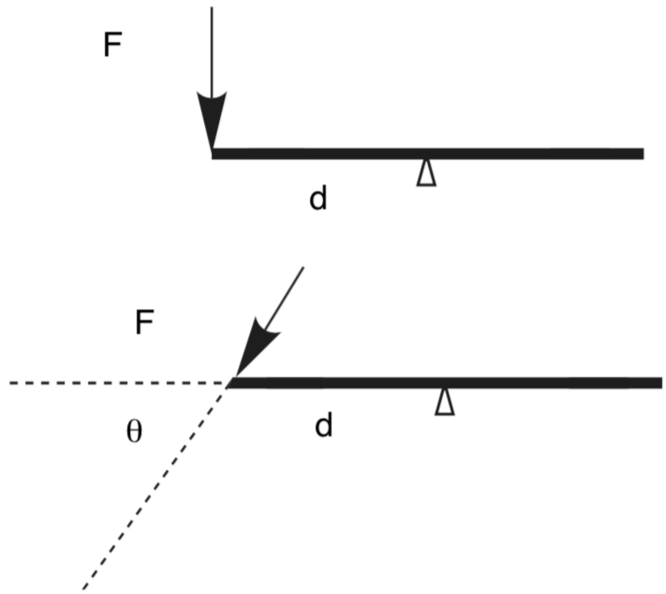

Figure 99:

We find that the effect of the force is not dependent only on its magnitude but also its direction (we should not be surprised since it is a vector). The tendency to rotate decreases with the angle θ between the force direction and the direction of the position vector from the axis of rotation to the point of application of the force.

It decreases from 1 to 0 as θ changes from $\frac{\pi}{2}$ to 0.

That suggest a functional dependence containing $\sin\theta$.

Finally, if we do the two experiments shown in figure 100

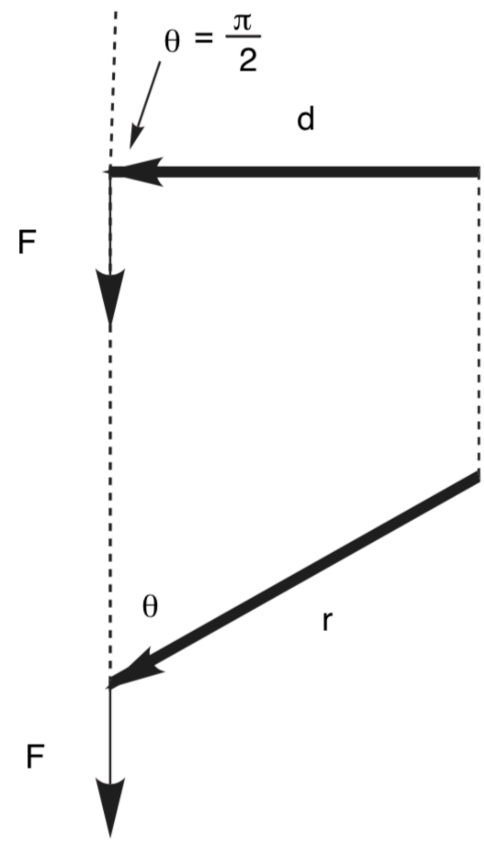

Figure 100:

The tendency to rotate is identical.

To account for the results of all of these experiments, we define

torque $= \boldsymbol{\tau} =$ tendency of a force to produce rotation

τ = (magnitude of the force)×(perpendicular distance from axis to force line) = $Fd = Fr\sin\theta$

In this simple case(rotation in a plane), we must associate a sign with the torque according to which way the system tends to rotate, say, by convention,

if system tends to rotate clockwise, then sign = -
if system tends to rotate counterclockwise, then sign = +

Finally, in this simple case, we assume that the torques calculated in this way add algebraically to give a total torque.

If the net torque is > 0, then the system rotates clockwise and if the net torque is < 0, then the system rotates counterclockwise.

> **If the net torque is $= 0$, then the system is in equilibrium (no tendency to rotate)**

Can we find a single formula that concisely expresses all of these features and can represent the torque in general?

The answer is yes.

If we write

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\vec{\tau} = \text{ torque vector}$$

$$\vec{r} = \text{ radius vector from axis of rotation to point of application of force}$$

$$\vec{F} = \text{ force vector}$$

Certainly, this formula gives the correct magnitudes since

$$|\vec{\tau}| = |\vec{r} \times \vec{F}| = rF \sin\theta$$

$$\theta = \text{ angle between } \vec{r} \text{ and } \vec{F} \text{ as we rotate } \vec{r} \text{ into } \vec{F}$$

The sign convention chosen above corresponds to the direction of the torque vector. We are able to use only $\pm$ signs above because the $\vec{r}$ and $\vec{F}$ vectors are all in one plane and hence $\vec{\tau}$, which must be orthogonal to that plane(see figure 101), can only point in two directions.

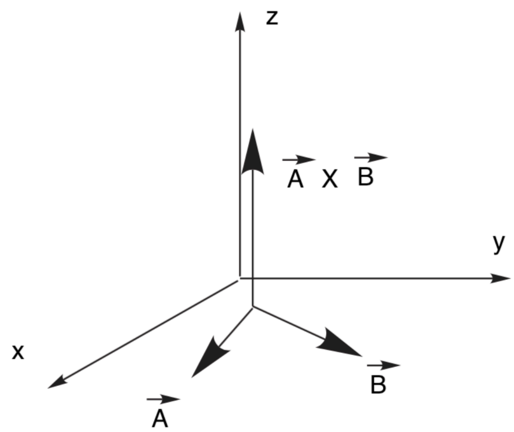

Figure 101:

Example using both methods - the beam balance - see figure 102

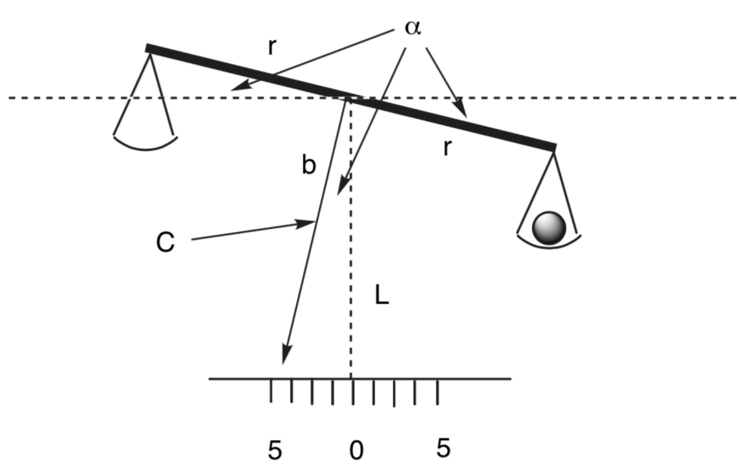

Figure 102:

Digression to center of gravity:

In order to find the line of action of the weight of an extended body we need to determine its point of application. Later we will derive a formula for this result, but for now we just do it experimentally. See figure 103.

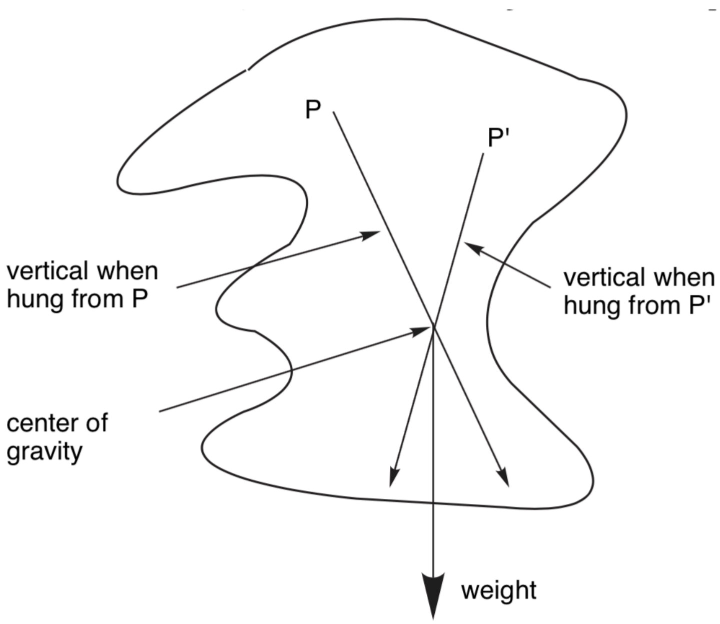

Figure 103:

The intersection point as shown above is the center of gravity. We assume that the point C is the center of gravity of the pointer-crossbar combination.

Without vectors we have:

$$+W_{pan} r\cos\alpha - (W_{pan} + Mg) r\cos\alpha + W_{bar} b\sin\alpha = 0$$

$$\tan\alpha = \frac{W_{pan} r}{W_{bar} b}$$

With vectors we have:

Define(with reference to directions in figure 104)

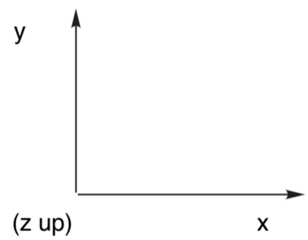

Figure 104:

$$\vec{W}_{pan} = -W_{pan}\hat{j} \qquad \vec{r}_{left} = r\cos\alpha\hat{i} + r\sin\alpha\hat{j}$$
$$\vec{W}_{bar} = -W_{bar}\hat{j} \qquad \vec{r}_{C} = -b\sin\alpha\hat{i} - b\cos\alpha\hat{j}$$
$$(W_{pan} + W_{ball}) = -(W_{pan} + Mg)\hat{j} \qquad \vec{r}_{right} = r\cos\alpha\hat{i} - r\sin\alpha\hat{j}$$

$$\begin{aligned}\vec{\tau} &= \vec{r}_{left} \times \vec{W}_{pan} + \vec{r}_{C} \times \vec{W}_{bar} + \vec{r}_{right} \times (W_{pan} + W_{ball}) \\ &= W_{pan} r\cos\alpha\hat{k} + W_{bar} b\sin\alpha\hat{k} - (W_{pan} + Mg) rcos\alpha = 0\end{aligned}$$

In this simple case the vector method is more complicated but as the problems get more complicated and we get into motion in three dimensions rather than a plane, the vector method will be much easier to use and understand the results.

Example

A uniform ladder AB with a weight W and a length L rests against a wall, making an angle of 60° with the floor. Find the forces on the ladder at A and B. Assume no friction (no vertical force)at the wall. See figure 105.

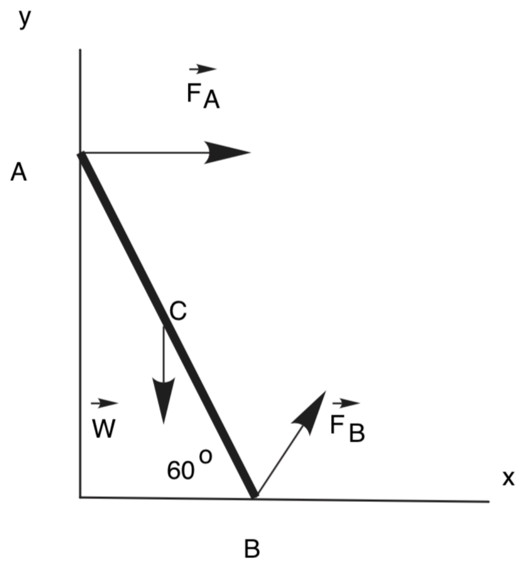

Figure 105:

Solution:

$$\sum \vec{F} = 0 \rightarrow \sum F_x = 0 = \sum F_y$$
$$\sum \vec{\tau} = 0 \rightarrow \sum \vec{\tau}_B = 0 \qquad \text{(about any axis)}$$

Therefore, we have

$$F_{Ax} + F_{Bx} = 0$$
$$F_{By} - W = 0$$
$$W\frac{L}{2}\sin 30° - F_{Ay}L\sin 30° = 0$$

or

$$F_{Ax} = -F_{Bx}$$
$$F_{By} = W$$
$$2W - \sqrt{3}F_{Ax} - F_{Ay} = 0 = W - \sqrt{3}F_{Ax} \rightarrow F_{Ax} = \frac{W}{\sqrt{3}} = -F_{Bx}$$

5. Momentum

Introduction

The Newton's 2nd law we have been using is of the form:

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} \tag{5.1}$$

This might lead us to believe that the most important **dynamical** quantity is velocity and that mass is just some constant parameter for any system that indicates its resistance to a force (its **inertia**).

This is not the case however. We were led to write down the above form of the 2nd law only because we were dealing with systems that all had a common property, namely, that their **mass remained constant** during the experiment.

In fact, the above form of the 2nd law is a special case when the mass $m =$ constant of the form of the law shown below:

$$\vec{F} = \frac{d(m\vec{v})}{dt} = \frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} + \vec{v}\frac{dm}{dt} \tag{5.2}$$

where we have introduced a **new dynamical quantity**

$$\vec{p} = m\vec{v} = \text{ linear momentum} \tag{5.3}$$

It is, in fact, the important **dynamical** quantity here and actually, in general, in all of physics. We cannot generalize the old form of the 2nd law to more complex systems but the new form will have no such limitations. The lesson learned here is that it is important to home in on the correct variables when developing a new theory.

Dynamics of a System of Particles

We now consider a system of **interacting** particles. What can we say about the properties of such a system in general?

First, we are free to choose the boundaries of the system, that is, which particles to include in our defined system, but once we choose the set of particles in the system we must be consistent about following the time development or motion of only those particles so that the definition of the system involved **remains the same** throughout our investigation.

Now consider figure 106:

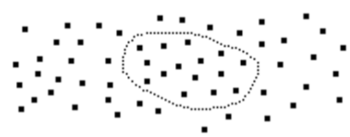

Figure 106:

We define the **system** = **particles inside dotted line** at one instant of time and the **outside** = **rest of universe**.

We assume that the particles of our system not only interact with the other particles **inside** the system, but also with those particles **outside** the system. We assume that our system has N interacting particles with masses

$$m_j \quad j = 1, 2, 3, ..., N$$

The position of the j^{th} particle is $\vec{r}_j$ the force on it is $\vec{F}_j$ and its momentum is

$$\vec{p}_j = m_j \vec{v}_j = m_j \dot{\vec{r}}_j = m_j \frac{d\vec{r}_j}{dt}$$

as shown in figure 107:

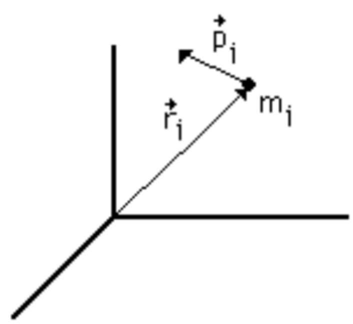

Figure 107:

The equation of motion for the j^{th} particle is (from Newton's 2nd law)

$$\vec{F}_j = \frac{d\vec{p}_j}{dt} \tag{5.4}$$

The force on the particle j can be split (a superposition) into two terms

$$\vec{F}_j = \vec{F}_j^{internal} + \vec{F}_j^{external} \tag{5.5}$$

where $\vec{F}_j^{internal}$, the internal force on the particle j, is the force due to all the other particles **in the system**, and $\vec{F}_j^{external}$, the external force on particle j, is the force due to all "outside" particles and whatever else may be around **outside the system** like electric, magnetic and gravitational fields. The equation of motion then becomes

$$\vec{F}_j = \vec{F}_j^{internal} + \vec{F}_j^{external} = \frac{d\vec{p}_j}{dt} \tag{5.6}$$

Let us add all these equations together (there are N of them) **vectorially**. We get

$$\sum_j \vec{F}_j^{internal} + \sum_j \vec{F}_j^{external} = \sum_j \frac{d\vec{p}_j}{dt} = \sum_j \vec{F}_j^{internal} + \vec{F}^{external} \tag{5.7}$$

where

$$\vec{F}^{external} = \sum_j \vec{F}_j^{external}$$

is the total external force acting on the internal system. The other term on the LHS of the equation, $\sum_j \vec{F}_j^{internal}$, is the sum of all internal forces acting on the system (due to all of the particles **within** the system).

Newton's 3rd law states that the forces between any two particles are equal and opposite and thus their sum is = 0. This means that

$$\sum_j \vec{F}_j^{internal} = 0$$

since all the internal forces cancel in pairs. The equation of motion then simplifies to

$$\vec{F}^{external} = \sum_j \frac{d\vec{p}_j}{dt} \tag{5.8}$$

We can now rewrite the RHS of the equation as

$$\vec{F}^{external} = \sum_j \frac{d\vec{p}_j}{dt} = \frac{d}{dt}\sum_j \vec{p}_j = \frac{d\vec{P}}{dt}$$

where

$$\sum_j \vec{p}_j = \vec{P} = \text{ total linear momentum of the system}$$

So, finally we have

$$\vec{F}^{external} = \frac{d\vec{P}}{dt} \tag{5.9}$$

or the **total external force acting on a system = the time rate of change of the total linear momentum of the system**.

This is valid independent of the details of the interactions, i.e., how the total external force is made up.

Bola example

Suppose we have 3 masses all tied together into a **"bola"** as shown below.

What happens in the two experiments below:

(1) tape them all together into one particle and throw it

(2) whirl them about(holding one mass) and throw it

In case (1) we have a single particle of $M = 3m$ with external force $\vec{F}_{ext} = M\vec{g}$ and hence we have an equation of motion

$$\vec{F}_{ext} = M\vec{g} = \frac{d\vec{P}}{dt}$$

In case (2) we have a 3 particles each of mass m each with external force $\vec{F}_{ext} = m\vec{g}$ and hence we have an equation of motion

$$\vec{F}_{ext} = m\vec{g} + m\vec{g} + m\vec{g} = M\vec{g} = \frac{d\vec{P}}{dt}$$

This is the **same** equation of motion for the **"entire" system** as before.

Anyone throwing a bola knows that they must forget the whirling masses and throw it as if it were a single particle.

What does that single particle do?

Digression: The Concept of the Center of Mass

We have (dropping the label "external")

$$\vec{F} = \frac{d\vec{P}}{dt} \tag{5.10}$$

As we have seen, a system of particles and a single particle of mass = total mass of the system have the **same** equation of motion.

Let us push on this further to see what it really implies. Suppose we define a new vector $\vec{R}$ (see figure 108)

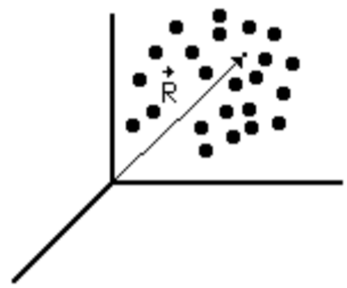

Figure 108:

such that

$$\vec{F} = M\frac{d^2\vec{R}}{dt^2} = \frac{d\vec{P}}{dt} = \frac{d}{dt}\sum_j \vec{p}_j = \frac{d}{dt}\sum_j m_j\frac{d\dot{\vec{r}}_j}{dt} = \frac{d^2}{dt^2}\sum_j m_j\vec{r}_j \tag{5.11}$$

This then implies that

$$\frac{d^2}{dt^2}\left(\vec{R} - \frac{1}{M}\sum_j m_j\vec{r}_j\right) = 0 \tag{5.12}$$

This is true if we define (we are free to define $\vec{R}$ in any way we wish at this point) $\vec{R}$ by

$$\vec{R} = \frac{1}{M}\sum_j m_j\vec{r}_j \tag{5.13}$$

$\vec{R}$ is then a vector from the origin to some point (**within** the system of particles). It is called the **center of mass (CM)** of the system.

Our equations then state that the system behaves as if all the mass M was concentrated at a single point and all the external forces act at that point.

That is why everything worked in our earlier discussions, where we were able to treat all of the bodies involved as point particles of mass = total mass of all the particles in the body.

Without knowing it, we were implicitly putting the point particle at the CM and saying that the total external force acted there!

This works for **"rigid"** bodies where all the particles involved have fixed separations. It does not work in quite so simple a manner for a general system of particles, although the CM in both cases satisfies the same equation, as we shall see later.

We can now understand why the bola behaved as it did in the two experiments. It is also clear that this simple equation only tells us about a part of the motion, namely, the translation of the entire body or the translational motion of its **CM**.

It does not describe the body's orientation in space or the rotational motion **about the center of mass** .... or what the three masses in the bola are actually doing in detail.

The **CM** will follow a parabola as any mass M would do when thrown in a uniform gravitational field!

In other words, as far as the translational motion of the **CM** is concerned, however, the equation

$$\vec{F} = M\frac{d^2\vec{R}}{dt^2}$$

is the whole story. It is valid for any system of particles no matter how the individual particles are moving with respect to(**wrt**) each other or how they are interacting.

Example - We now consider a **massless** stick with 2 masses on the ends as shown in the figure 109.

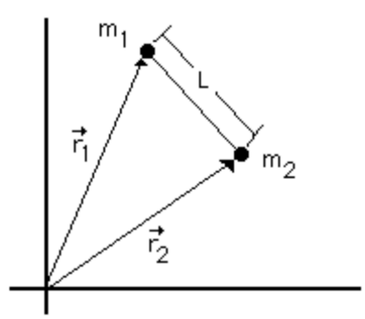

Figure 109:

The position vector of the CM is given by

$$\vec{R} = \frac{1}{M}\sum_j m_j\vec{r}_j = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

as shown in the figure 110.

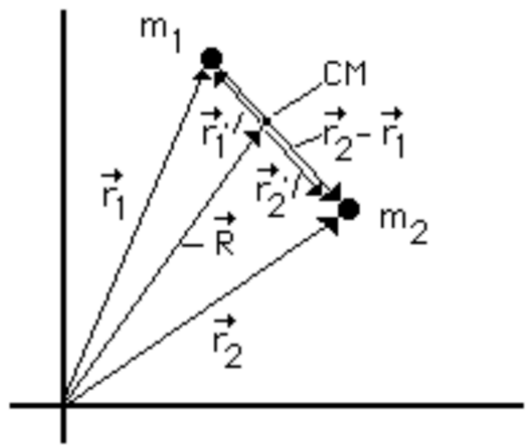

Figure 110:

Clearly, the CM lies on the **line joining the two masse** since

$$\vec{r}_2' = \vec{r}_2 - \vec{R} = \vec{r}_2 - \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = \frac{m_1(\vec{r}_2 - \vec{r}_1)}{m_1 + m_2}$$

$$\vec{r}_1' = \vec{r}_1 - \vec{R} = \vec{r}_1 - \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = \frac{m_2(\vec{r}_1 - \vec{r}_2)}{m_1 + m_2}$$

where the **primed** vectors use the CM as an origin. These equations imply that $\vec{r}_1'$ and $\vec{r}_2'$ lie along the line joining m_1 and m_2. We also note that

$$|\vec{r}_1'| = \frac{m_2|\vec{r}_1 - \vec{r}_2|}{m_1 + m_2} = \frac{m_2}{m_1 + m_2}L \ , \ |\vec{r}_2'| = \frac{m_1|\vec{r}_2 - \vec{r}_1|}{m_1 + m_2} = \frac{m_1}{m_1 + m_2}L$$

The total external force on the system is

$$\vec{F} = (m_1 + m_2)\vec{g} \tag{5.14}$$

Therefore the equation of motion is

$$\vec{F} = M\frac{d^2\vec{R}}{dt^2} = (m_1 + m_2)\frac{d^2\vec{R}}{dt^2} = (m_1 + m_2)\vec{g} \rightarrow \frac{d^2\vec{R}}{dt^2} = \vec{g} \tag{5.15}$$

or the CM follows the **same parabolic trajectory** as that of a single mass in a uniform gravitational field as shown in figure 111.

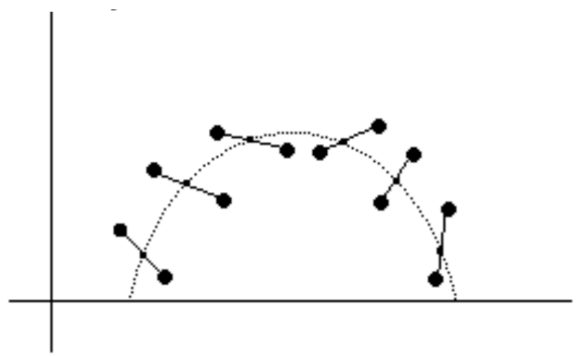

Figure 111:

Finding the CM of a system of point masses is easy(just the sum above), but finding the CM of a real extended body is not always so straightforward.

With the help of simple calculus, however, it is not hard to accomplish. The process goes as follows:

We divide the body into N mass elements (think of particles glued together) with $\vec{r}_j$ the position vector of the j^{th} element and m_j its mass. Then we have the **approximate** result

$$\vec{R} = \frac{1}{M} \sum_{j=1}^{N} \vec{r}_j \tag{5.16}$$

This is an **approximate** result because any real body cannot be represented by particles of **finite size glued together** (lots of holes that are not really there).

If we take the limit of $N \to \infty$ and the size of each particle approaching 0, then this formula becomes exact. The limiting process, as you learned in elementary calculus, turns the sum into an integral, that is,

$$\vec{R} = \lim_{N \to \infty} \frac{1}{M} \sum_{j=1}^{N} m_j \vec{r}_j = \frac{1}{M} \int \vec{r} dm \tag{5.17}$$

where dm is an **infinitesimal amount** of mass. We can visualize this in **3-dimensions** as shown in figure 112

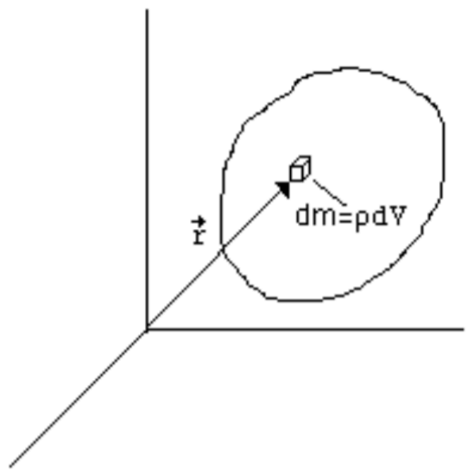

Figure 112:

we have

$$\vec{R} = \frac{1}{M}\int \vec{r}dm = \frac{1}{M}\int \rho(\vec{r})\vec{r}dV \tag{5.18}$$

where $\rho(\vec{r})$ = the mass density function, dV = an infinitesimal volume element (see figure 112) and $dm = \rho(\vec{r})dV$ is the element of mass.

This is a **"volume"** integral. We will only look a very special bodies (with lots of symmetry) so that the integrals are always 1-dimensional and you can use ordinary calculus to evaluate them.

Examples:

1-Dimensional Object —- CM of a Rod

A very thin rod of length L has a density $\lambda(x)$ (mass per unit length). Using the geometry in figure 113

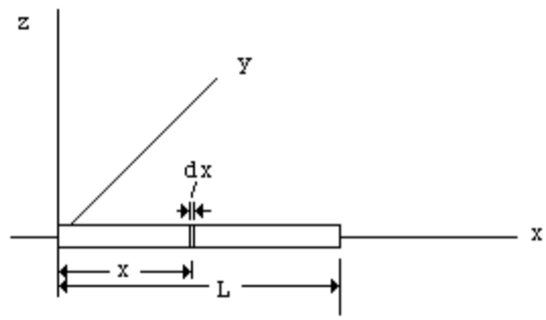

Figure 113:

the element of mass is chosen as $dm = \lambda(x)dx$. The position of the CM (clearly along the x-axis) is then given by

$$X = \frac{1}{M}\int_0^L xdm = \frac{1}{M}\int_0^L x\lambda(x)dx$$

where

$$M = \int_0^L dm = \int_0^L \lambda(x)dx$$

Case #1: Uniform Rod where $\lambda(x) = \text{constant} = \lambda_0$.

We then have

$$X = \frac{1}{M}\int_0^L xdm = \frac{1}{M}\int_0^L x\lambda(x)dx = \frac{\lambda_0}{M}\int_0^L xdx = \frac{\lambda_0}{2M}L^2$$

and

$$M = \int_0^L dm = \int_0^L \lambda(x)dx = \lambda_0\int_0^L dx = \lambda_0 L$$

so that

$$X = \frac{\lambda_0}{2M}L^2 = \frac{\lambda_0}{2\lambda_0 L}L^2 = \frac{L}{2}$$

as we expected!

Case #2: Non-Uniform Rod where $\lambda(x) = \lambda_0\frac{x}{L}$.

We then have

$$X = \frac{1}{M}\int_0^L xdm = \frac{1}{M}\int_0^L x\lambda(x)dx = \frac{\lambda_0}{ML}\int_0^L x^2dx = \frac{\lambda_0}{3M}L^2$$

and

$$M = \int_0^L dm = \int_0^L \lambda(x)dx = \frac{\lambda_0}{L}\int_0^L xdx = \frac{\lambda_0}{2}L$$

so that

$$X = \frac{\lambda_0}{3M}L^2 = \frac{\lambda_0}{3\frac{\lambda_0}{2}L}L^2 = \frac{2}{3}L$$

2-Dimensional Object — CM of a Triangular Sheet

Consider the geometry shown in figure 114.

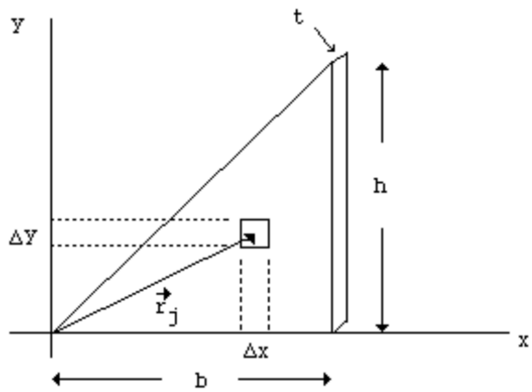

Figure 114:

which is a 2-dimensional uniform (constant density) triangular sheet of mass M and dimensions as shown. If we divide the sheet into small rectangular areas of sides Δx and Δy as shown, then the volume of each element is $\Delta V = t\Delta x\Delta y$, where t is the thickness of the plate, and

$$\vec{R} = \frac{1}{M}\sum_{j=1}^{N} m_j\vec{r}_j = \frac{1}{M}\sum_{j=1}^{N}(\rho_j\Delta V)\vec{r}_j = \frac{1}{M}\sum_{j=1}^{N}\rho_j t\Delta x\Delta y\vec{r}_j$$

where j is the label for one of the volume elements and ρ_j is the density at that point. Since the sheet is uniform we have

$$\rho_j = \frac{M}{V} = \frac{M}{(area)t} = \frac{M}{At}$$

where A is the area of the sheet.

We can formally write down this result as follows.

We do the sum by first summing over the Δx's and then over the Δy's instead of over a single index j. This gives a double sum which can be converted into a **"double"** or 2-dimensional integral by taking the limit

$$\vec{R} = \lim_{\Delta x, \Delta y \to 0} \frac{1}{M} \sum_{j=1}^{N} \rho_j \Delta x \Delta y \vec{r}_j = \frac{1}{A} \iint \vec{r} dxdy$$

Now let $\vec{r} = x\hat{e}_x + y\hat{e}_y$ be the position vector of an element $dxdy$. then writing $\vec{R} = X\hat{e}_x + Y\hat{e}_y$, we have

$$\vec{R} = X\hat{e}_x + Y\hat{e}_y = \frac{1}{A} \iint (x\hat{e}_x + y\hat{e}_y) dxdy = \frac{1}{A}\hat{e}_x \iint xdxdy + \frac{1}{A}\hat{e}_y \iint ydxdy$$

Hence, the coordinates of the CM are given by

$$X = \frac{1}{A} \iint xdxdy \quad , \quad Y = \frac{1}{A} \iint ydxdy$$

What does the x-integral say?

It tells us to take each element, multiply its area by its x-coordinate and sum the results.

Let us do this in stages.

First consider the elements in a vertical strip parallel to the y-axis as shown in figure 115.

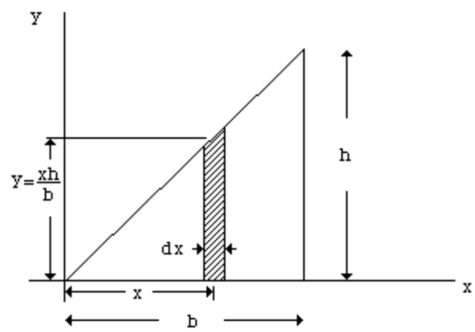

Figure 115:

The strip runs from $y = 0$ to $y = \frac{xh}{b}$. Each element in the strip has the same x-coordinate and the contribution of each strip to the double integral is

$$\frac{1}{A} x dx \int_0^{xh/b} dy = \frac{h}{bA} x^2 dx$$

Finally, we sum the contributions from all such strips $x = 0$ to $x = b$ to find

$$X = \frac{h}{bA} \int_0^b x^2 dx = \frac{hb^2}{3A}$$

Since $A = bh/2$ we have

$$X = \frac{2}{3} b$$

Writing out the steps we just carried out in one formula we have

$$X = \frac{1}{A} \int_0^b \left(\int_0^{xh/b} dy \right) x dx = \frac{h}{bA} \int_0^b x^2 dx = \frac{hb^2}{3A} = \frac{2b}{3}$$

Similarly

$$Y = \frac{1}{A} \int_0^b \left(\int_0^{xh/b} y dy \right) dx = \frac{h^2}{2b^2 A} \int_0^b x^2 dx = \frac{h^2 b}{6A} = \frac{h}{3}$$

so that

$$\vec{R} = \frac{2b}{3} \hat{e}_x + \frac{h}{3} \hat{e}_y$$

How could we operationally find the CM of such a plate?

Demonstration of plate and plumb bob method

Another Interesting Example

CM Motion

Suppose that a rectangular box is held with one corner resting on a **frictionless** table and it is gently released. It falls in a complex tumbling motion, which we do not know how to figure out yet.

However, we can answer the question: What is the trajectory of the CM?

The external forces acting on the box are gravity and the normal force from the table.

Neither has a horizontal component.

This means that

$$F_x = 0 = ma_x \rightarrow a_x = 0$$

which means that the CM **does not move horizontally**.

It falls vertically as shown in figure 116!

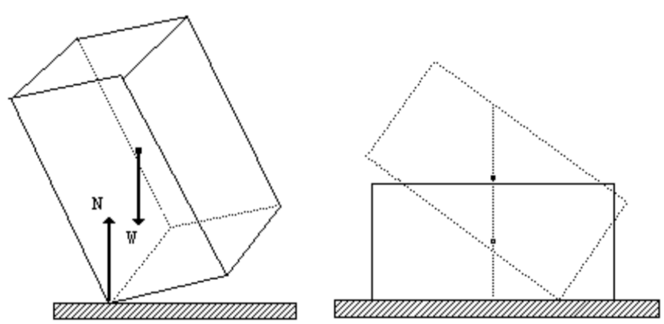

Figure 116:

We now discover our first **conservation law**.

Conservation of Momentum

We have in general for any system of particles

$$\vec{F} = \frac{d\vec{P}}{dt}$$

Suppose the system is **isolated**, which means that the total external force is zero.

We then have

$$\vec{F} = \frac{d\vec{P}}{dt} = 0$$

or the **total momentum is a constant vector (constant in magnitude and direction)**.

Thus no matter how strong the internal interactions, **the total momentum of an isolated system is a constant**.

This is called the

Law of Conservation of Momentum

As we shall see, this law will give us very powerful insights into the behavior of complicated systems.

Remember Newton's 1st law..... We can now restate it as follows:

For an isolated body the linear momentum is constant.

Let us now look back at Newton's three laws.

The 3rd law now reads.

Considering only internal forces, the total momentum of any pair of particles is a constant.

This means that

$$\vec{p}_1 + \vec{p}_2 = \text{ constant } \rightarrow \Delta\vec{p}_1 + \Delta\vec{p}_2 = 0 \rightarrow \Delta\vec{p}_1 = -\Delta\vec{p}_2 \qquad (5.19)$$

$$\frac{\Delta\vec{p}_1}{\Delta t} = -\frac{\Delta\vec{p}_2}{\Delta t} \text{ in any time interval } \Delta t \qquad (5.20)$$

Defining

$$\frac{\Delta\vec{p}_1}{\Delta t} = \vec{F}_{12} \quad , \quad \frac{\Delta\vec{p}_2}{\Delta t} = \vec{F}_{21} \tag{5.21}$$

Note that this last equation is Newton's 2nd law!

Then we have

$$\vec{F}_{12} = -\vec{F}_{21} \qquad \text{3rd law} \tag{5.22}$$

For a single isolated particle, the momentum is a constant.

$$\Delta\vec{p} = 0 \qquad \text{This is the 1st law} \tag{5.23}$$

So, all three laws are really included in the statement that momentum is conserved for an isolated system plus the definition of a force!

Example - Spring Gun Recoil

We have a loaded spring gun. Everything is initially at rest on a horizontal frictionless surface. See figure 117.

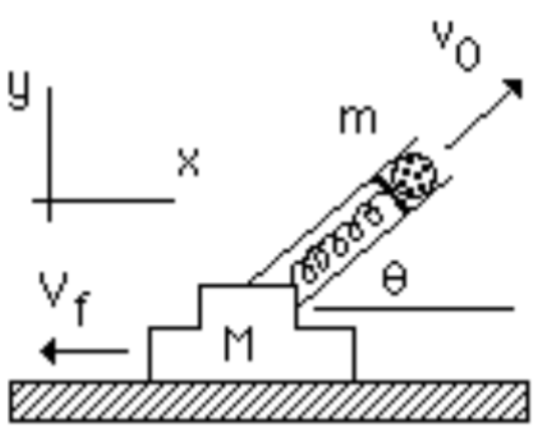

Figure 117:

What happens?

The **physical system** is the **gun + the ball**. Everything else is **external**. The **external forces** acting on the system are gravity and the normal force from the table. Both of these forces are in the y-direction, hence the external force in the x-direction = 0. This implies that

$$\frac{dP_x}{dt} = 0 \rightarrow P_x = \text{ constant } \rightarrow P_{x,initial} = P_{x,final}$$

The initial time is before we fire the gun, so $P_{x,initial} = 0$ (system is at rest). After the ball leaves the gun, the gun recoils with speed V_f to the left relative to the ground (**an inertial frame**). Therefore,

$$P^{ball}_{x,initial} = 0 \;\;,\;\; P^{gun}_{x,initial} = 0 \;\;,\;\; P^{gun}_{x,final} = -MV_f$$

Now at the instant the ball leaves the gun it has a speed v_0 relative to the gun at an angle θ with the horizontal. Therefore relative to the ground it has the velocity in the x-direction

$$v_x = v_0 \cos\theta - V_f$$

and a momentum

$$P^{ball}_{x,final} = mv_x = m(v_0 \cos\theta - V_f)$$

Total momentum conservation then gives

$$P_{x,initial} = P_{x,final}$$

$$0 = m(v_0 \cos\theta - V_f) - MV_f$$

or

$$V_f = \frac{mv_0 \cos\theta}{m + M}$$

What if we tried to do this using Newton's laws instead ......?

Let $\vec{v}(t)$ be the velocity of the ball and $\vec{V}(t)$ be the velocity of the gun. While the ball is in the gun it is acted on by gravity, the spring and friction forces inside the barrel. Let the net force on the ball be $\vec{f}(t)$. The x-equation of motion of the ball is

$$m\frac{dv_x}{dt} = f_x(t)$$

Integrating we get

$$mv_x(t) - mv_x(0) = \int_0^t f_x(t)dt$$

The external forces are all vertical and thus the horizontal force f_x on the ball is due entirely to the gun. Newton's 3rd law say the ball exerts an equal and opposite force $-f_x$ on the gun. The horizontal equation of motion of the gun is then

$$M\frac{dV_x}{dt} = -f_x(t)$$

Integrating we get

$$MV_x(t) - MV_x(0) = -\int_0^t f_x(t)dt$$

Eliminating the integral to get

$$MV_x(t) + mv_x(t) = MV_x(0) + mv_x(0)$$

so we have just **rediscovered** conservation of momentum!!!!.

What happens to the CM during all of this?

The horizontal velocity of the CM is given by

$$x_{cm} = \frac{Mx_{gun} + mx_{ball}}{M+m} \to \dot{x}_{cm} = \frac{M\dot{x}_{gun} + m\dot{x}_{ball}}{M+m}$$

$$v_{x,cm}(t) = \frac{M\dot{x}_{gun} + m\dot{x}_{ball}}{M+m} = \frac{P_x}{M+m} = 0$$

since the system was initially at rest and momentum is conserved.

Therefore x_{cm} is a constant.

There is no external force in the x-direction and so the CM is not accelerated during the interactions. It was at rest to start with and remains at rest!

The law of conservation of momentum is universal ... it holds everywhere in physics ... not just in mechanics.

It is more fundamental than Newton's 3rd law which relies on the concept of force.

Center of Mass Coordinates

The correct choice of coordinate system can often greatly simplify the solution of a problem.

One possible choice in many problems is the CM coordinate system ... where the origin is attached to the CM, where the position vector of the CM (see figure 118)

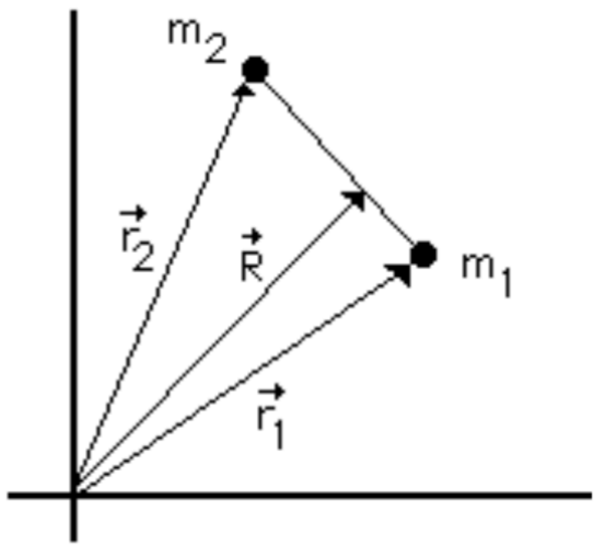

Figure 118:

is given by

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = \vec{R}_{CM,LAB} \tag{5.24}$$

or the CM position in the LAB frame.

Now attaching an $x'y'z'$ coordinate system to the CM, the CM position vectors of the particles in that frame are then given by

$$\vec{r}_1' = \vec{r}_1 - \vec{R} \quad , \quad \vec{r}_2' = \vec{r}_2 - \vec{R} \tag{5.25}$$

as shown in figure 119.

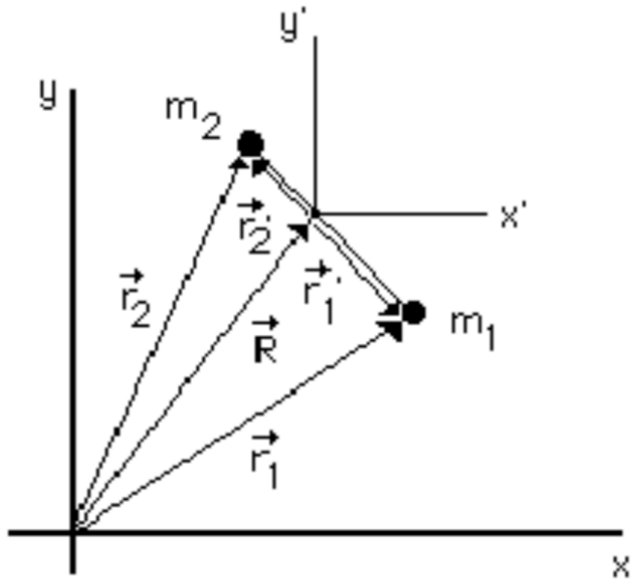

Figure 119:

We note a few important properties of these new coordinates.

For an isolated 2-body system, the total momentum is a constant and hence

$$\frac{d\vec{R}}{dt} = \frac{m_1 \frac{d\vec{r}_1}{dt} + m_2 \frac{d\vec{r}_2}{dt}}{m_1 + m_2} = \frac{\vec{P}}{m_1 + m_2} = \text{ constant} \tag{5.26}$$

or the CM moves with a **constant(uniform)** velocity. In the CM system

$$\vec{R}_{CM,LAB} = 0 = \frac{m_1 \vec{r}_1' + m_2 \vec{r}_2'}{m_1 + m_2} \rightarrow m_1 \vec{r}_1' + m_2 \vec{r}_2' = 0 \tag{5.27}$$

so that if the motion of one particle is known, then the motion of the other follows directly(the motions are **correlated**).

The Push Me - Pull You

Two identical blocks(mass = m) slide with no friction on a horizontal surface are connected by a spring (spring constant k and unstretched length L).

Initially they are at rest.

At $t = 0$ the block is hit sharply giving it an instantaneous velocity v_0 to the right (see figure 120).

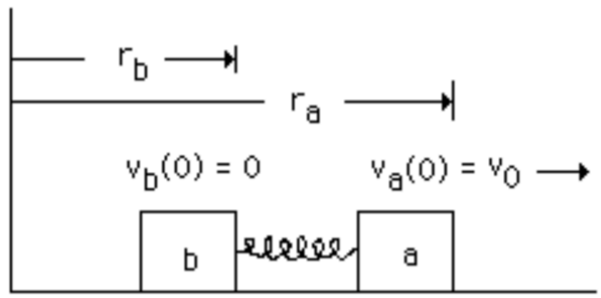

Figure 120:

What happens?

Since there is no friction there are no external forces in the horizontal direction and hence total momentum is conserved in the horizontal direction.

This means that the **CM mass moves uniformly** and thus **defines an inertial frame**.

We transform, therefore, to CM frame of reference as shown in figure 121.

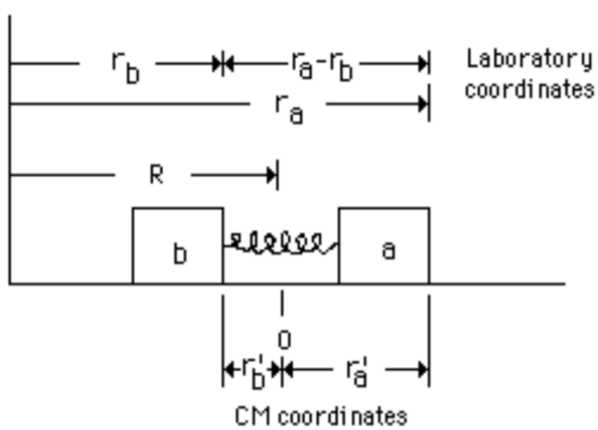

Figure 121:

The CM is at

$$R = \frac{m_a r_a + m_b r_b}{m_a + m_b} = \frac{1}{2}(r_a + r_b)$$

i.e., R is ALWAYS half way between the masses (as we expect for equal masses). The CM coordinates are

$$r'_a = r_a - R = \frac{1}{2}(r_a - r_b) \quad , \quad r'_b = r_b - R = -\frac{1}{2}(r_a - r_b) = -r'_a$$

The instantaneous length of the spring is

$$r_a - r_b - L = r'_a - r'_b - L$$

and the magnitude of the spring force is

$$k(r'_a - r'_b - L)$$

Note that the equilibrium length is the same in both frames (it is a displacement).

The equations of motion in the CM frame are

$$m\ddot{r}'_a = -k(r'_a - r'_b - L) \quad , \quad m\ddot{r}'_b = +k(r'_a - r'_b - L)$$

Subtracting the equations we get the equation for the **relative** (as if we were sitting on b) motion of the masses

$$m(\ddot{r}'_a - \ddot{r}'_b) = -2k(r'_a - r'_b - L)$$

Letting $u = r'_a - r'_=$ position of a wrt b. We get

$$m\ddot{u} + 2ku = 2kL$$

Letting $u = y + L$ we have

$$\ddot{y} + \omega^2 y = 0 \quad , \quad \omega = \sqrt{\frac{2k}{L}}$$

This is the equation of SHM and we know its solution

$$y(t) = A\sin\omega t + B\cos\omega t \quad , \quad u(t) = L + A\sin\omega t + B\cos\omega t$$

The initial condition $u(0) = L$ (spring is unstretched) says that we must choose $B = 0$. Also we have (initially)

$$\dot{u}(0) = \dot{r}'_a(0) - \dot{r}'_b(0) = \dot{r}_a(0) - \dot{r}_b(0) = v_a(0) - b_b(0) = v_a(0) = v_0 = A\omega$$

so that he solution is finally

$$u(t) = L + \frac{v_o}{\omega} \sin \omega t$$

Now

$$\dot{u} = \dot{r}'_a - \dot{r}'_b = v'_a - v'_b$$

and in the CM frame (where $mr'_a + mr'_b = 0$)

$$mv'_a + mv'_b = 0 = m(v'_a + v'_b) \quad \text{or} \quad v'_a = -v'_b = \frac{1}{2}\dot{u} = \frac{v_0}{2} \cos \omega t$$

The laboratory velocities are

$$v_a = \dot{R} + v'_a \quad , \quad b_b = \dot{R} + v'_b$$

since

$$\vec{r}_a = \vec{R} + \vec{r}'_a \quad \text{and} \quad \vec{r}_b = \vec{R} + \vec{r}'_b$$

Since $\dot{R}$ = constant (it equals its initial value), we have

$$\dot{R} = \frac{1}{2}(v_a(0) + v_b(0)) = \frac{1}{2}v_0$$

So

$$v_a = \frac{v_0}{2}(1 + \cos \omega t) \quad , \quad v_b = \frac{v_0}{2}(1 - \cos \omega t)$$

The masses move to the right on average, but they alternately come to rest in a push-me pull-you fashion!!!

Example

A circus acrobat of mass M leaps straight up with initial velocity v_0 from a trampoline. As she rises up, she take a trained monkey of mass m off a perch at a height h above the trampoline. See figure 122.

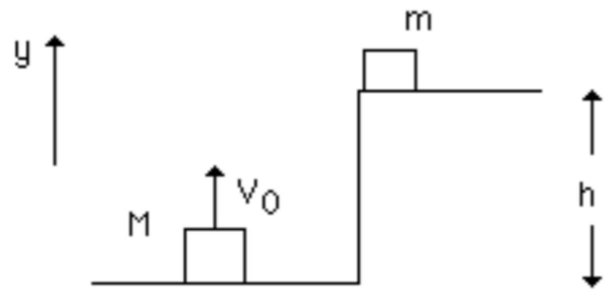

Figure 122:

What is the maximum height attained by the pair?

Before she grabs the monkey we must calculate what happens in getting to height h:

$$\frac{dv}{dt} = a = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy} \rightarrow vdv = ady = -gdy$$

$$\int_{v_0}^{v_f} vdv = \frac{1}{2}(v_f^2 - V_0^2) = -g\int_{y_0}^{y_f} dy = -g(y_f - y_0)$$

$$v^2(h) - v_0^2 = -2g(h-0) = -2gh \rightarrow v(h) = \sqrt{v_0^2 - 2gh}$$

Grabbing monkey → at h momentum is conserved:

$$P_{initial} = Mv = P_{final} = (M+m)v'$$

$$v' = \frac{Mv}{M+m}$$

is the new initial velocity of the pair. Let h' = extra height, then

$$v_{max\,height}^2 - v_{after\,pickup}^2 = 0 - v'^2 = -2g(y_{max} - h) = -2gh'$$

$$h' = \frac{v'^2}{2g} = \frac{1}{2g}\left(\frac{M}{M+m}\right)^2 (v_0^2 - 2gh)$$

Impulse

We have the differential form of Newton's law

$$\vec{F} = \frac{d\vec{P}}{dt} \tag{5.28}$$

We can turn this into an integral form as

$$\vec{I}(t) = \text{ impulse } = \int_0^t \vec{F} dt = \int_0^t d\vec{P} = \vec{P}(t) - \vec{P}(0) \tag{5.29}$$

that is, the change in the momentum of the system = the impulse.

This implies that small forces acting for long times can produce the same effect as large forces acting for short times.

Example - Rubber Ball Rebound

A rubber ball of mass 0.2 kg falls to the floor. The ball hits with a speed of 8 m/s and rebounds with approximately the same speed.See figure 123.

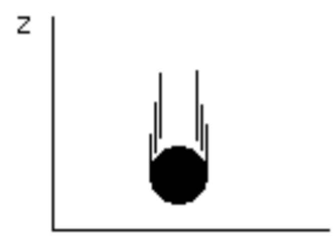

Figure 123:

Experiment find that the ball is in contact with the floor for 10^{-3} s.

What is force exerted by floor on the ball?

We have

$$\int_0^{\Delta t} \vec{F} dt = \vec{P}(\Delta t) - \vec{P}(0) = \int_0^{\Delta t} (F_{floor} - mg) dt = (mv_{hit} - (-mv_{hit}))$$

$$\int_0^{\Delta t} F_{floor} dt - mg\Delta t = 2mv_{hit}$$

$$\int_0^{\Delta t} F_{av,floor} dt = F_{av,floor} \Delta t = mg\Delta t + 2mv_{hit}$$

$$F_{av,floor} = mg + \frac{2mv_{hit}}{\Delta t}$$

A plot of F versus time is shown in figure 124.

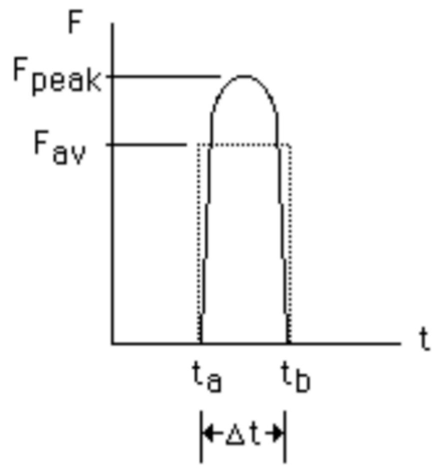

Figure 124:

Finally, we can calculate some numbers.

$$F_{av,floor} = mg + \frac{2mv_{hit}}{\Delta t} = (0.2)(9.8) + \frac{2(0.2)(8)}{10^{-3}} = (1.96 + 3200) \text{ Newtons}$$

This shows that a very quick collision such as a hammer on a nail can, in this way, produce a much larger force on the nail than any person could exert directly.

Momentum and Flow of Mass

Systems where mass flows into and out of the system are very difficult to deal with.

The main difficulty arises when one loses sight of how the system was originally defined.

It is very important to keep track of all parts of the system no matter where they might be at any instant of time.

This means that the mass of the system as a whole cannot change during the process under investigation.

The mass can redistribute itself within the system however.

No new equations are involved in this discussion so we will proceed by doing examples that illustrate the inherent difficulties and how to deal with them.

Example - Spacecraft and Dust Cloud

A spacecraft moves through space with constant velocity $\vec{v}$.

The spacecraft encounters a stream of dust particles which embed themselves into the spacecraft at a rate dm/dt.

The dust has a velocity $\vec{u}$ just before its hits the spacecraft.

At time t the total mass of the spacecraft is $M(t)$.

What external force $\vec{F}$ is necessary to keep the spacecraft moving uniformly (normally $\vec{F}$ comes from the spacecraft's own engines ...we will simplify matters by thinking of it as a true external force)? See figure 125.

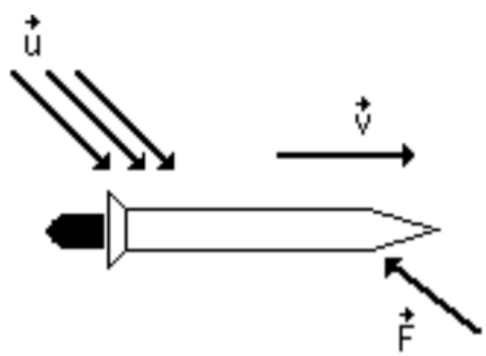

Figure 125:

We focus our attention on a short time interval between t and $t + \Delta t$.

Figure 126 show the system at the beginning and the end of the interval.

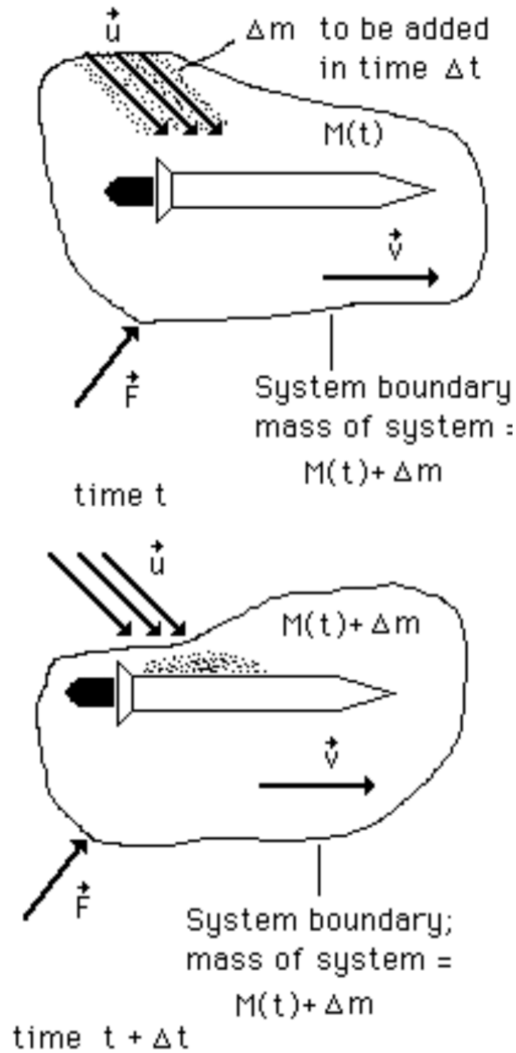

Figure 126:

Δm = amount of mass added during next Δt. The system consists of $M(t) + \Delta m$

The initial momentum is $\vec{P}(t) = M(t)\vec{v} + (\Delta m)\vec{u}$.

The final momentum is $\vec{P}(t + \Delta t) = M(t)\vec{v} + (\Delta m)\vec{v}$.

The change in momentum is

$$\Delta\vec{P} = \vec{P}(t+\Delta t) - \vec{P}(t) = (\vec{v} - \vec{u})\Delta m$$

Therefore the time rate of change of momentum is approximately

$$\frac{\Delta\vec{P}}{\Delta t} = \frac{\vec{P}(t+\Delta t) - \vec{P}(t)}{\Delta t} = (\vec{v} - \vec{u})\frac{\Delta m}{\Delta t}$$

Taking the limit as $\Delta t \to 0$ we get

$$\frac{d\vec{P}}{dt} = (\vec{v} - \vec{u})\frac{dm}{dt} = \vec{F}$$

Note that $\vec{F}$ can be either positive or negative depending on the direction of the stream of dust. If $\vec{v} = \vec{u}$ the momentum of the system is constant and $\vec{F} = 0$.

This procedure seems overly formal. If the mass of the system is changing why don't we just use

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt} + \vec{v}\frac{dm}{dt} = \vec{v}\frac{dm}{dt}$$

since $\vec{v}$ = constant, which, of course, is incorrect.

The difficulty is there are more than one contribution to the momentum and the only way in many problems to keep track correctly is this overly formal method.

Example - Freight Car and Hopper

Sand falls from a stationary hopper onto a freight car which is moving with velocity v. See figure 127

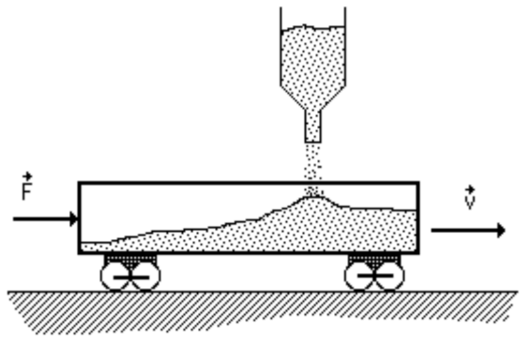

Figure 127:

The sand falls at the rate dm/dt.

How much force is needed to keep the freight car moving at speed v? The initial horizontal speed of the sand $= 0$.

Thus we have in the horizontal direction

$$\frac{dP}{dt} = v - u\frac{dm}{dt} = v\frac{dm}{dt} = F$$

Does this make sense?

What happens to each grain of sand as it transfers to the car?

Example - Leaky Freight Car

The same freight car is leaking sand at a rate dm/dt.

What force is needed to keep it moving with constant speed?

The mass of the car is decreasing.

The velocity of the sand leaving the car is the same as that of the car and hence when it leaves there is no momentum change for the SYSTEM.

Thus, the force $= 0$.

What happens when the sand hits the ground?

Example

An empty freight car of mass M starts from rest under an applied force F.

At the same time, sand begins to run into the car at a steady rate b from a hopper at rest along the track. See figure 128.

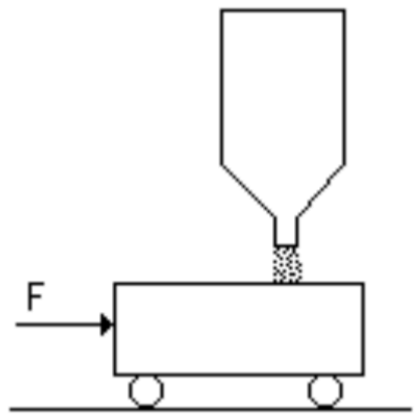

Figure 128:

Find the speed when a mass m of sand has been transferred.

System = car + sand to enter in next Δt.

$$b = \frac{dM}{dt} \rightarrow M = M_0 + bt$$

We only consider the horizontal direction.

$$P(t) = M(t)v \quad , \quad P(t+\Delta t) = (M(t) + b\Delta t)(v + \Delta v)$$

$$\Delta P = P(t+\Delta t) - P(t) = bv\Delta t + M(t)\Delta v \rightarrow \frac{\Delta P}{\Delta t} = bv + M(t)\frac{\Delta v}{\Delta t}$$

$$F = \frac{dP}{dt} = bv + M(t)\frac{dv}{dt} = bv + (M_0 + bt)\frac{dv}{dt}$$

$$\frac{dt}{M_0 + bt} = \frac{dv}{F - bv} \rightarrow \ln\frac{M_0 + bt}{M_0} = \ln\frac{F - bv}{F}$$

$$\frac{M_0 + bt}{M_0} = \frac{F - bv}{F} \rightarrow v = \frac{Ft}{M_0 + bt}$$

What does that look like?

Now for **two-line solution** ...

$$I = \int_0^t F dt = Ft = \Delta P = P(t) - P(0) = m(t)v(t) - 0$$

$$v(t) = \frac{Ft}{M_0 + bt}$$

The impulse from the sand is perpendicular to the direction of motion and does not contribute!

Example

Material is blown into cart A from cart B at a rate b kg/s.

The material leaves the chute vertically downward, so that it has the same horizontal velocity as cart B, u .

At the moment of interest cart A has mass M and velocity v, as in figure 129.

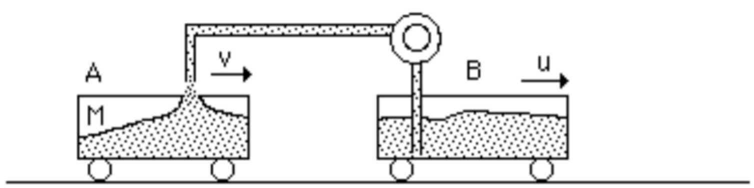

Figure 129:

What is the acceleration of A?

Assume Δm = mass that will arrive at cart A in next Δt. Then system = $M + \Delta m$. Consider only the horizontal motion

$$P + i = Mv + (\Delta m)u \quad , \quad P_f = (M + \Delta m)(v + \Delta v)$$

No external forces means momentum is constant (horizontal direction).

$$0 = P_f - P_i = M\Delta v + \Delta m(v - u) \rightarrow M\frac{dv}{dt} = \frac{dm}{dt}(u - v)$$

But

$$\frac{dM}{dt} = \frac{dm}{dt} = b$$

so,

$$\frac{dv}{dt} = \frac{b}{M}(u - v)$$

or

$$\frac{dv}{u - v} = \frac{b}{M}dt = \frac{dM}{M}$$

Therefore

$$\int_{v_0}^{v} \frac{dv}{u - v} = \int_{M_0}^{M_0+bt} \frac{dM}{M} \rightarrow -\ln\frac{u - v(t)}{u - v_0} = \ln\frac{M_0 + bt}{M_0}$$

$$\frac{u - v_0}{u - v(t)} = \frac{M_0 + bt}{M_0} \rightarrow v(t) = u - \frac{M_0(u - v_0)}{M_0 + bt} \rightarrow a = \frac{dv}{dt} = \frac{bM_0(u - v_0)}{(M_0 + bt)^2}$$

Example N men, each with mass m, stand on a railway flatcar of mass M. They jump off one end of the flatcar with velocity u relative to the flatcar.

The car rolls in the opposite direction without friction.

(a) What is the final velocity of the flatcar if all the men jump off at the same time?

In each case the flatcar is at its final velocity as the jumper parts company; no acceleration after the jumper leaves.

See figure 130.

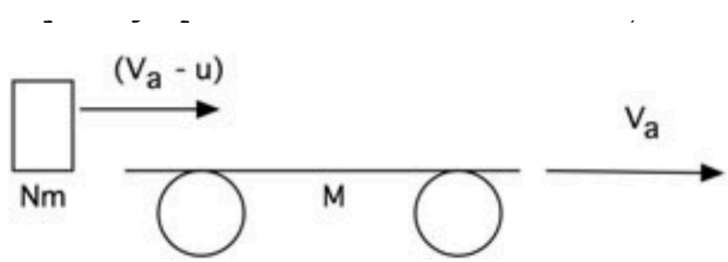

Figure 130:

Remember velocity of jumper is relative to flatcar.

$$p_i = 0$$

$$p_f = MV_a + Nm(V_a - u)$$

Therefore,

$$V_a = \left(\frac{Nm}{Nm + M}\right)u$$

(b) What is the final velocity of the flatcar if they jump off one at at time?

See figure 131.

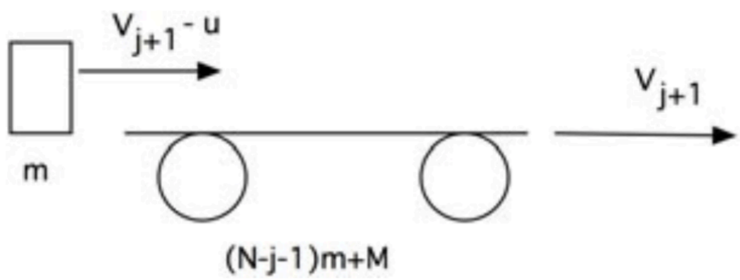

Figure 131:

Assume j men have already jumped so that the speed of the flatcar is V_j.

$$p_i = ((N - j)m + M)V_j$$
$$p_f = ((N - j - 1)m_M)V_{j+1} + m(V_{j+1} - u)$$
$$V_{j+1} = \frac{m}{(N - j)m + M}u + V_j$$

Therefore,

$$V_b = \left[\frac{m}{Nm + M} + \frac{m}{(N-1)m + M} + \cdots + \frac{m}{m + M}\right]u$$

(c) Which case gives the larger velocity? Why?

Now,

$$V_a = \left[\frac{m}{Nm + M} + \frac{m}{Nm + M} + \cdots + \frac{m}{Nm + M}\right]u < V_b$$

The Motion of a Rocket

Let us use the concept of momentum to understand how a rocket behaves.

A rocket accelerates by expelling gas at a very high velocity; the reaction force of the gas on the rocket accelerates the rocket in the opposite direction.

To analyze the motion of the rocket we consider the system of the rocket and fuel during the time interval t to $t + \Delta t$.

Between t and $t + \Delta t$ a mass of fuel Δm is burned and expelled as gas with a velocity $\vec{u}$ relative to the rocket.

The exhaust velocity $\vec{u}$ is independent of the velocity of the rocket (i.e., it depends on the type of chemical reaction, etc).

See figure 132.

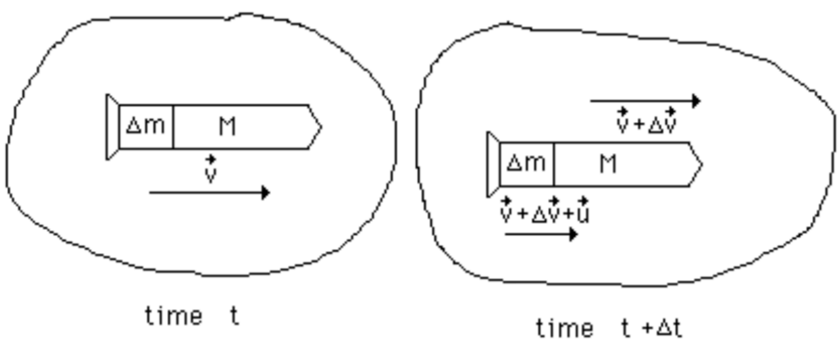

Figure 132:

All velocities here are with respect to an inertial frame (not attached to the rocket).

The initial momentum is

$$\vec{P}(t) = (M + \Delta m)\vec{v}$$

The final momentum is

$$\vec{P}(t + \Delta t) = M(\vec{v} + \Delta\vec{v}) + \Delta m(\vec{v} + \Delta\vec{v} + \vec{u})$$

Here we have chosen the velocity of Δm to be $vecv + \Delta\vec{v} + \vec{u}$. We could have chosen it to be $\vec{v} + \vec{u}$. The end result in the limit of $\Delta t \to 0$ is the same.

The change in momentum is

$$\Delta\vec{P} = \vec{P}(t + \Delta t) - \vec{P}(t) = M\Delta\vec{v} + (\Delta m)\vec{u}$$

which gives

$$\frac{d\vec{P}}{dt} = M\frac{d\vec{v}}{dt} + \vec{u}\frac{dm}{dt}$$

Here we have assumed that $M(t)+m(t)$ = constant where $M(t)$ is the amount of mass remaining in the rocket at time t and $m(t)$ is the amount of mass expelled by time t.

In most normal rockets $\vec{u}$ is opposite to $\vec{v}$.

Since $M(t) + m(t)$ = constant, we have

$$\frac{dm}{dt} = -\frac{dM}{dt}$$

or

$$\frac{d\vec{P}}{dt} = M\frac{d\vec{v}}{dt} - \vec{u}\frac{dM}{dt} \rightarrow \text{ rocket equation}$$

Note that it is NOT

$$\frac{d(M(t)\vec{v}(t))}{dt}$$

Rocket in Free Space

In this case, $\vec{F} = 0$ and

$$\vec{F} = \frac{d\vec{P}}{dt} = M\frac{d\vec{v}}{dt} - \vec{u}\frac{dM}{dt} = 0 \rightarrow M\frac{d\vec{v}}{dt} = \vec{u}\frac{dM}{dt}$$

$$\frac{d\vec{v}}{dt}dt = \vec{u}\frac{dM}{M} \rightarrow \vec{v} = \vec{u}\frac{dM}{M}$$

Integrating we get

$$\vec{v}_f - \vec{v}_0 = \vec{u}\ln\frac{M_f}{M_0} = -\vec{u}\ln\frac{M_0}{M_f}$$

If $\vec{v}_0 = 0$, then

$$\vec{v}_f = -\vec{u} \ln \frac{M_0}{M_f}$$

The final velocity is **independent** of how the mass is released ... fast or slow ... The only important quantity is the exhaust velocity.

Rocket in a Gravitational Field

$$\vec{F} = \frac{d\vec{P}}{dt} = M\vec{g} = M\frac{d\vec{v}}{dt} - \vec{u}\frac{dM}{dt}$$

where $\vec{u}$ and $\vec{g}$ are both directed downwards. We have

$$\vec{u}\frac{dM}{M} + \vec{g}dt = d\vec{v}$$

$$\vec{v}_f - \vec{v}_0 = \vec{u} \ln \frac{M_f}{M_0} + \vec{g}(t - t_0)$$

If $\vec{v}_0 = 0$, $t_0 = 0$, then

$$\vec{v}_f = \vec{u} \ln \frac{M_f}{M_0} + \vec{g}t$$

or choosing upwards as positive

$$v_f = u \ln \frac{M_0}{M_f} - gt$$

Now there is a premium attached to burning the fuel rapidly.

The shorter the burn time, the greater the velocity.

That is why takeoffs are so spectacular!

Example - A rocket ascends from rest in a uniform gravitational filed by ejecting exhaust with constant speed u. Assume that the rate at which mass is expelled is given by

$$\frac{dm}{dt} = -\gamma m$$

where m is the instantaneous mass of the rocket and γ is a constant, and that the rocket is retarded by air resistance with a force mbv, where b is a constant.

Find $v(t)$.

The rocket equation gives:

$$m\frac{dv}{dt} + u\frac{dm}{dt} = -mg - mbv$$

$$\frac{dv}{dt} - \gamma u = -g - bv$$

$$\frac{dv}{dt} + bv = \gamma u - g$$

The most general solution to this equation is the sum of two solutions

$$v = v_h + v_p$$

where

$$v_h = \text{ homogeneous solution } \rightarrow \frac{dv_h}{dt} + bv_h = 0$$

$$v_p = \text{ particular solution } \rightarrow \frac{dv_p}{dt} + bv_p = \gamma u - g$$

Since the particular solution is unique we only need to guess it.

We have

$$v_p = \frac{\gamma u - g}{b}$$

We solve for the homogeneous solution by separating the variables

$$\frac{dv_h}{dt} + bv_h = 0$$

$$\frac{dv_h}{v_h} = -bdt \rightarrow \int_{v_h(0)}^{v_h(t)} \frac{dv_h}{v_h} = -b\int_o^t dt \rightarrow \ln\frac{v_h(t)}{v_h(0)} = -bt \rightarrow v_h(t) = v_h(0)e^{-bt}$$

Therefore, the solution is

$$v(t) = \frac{\gamma u - g}{b} + Ae^{-bt}$$

We determine the constant A by using the initial condition $v(0) = 0$ to get

$$0 = \frac{\gamma u - g}{b} + A \rightarrow A = -\frac{\gamma u - g}{b}$$

so that

$$v(t) = \frac{\gamma u - g}{b}(1 - e^{-bt})$$

After a long time the velocity becomes a constant

$$v(t_{large}) = \frac{\gamma u - g}{b} = \text{ terminal velocity}$$

Momentum Transport

If you get blasted by a stream of water from a hose you feel a push.

The push comes from the momentum transfer as the water bounces off of you.

Let us see exactly how this occurs.

Picture the water stream as a series of drops, each of mass m, a distance L apart and traveling with velocity v_0.

See figure 133.

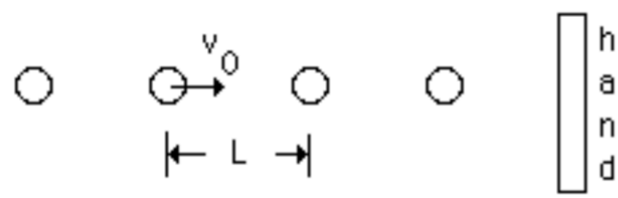

Figure 133:

Assume the drops collide with your hand and then just drop straight down.

As each drop hits there is a large force for a short time.

We must have

$$I_{droplet} = \int_{1\,collision} F dt = \Delta p = m(v_f - v_0) = -mv_0$$

$$-I_{droplet} = I_{hand} = mv_0$$

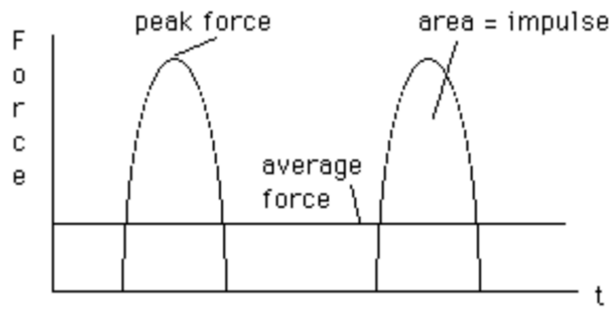

Figure 134:

in the same direction as the velocity of the drops.

The impulse = the area under one of the peaks as in figure 134.
If there are many drops per second you do not feel the individual shocks but instead feel the average force as shown.

The area under the average force during one period T (T = time between drops) = area under a single peak.

$$F_{av}T = \int_{1\,collision} F dt$$

Now $T = L/v_0$ and $\int_{1\,collision} F dt = mv_0$ so the average force is

$$F_{av} = \frac{mv_0}{T} = \frac{m}{L}v_0^2$$

Another way:

Consider length D of the stream just about to hit the surface. See figure 135.

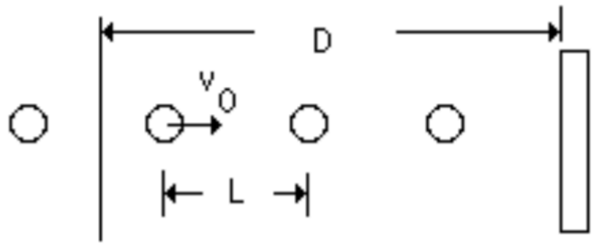

Figure 135:

The number of drops in D is D/L. Each has momentum mv_0, so the total momentum is $\Delta p = (Dmv_0)/L$.

All of these drops strike the wall in time $\Delta t = D/v_0$. Thus, the average force is

$$F_{av} = \frac{\Delta p}{\Delta t} = \frac{m}{L}v_0^2$$

To apply this model to a fluid, consider a stream moving with speed v. See figure 136.

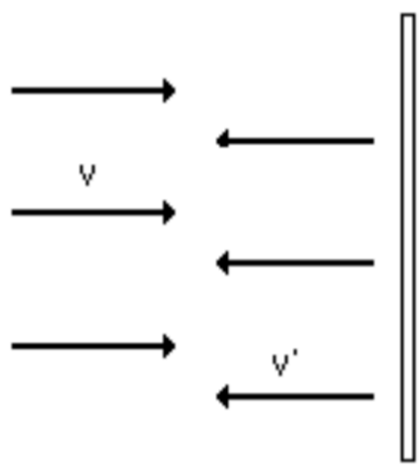

Figure 136:

If the mass per unit length is $\lambda = m/L$, the momentum per unit length is λv

and the rate at which the stream transports momentum to the surface is

$$\frac{dp}{dt} = (\text{momentum / length})(\text{length / sec}) = \lambda v^2$$

If the stream comes to rest at the surface, the force is $F = \lambda v^2$.

If the stream rebounds then we have a different situation (see figure 136).

Now $\frac{dm}{dt} = \lambda v$ = rate mass arriving at surface and the rate of mass leaving is $\lambda' v'$.

Since mass does not accumulate at the surface or get lost anywhere we must have $\lambda v = \lambda' v'$.

The total force on the surface due to both contributions is

$$\begin{aligned} F &= \frac{dP}{dt} = \frac{d(p+p')}{dt} = \frac{dp}{dt} + \frac{dp'}{dt} = v'\frac{dm'}{dt} + v\frac{dm}{dt} \\ &= \lambda' v' v' + \lambda v v = \lambda v(v+v') \end{aligned} \tag{5.30}$$

Special Cases:

no rebound

$$\rightarrow v' = 0 \rightarrow F = \lambda v^2 \rightarrow \text{ same result as before}$$

perfect rebound

$$\rightarrow v' = v \rightarrow F = 2\lambda v^2$$

Example

Water shoots out of a fire hydrant having nozzle diameter D with nozzle speed V_0. See figure 137.

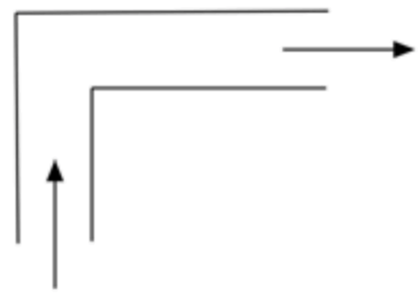

Figure 137:

What is the reaction force on the hydrant?

$$\lambda = \text{ mass/unit-length } = \frac{\pi D^2}{4}\rho \cdot 1 = \frac{\pi D^2}{4}\rho$$

$$\rho = \text{ density of the water}$$

$$\Delta p_{in} = \Delta p_{out} = \lambda V_0^2 \Delta t$$

As the figure 138 shows

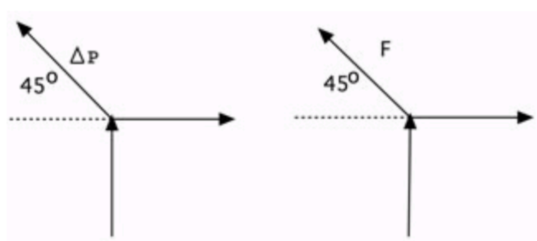

Figure 138:

$\Delta \vec{p}$ transferred to the hydrant is at 45° and has magnitude $\sqrt{2}\lambda V_0^2 \Delta t$.

Therefore, the force on the hydrant is

$$\frac{\Delta p}{\Delta t} = \sqrt{2}\lambda V_0^2 = \frac{\sqrt{2}}{4}\pi D^2 \rho V_0^2$$

6. Work and Energy

Part 1

Let us look back at some systems that we have already investigated to see if we can learn more about the physics that is taking place.

The theorist often does this to get a better understanding and many times this second look leads to new insights and occasionally brings about a paradigm shift in how we think about the physics involved.

Earlier we carried out the following alternative mathematical way of dealing with the kinematics equations.....(we consider 1-dimension only to start with).

If we know the force as a function of time $F(t)$ then we can directly integrate Newton's 2nd law to yield the velocity as follows

$$m\frac{dv}{dt} = F(t) \to dv = \frac{1}{m}F(t)dt \tag{6.1}$$

$$\int_{v_0}^{v(t)} dv = \frac{1}{m}\int_{t_0}^{t} F(t)dt \to v(t) = v_0 + \frac{1}{m}\int_{t_0}^{t} F(t)dt \tag{6.2}$$

which yields $v(t)$.

If we know the force as a function of position $F(x)$, this is a more complicated problem(formal diffEQ theory can handle it, but we do not know all of that stuff yet). In this case we handle things as follows:

$$m\frac{dv}{dt} = F(x) \to \frac{dv}{dx}\frac{dx}{dt} = mv\frac{dv}{dx} = F(x) \tag{6.3}$$

$$m\int_{v_0}^{v(x)} vdv = \int_{x_0}^{x} F(x)dx \to \frac{1}{2}mv^2(x) - \frac{1}{2}mv_0^2 = \int_{x_0}^{x} F(x)dx \tag{6.4}$$

which yields $v(x)$.

We have used the chain rule of calculus above

$$\frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} \tag{6.5}$$

This is an important trick that we will use many time during this course.

Let us see what happens in some special cases (that is how we really learn what is going on).

Mass Thrown Upward in a Uniform Gravitational Field

Mass m is thrown upward with initial velocity v_0.

How high does it rise?

Assume z positive is upward.

We then have

$$F = F_z = -mg.$$

We get

$$\frac{1}{2}mv^2(z) - \frac{1}{2}mv_0^2 = \int_{z_0}^{z} F(z)dz = -mg\int_{z_0}^{z} dz = -mg(z - z_0)$$

At the maximum height, $v(z_{max}) = 0$, which yields the result

$$z_{max} = z_0 + \frac{v_0^2}{2g}$$

Note that this solution method makes **NO** reference to time at all.

To get the same result by integrating with respect to t, we would have had to eliminate t from our equations, which is generally much more complicated.

We note for later discussion the intriguing result

$$\frac{1}{2}mv^2(z) + mgz = \frac{1}{2}mv_0^2 + mgz_0 = \text{ constant during the motion}$$

during the motion.

We get this result by rearranging the equation derived above.

Solving the Equations of SHM

First let us solve this problem using the force as a function of position $F(x)$

method.

We consider a mass M attached to a spring(k). Let x be measured from equilibrium so that $F = -kx$. We then have

$$\frac{1}{2}Mv^2(x) - \frac{1}{2}Mv_0^2 = \int_{x_0}^{x} F(x)dx = -k\int_{x_0}^{x} xdx = -\frac{1}{2}kx^2 + \frac{1}{2}kx_0^2 \tag{6.6}$$

Suppose $v_0 = 0$ (release at rest) and $x_0 \neq 0$ (displaced from equilibrium), then

$$\frac{1}{2}Mv^2(x) = -\frac{1}{2}kx^2 + \frac{1}{2}kx_0^2 \rightarrow v^2(x) = -\frac{k}{M}x^2 + -\frac{k}{M}x_0^2 \tag{6.7}$$

$$v(x) = \frac{dx}{dt} = \sqrt{\frac{k}{M}}\sqrt{x_0^2 - x^2} \tag{6.8}$$

Separating variables and integrating we obtain

$$\frac{dx}{\sqrt{x_0^2 - x^2}} = \sqrt{\frac{k}{M}}dt \rightarrow \int_{x_0}^{x} \frac{dx}{\sqrt{x_0^2 - x^2}} = \sqrt{\frac{k}{M}}\int_0^t dt \tag{6.9}$$

$$\sin^{-1}\frac{x}{x_0}\bigg|_{x_0}^{x} = \omega\int_0^t dt = \omega t \quad , \quad \omega^2 = \frac{k}{M} \tag{6.10}$$

$$\sin^{-1}\frac{x}{x_0} - \sin^{-1} = \sin^{-1}\frac{x}{x_0} - \frac{\pi}{2} = \omega t \tag{6.11}$$

$$\sin^{-1}\frac{x}{x_0} = \omega t + \frac{\pi}{2} \rightarrow x = x_0\sin\left(\omega t + \frac{\pi}{2}\right) = x_0\cos\left(\omega t\right) \tag{6.12}$$

We just need to use integration in a clever manner and no diffEQ!!

We note again the intriguing relation (for later discussion).

$$\frac{1}{2}Mv^2(x) + \frac{1}{2}kx^2 = \frac{1}{2}Mv_0^2 + \frac{1}{2}kx_0^2 = \text{ constant during the motion} \tag{6.13}$$

The Work-Energy Theorem in One Dimension

We now define two new quantities

(1) kinetic energy = $K = \frac{1}{2}mv^2$

(2) work done by force $F(x)$ on particle as it moves from point a to point b is

$$W_{ba} = \int_{x_a}^{x_b} F(x)dx \tag{6.14}$$

As we have seen in all of the above special case examples, we then have the relationship (called the **Work-Energy Theorem**) between these two quantities

$$W_{ba} = \int_{x_a}^{x_b} F(x)dx = K_b - K_a = (\Delta K)_{ba} \tag{6.15}$$

which we assume to be generally true(a theorem).

Example - Vertical Motion in an Inverse Square Field

A mass m is shot **vertically upward** from the surface of the Earth with an initial speed v_0.

What is maximum height (altitude) reached? What initial speed will allow the mass to escape(called escape velocity) Earth completely, that is, get to $r = \infty$? The situation is represented by figure 139:

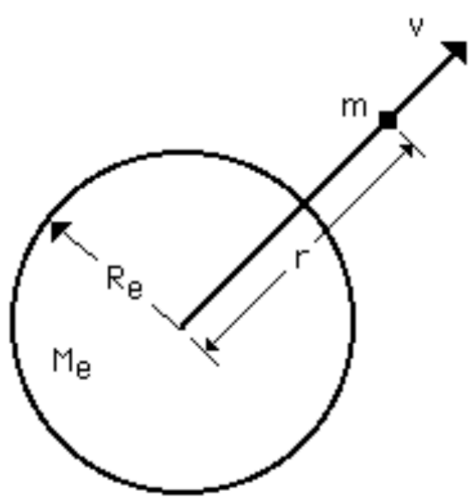

Figure 139:

The motion is 1-dimensional (radial direction) in this case. The radial force on m is

$$F_r = -\frac{GM_e m}{r^2}$$

We then have

$$\Delta K = K(r) - K(R_e) = \int_{R_e}^{r} F_r(r)dr = W_{rR_e}$$

$$\frac{1}{2}mv^2(r) - \frac{1}{2}mv_0^2 = -GM_e m \int_{R_e}^{r} \frac{1}{r^2}dr = GM_e m\left(\frac{1}{r} - \frac{1}{R_e}\right)$$

The maximum height corresponds to $v(r) = 0$, which gives

$$-\frac{1}{2}mv_0^2 = GM_e m\left(\frac{1}{r_{max}} - \frac{1}{R_e}\right) = gR_e^2\left(\frac{1}{r_{max}} - \frac{1}{R_e}\right)$$

where

$g = \dfrac{GM_e}{R_e^2} =$ acceleration due to earth's gravitational field at the Earth's surface.

Finally we get

$$r_{max} = \frac{R_e}{1 - \frac{v_0^2}{2gR_e}}$$

The escape velocity is found by setting $r_{max} = \infty$ to get

$$v_0(r_{max} \to \infty) = v_{escape} = \sqrt{2gR_e}$$

Now let us put in some numbers:

$$v_{escape} = \sqrt{2gR_e} = \sqrt{2(9.8\ \text{m/s}^2)(6.4 \times 10^6\ \text{m})} = 1.1 \times 10^4\ \text{m/s}$$

and the kinetic energy ($\frac{1}{2}mv_0^2$) needed to eject a 50 kg spacecraft is

$$K_{escape} = \frac{1}{2}Mv_{escape}^2 = (0.5)(50\ \text{kg})(1.1 \times 10^4\ \text{m/s})^2 = 3.025 \times 10^9\ \text{J(oules)}$$

Again we note the **intriguing relationship** (for later discussion).

$$\frac{1}{2}mv^2(r) - \frac{GM_e m}{r} = \frac{1}{2}mv_0^2 - \frac{GM_e m}{R_e} = \text{constant}$$

Integrating the Equation of Motion in Several Dimensions

In general 3-dimensional motion we have the vector equation

$$\vec{F}(\vec{r}) = m\frac{d\vec{v}}{dt} \tag{6.16}$$

Let us generalize our 1-dimensional procedure to this deal with this case.

We consider what happens when the particle moves through a displacement $\Delta\vec{r}$ (i.e., during the actual motion).

The displacement is infinitesimal so we can assume that the force is constant during the displacement.

Let us consider figure 140.

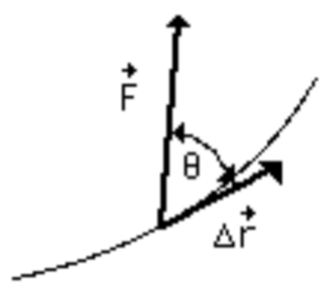

Figure 140:

so that

$$\begin{aligned}\vec{F}(\vec{r})\cdot\Delta\vec{r} = F(\Delta r)\cos\theta = m\frac{d\vec{v}}{dt}\cdot\Delta\vec{r} &= m\left(\frac{d\vec{v}}{dt}\cdot\vec{v}\right)\Delta t \\ &= m\left(\frac{1}{2}\frac{d\vec{v}}{dt}\cdot\vec{v} + \frac{1}{2}\vec{v}\cdot\frac{d\vec{v}}{dt}\right) = \frac{1}{2}m\frac{d}{dt}\left(\vec{v}\cdot\vec{v}\right) = m\Delta t\frac{d}{dt}\left(\frac{1}{2}v^2\right) \\ &= \frac{1}{2}m\Delta t\frac{dv^2}{dt}\end{aligned}$$

Now divide the entire trajectory (see figure 141) from the initial position $\vec{r}_a$ to the final position $\vec{r}_b$ into N short segments of length $\Delta\vec{r}_j$

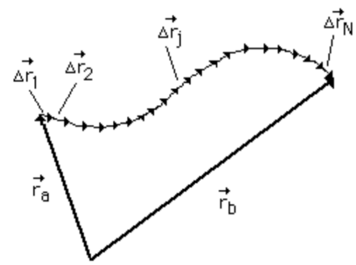

Figure 141:

where j is an index numbering the segments.

For each segment we have according to the derivation above

$$\vec{F}(\vec{r}_j)\cdot\Delta\vec{r}_j = \frac{1}{2}m\Delta t\frac{dv_j^2}{\Delta t_j}$$

where $\vec{r}_j$ is the position of segment j, $\vec{v}_j$ is the velocity the particle has there, and Δt_j is the time it spends traversing it. Adding all of the these terms for the N segments we get

$$\sum_{j=1}^{N}\vec{F}(\vec{r}_j)\cdot\Delta\vec{r}_j = \sum_{j=1}^{N}\frac{1}{2}m\Delta t\frac{dv_j^2}{dt}\Delta t_j$$

Now we take the limit where the length of each segment approaches zero and the number of segments approaches infinity so that the sums become integrals and we get

$$\begin{aligned}\text{total work } &= \int_{\vec{r}_a}^{\vec{r}_b}\vec{F}(\vec{r})\cdot d\vec{r} = \frac{1}{2}m\int_{t_a}^{t_b}\frac{dv^2}{dt}dt = \frac{1}{2}m\int_{v_a}^{v-b}dv^2 \\ &= \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2 = \Delta K \qquad (6.17)\end{aligned}$$

Example - A Path-Dependent Line Integral

Now let us do a path-dependent (work) integral. Let

$$\vec{F} = 5xy\hat{i} + 5y^2\hat{j}$$

and consider the integral from (0,0) to (0,1) along paths labelled 1(covers 3 sides) and 2(covers 1 side) as shown in figure 142.

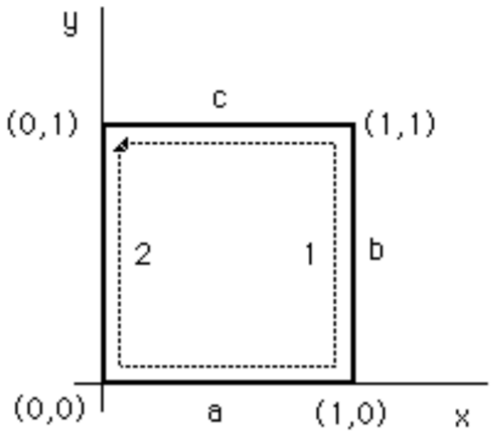

Figure 142:

This is an example of a so-called **nonconservative force**. We can write these integrals as follows:

Path 1:

$$\int_1 \vec{F} \cdot d\vec{r} = \int_a \vec{F} \cdot d\vec{r} + \int_b \vec{F} \cdot d\vec{r} + \int_c \vec{F} \cdot d\vec{r}$$

$$\int_1 \vec{F} \cdot d\vec{r} = \int_{0,0}^{1,0} F_x dx + \int_{(1,0}^{1,1} F_y dy + 5 \int_{1,1}^{0,1} F_x dx$$

$$\int_1 \vec{F} \cdot d\vec{r} = 5 \int_{0,0}^{1,0} xydx + 5 \int_{(1,0}^{1,1} y^2 dy + \int_{1,1}^{0,1} xydx$$

$$\int_1 \vec{F} \cdot d\vec{r} = 5 \int_{0,0}^{1,0} x(0)dx + 5 \int_{(1,0}^{1,1} y^2 dy + \int_{1,1}^{0,1} x(1)dx$$

$$\int_1 \vec{F} \cdot d\vec{r} = 0 + \frac{5}{3} - \frac{5}{2} = -\frac{5}{6}$$

Path 2:

$$\int_2 \vec{F} \cdot d\vec{r} = \int_{0,0}^{0,1} F_y dy = 5 \int_{0,0}^{0,1} y^2 dy = \frac{5}{3}$$

Clearly, the work done by this applied nonconservative force is **DIFFERENT** along the two paths.

Potential Energy

Several times we noted in various examples that the quantity

$$\frac{1}{2}mv^2 + U(\vec{r}) = \text{ constant} \tag{6.18}$$

where $U(\vec{r})$ = some function of position only.

It turns out that all of these examples involved so-called **conservative** forces and the above relationship is a special case of a more general result for conservative forces → **work integral independent of path of integration**.

For conservative forces we find that

$$\int_{\vec{r}_a}^{\vec{r}_b} \vec{F} \cdot d\vec{r} = \text{ function}(\vec{r}_b) - \text{ function}(\vec{r}_a) = -U(\vec{r}_b) + U(\vec{r}_a) \tag{6.19}$$

where $U(\vec{r})$ is a function defined as above, known as the **potential energy** function.

Clearly, in this definition, the integral depends only on the endpoints and not on the path.

The work-energy theorem then gives

$$W_{ba} = \int_{\vec{r}_a}^{\vec{r}_b} \vec{F} \cdot d\vec{r} = K_b - K_a = -U_b + U_a \tag{6.20}$$

or

$$K_b + U_b = K_a + U_a \tag{6.21}$$

Since the LHS depends only on what is happening at b and the RHS depends only on what is happening at a, the quantity

$$K + U = \text{ constant } = E = \text{ total mechanical energy of the system} \tag{6.22}$$

or in general, **the total energy is constant** for conservative forces.

Thus, for conservative forces the total energy is independent of the position of the particle .... it remains constant .... it is **conserved**.

This is a **"derived"** law.

It has **no more physical content** than Newton's 2nd law from which it was derived.

A peculiar property of energy is that the value of E is to a certain extent arbitrary. Only changes in E have physical significance(not the actual value).

This comes about because the relation

$$-\int_{\vec{r}_a}^{\vec{r}_b} \vec{F} \cdot d\vec{r} = U_b - U_a \tag{6.23}$$

defines only the **difference** in potential energy between a and b and not the potential energy itself.

So if I change the definition of U by adding a constant, the work equation does not change and hence the energy is still constant (although a different constant).

We will take advantage of this feature in many ways.

It is called setting the potential **reference level** or potential **zero level**.

The zero level can be chosen arbitrarily so as to make the calculations as easy as possible.

We will see this in examples.

This is an example of a general property of physical systems called **"gauge invariance"** which is a powerful principle in electromagnetism, quantum physics, particle physics, and general relativity.

Potential Energy of a Uniform Force Field

For $\vec{F} = -mg\hat{z}$ (z positive upwards) we found earlier that

$$\frac{1}{2}mv^2(z) - \frac{1}{2}mv^2(z_0) = \int_{z_0}^{z} F(z)dz = -mg\int_{z_0}^{z} dz = -mg(z-z_0) = -U(z) + U(z_0)$$

so the gravitation potential energy is

$$U(z) = mgz \tag{6.24}$$

Where is the zero level in this case?

Example using energy conservation

Project a mass m vertically upward with $v(0) = v_0$, then

$$\frac{1}{2}mv^2(h) - \frac{1}{2}mv_0^2 = -mg(h-0) = -mgh \rightarrow v(h) = \sqrt{v_0^2 - 2gh}$$

as we found earlier.

For $\vec{F} = F_0\hat{n}$ (this is a constant force), we found earlier that

$$\frac{1}{2}mv^2(b) - \frac{1}{2}mv^2(a) = \int_a^b \vec{F}\cdot d\vec{r} = \vec{F}\cdot\int_a^b d\vec{r} = \vec{F}\cdot(\vec{r}_b - \vec{r}_a) = -U(b) + U(a)$$

so that the potential energy of a constant force is

$$U(\vec{r}) = -\vec{F}\cdot\vec{r} \tag{6.25}$$

Does that agree with the last example?

Potential Energy of an Inverse Square Force

For the force

$$\vec{F} = -\frac{GM_em}{r^2}\hat{r} = -\frac{mgR_e^2}{r^2}\hat{r}$$

we found earlier that

$$\begin{aligned}\int_{R_e}^r F_r(r)dr &= \frac{1}{2}mv^2(r) - \frac{1}{2}mv^2(R_e) = -GM_em\int_{R_e}^r \frac{1}{r^2}dr \\ &= GM_em\left(\frac{1}{r} - \frac{1}{R_e}\right) = mgR_e^2\left(\frac{1}{r} - \frac{1}{R_e}\right) = -U(r) + U(R_e) \quad (6.26)\end{aligned}$$

so the general gravitational potential energy is

$$U(r) = -\frac{GM_em}{r} = -\frac{mgR_e^2}{r} \tag{6.27}$$

Where is the zero level?

Potential Energy of a Spring

For the force $F = -kx$, we found earlier that

$$\frac{1}{2}Mv^2(x) - \frac{1}{2}Mv^2(x_0) = \int_{x_0}^{x} F(x)dx = -\frac{1}{2}kx^2 + \frac{1}{2}kx_0^2 = -U(x) + U(x_0)$$

so the spring potential energy is

$$U(x) = \frac{1}{2}kx^2 \tag{6.28}$$

Where is the zero level?

Relationship between the Potential Energy and the Force

For **conservative forces** we have defined the potential energy by the relation

$$U_b - U_a = -\int_{\vec{r}_a}^{\vec{r}_b} \vec{F} \cdot d\vec{r}$$

where the integral is over any path between a and b.

In many systems it is easier to figure out the potential energy than the force and hence it would be useful to have a relation that goes the other way

> .... if we are given the potential energy, then we calculate the force .....

We can see what happens by looking at a 1-dimensional system, such as a mass on a spring.

Let the force be $F(x)$.

We then have

$$U_b - U_a = -\int_{x_a}^{x_b} F(x)dx$$

Now consider the case where $x_a = x$ and $x_b = x + \Delta x$.

We then get

$$U(x + \Delta x) - U(x) = \Delta U = \int_{x}^{x+\Delta x} F(x)dx$$

If Δx is very small, then we can assume that $F(x)$ is constant over the range of integration and we get

$$\Delta U = -F(x) \int_x^{x+\Delta x} dx = -F(x)\Delta x \rightarrow F(x) = -\frac{\Delta U}{\Delta x}$$

If we take the limit $\Delta x \rightarrow 0$, then we have

$$F(x) = -\frac{dU(x)}{dx} \tag{6.29}$$

This result holds whenever the potential is a **function of only one variable**.

Examples

Uniform gravitational field

$$U(z) = mgz \quad , \quad F(z) = -\frac{dU}{dz} = -mg \tag{6.30}$$

Constant force

$$U(x) = -F_0 x \quad , \quad F(x) = -\frac{dU}{dx} = F_0 \tag{6.31}$$

Spring force

$$U(x) = \frac{1}{2} lkx^2 \quad , \quad F(x) - \frac{dU}{dx} = -kx \tag{6.32}$$

Inverse square force

$$U(r) = -\frac{GM_e m}{r} = -\frac{mgR_e^2}{r} \quad , \quad F(r) = -\frac{dU}{dr} = -\frac{GM_e m}{r^2} = -\frac{mgR_e^2}{r^2} \tag{6.33}$$

A rather amazing result!!

Earlier we found that whenever the potential is a **function of only one variable**, we have

$$F(x) = -\frac{dU(x)}{dx}$$

Let us expand a bit on our earlier discussion of generalizing this result to more dimensions.

Now consider the conservative force given by

$$\vec{F} = 2xy\hat{i} + x^2\hat{j}$$

We can show that this is a conservative force by direct integration.

In this case, we evaluate the path-dependent work integral over a closed path, that is, consider the integral from (0,0) to (1,1) along the path labelled 1 and back from (1,1) to (0,0) along the path labelled 2 (we are going along the path opposite to the arrow in this case) as shown in figure 143. This is a closed loop integration - start point = end point (indicated by a circle on the integration symbol.

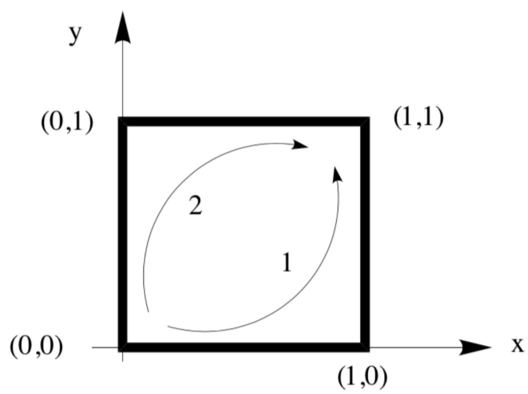

Figure 143:

For the nonconservative force $\vec{F} = xy\hat{\imath} + y^2\hat{\jmath}$ discussed earlier we find this integral is

$$\oint \vec{F}\cdot d\vec{r} = \int_1 \vec{F}\cdot d\vec{r} + \int_2 \vec{F}\cdot d\vec{r}$$

$$\int_1 \vec{F}\cdot d\vec{r} = \int_{0,0}^{1,0} F_x dx + \int_{1,0}^{1,1} F_y dy = \int_{0,0}^{1,0} xy dx + \int_{1,0}^{1,1} y^2 dy$$

$$\int_1 \vec{F}\cdot d\vec{r} = \int_{0,0}^{1,0} x(0) dx + \int_{1,0}^{1,1} y^2 dy = 0 + \frac{1}{3} = \frac{1}{3}$$

$$\int_2 \vec{F}\cdot d\vec{r} = \int_{1,1}^{0,1} F_x dx + \int_{0,1}^{0,0} F_y dy$$

$$\int_2 \vec{F}\cdot d\vec{r} = \int_{1,1}^{0,1} xy dx + \int_{0,1}^{0,0} y^2 dy$$

$$\int_2 \vec{F}\cdot d\vec{r} = \int_{1,1}^{0,1} x(1) dx + \int_{0,1}^{0,0} y^2 dy = -\frac{1}{2} - \frac{1}{3} = -\frac{5}{6}$$

$$\oint \vec{F} \cdot d\vec{r} = -\frac{1}{2} \neq 0$$

It is not zero!

This is characteristic of path-dependent or nonconservative forces.

For the conservative force $\vec{F} = 2xy\hat{i} + x^2\hat{j}$ above, however, we find

$$\oint \vec{F} \cdot d\vec{r} = \int_1 \vec{F} \cdot d\vec{r} + \int_2 \vec{F} \cdot d\vec{r}$$

$$\int_1 \vec{F} \cdot d\vec{r} = \int_{0,0}^{1,0} F_x dx + \int_{1,0}^{1,1} F_y dy = \int_{0,0}^{1,0} 2xy dx + \int_{1,0}^{1,1} x^2 dy$$

$$\int_1 \vec{F} \cdot d\vec{r} = \int_{0,0}^{1,0} 2x(0) dx + \int_{1,0}^{1,1} (1) dy = 0 + 1 = 1$$

$$\int_2 \vec{F} \cdot d\vec{r} = \int_{1,1}^{0,1} F_x dx + \int_{0,1}^{0,0} F_y dy$$

$$\int_2 \vec{F} \cdot d\vec{r} = \int_{1,1}^{0,1} 2xy dx + \int_{0,1}^{0,0} x^2 dy$$

$$\int_2 \vec{F} \cdot d\vec{r} = \int_{1,1}^{0,1} 2x(1) dx + \int_{0,1}^{0,0} (0) dy = -1 + 0 = -1$$

$$\oint \vec{F} \cdot d\vec{r} = 0$$

i.e., the integral is path-independent (equal and opposite in sign along the two paths) and thus the integral around the closed path is zero.

These are general results for conservative forces.

When you study Advanced Calculus or Electricity and Magnetism you will derive this general result and, in addition, derive the following more general results.

We start from the result

$$\vec{F}(\vec{r}) \cdot d\vec{r} = -dU(\vec{r})$$

which implies that

$$\vec{F}(\vec{r}) = -\nabla U(\vec{r}) = -\frac{\partial U}{\partial x}\hat{x} - \frac{\partial U}{\partial y}\hat{y} - \frac{\partial U}{\partial z}\hat{z}$$

Digression on Partial Derivative

Suppose we have a function $f(x,y,z)$.

Then the partial derivative of f with respect to x is defined by

$$\frac{\partial f(x,y,z)}{\partial x} = \left.\frac{df(x,y,z)}{dx}\right|_{y,z=\text{ constant}}$$

where

$$\nabla f \equiv \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

Example

$$f(x,y,z) = x^3yz + \sin(x)y + y^2z^3 \to \frac{\partial f(x,y,z)}{\partial x} = 3x^2yz + \cos(x)y$$

Now

$$\nabla \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k} = \text{ curl of } \vec{A}$$

For a conservative force $\vec{F}$, $\nabla \times \vec{F} = 0$ (everywhere).

For our examples above we have

$$\nabla \times \vec{F} = \nabla \times (2xy\hat{i} + x^2\hat{j}) = 0 \to \text{ conservative}$$

$$\nabla \times \vec{F} = \nabla \times (xy\hat{i} + y^2\hat{j}) = -x\hat{k} \neq 0 \text{ (everywhere)} \to \text{ nonconservative}$$

Thus the rule

$$F(x) = -\frac{dU(x)}{dx}$$

generalizes to

$$\vec{F} = -\nabla U = -\frac{\partial U}{\partial x}\hat{i} - \frac{\partial U}{\partial y}\hat{j} - \frac{\partial U}{\partial z}\hat{k} \tag{6.34}$$

For the conservative force above, this gives

$$U(x,y) = -x^2y \to -\frac{\partial U}{\partial x} = 2xy = F_x \quad , \quad -\frac{\partial U}{\partial y} = x^2 = F_y$$

This gives us the condition for a force to be conservative

$$\nabla \times \vec{F}(\vec{r}) = -\nabla \times \nabla U(\vec{r}) = 0 \tag{6.35}$$

Non-conservative Forces

Nonconservative forces like friction play important parts in the motion of many physical systems and cannot be neglected when we talk about energy.

Let see how we must modify the work-energy relation in this case.

If both conservative and nonconservative forces are present then we have

$$\vec{F} = \vec{F}^c + \vec{F}^{nc}$$

The work-energy theorem is true for **all types of forces**, so we have

$$\begin{aligned} W_{ba}^{total} &= \int_a^b \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}^C \cdot d\vec{r} + \int_a^b \vec{F}^{nc} \cdot d\vec{r} \\ &= -U_b + U_a + \int_a^b \vec{F}^{nc} \cdot d\vec{r} \end{aligned} \tag{6.36}$$

This finally gives

$$E_b - E_a = W_{ba}^{nc} = \int_a^b \vec{F}^{nc} \cdot d\vec{r} \tag{6.37}$$

Example - Block sliding down a rough inclined plane

A block of mass m slides down the plane as shown in figure 144.

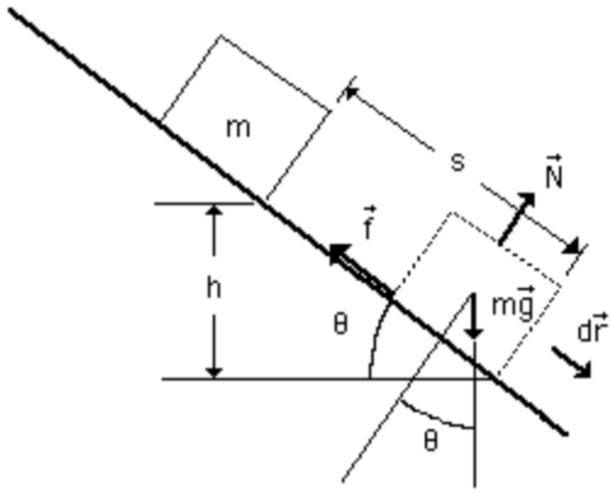

Figure 144:

How fast is it moving after falling a vertical distance h?

It starts from rest.

Define the zero level of gravitational potential energy at its final position. Thus,

$$U_a = mgh \quad , \quad U_b = 0$$

$$K_a = 0 \quad , \quad K_b = \frac{1}{2}mv^2$$

$$E_a = mgh \quad , \quad E_b = \frac{1}{2}mv^2$$

The nonconservative force is $f = \mu N = \mu mg\cos\theta$ (a constant).

Thus, the nonconservative work is

$$W_{ba}^{nc} = int_a^b \vec{F}^{nc} \cdot d\vec{r} = -f\int_a^b ds = -fs$$

since the frictional force is opposite to the displacement.

We then have

$$E_b - E_a = \frac{1}{2}mv^2 - mgh = W_{ba}^{nc} = -fs$$

$$v^2 = 2gh - 2\mu gs\cos\theta = 2gh - 2\mu g\frac{h}{\sin\theta}\cos\theta$$

$$v = \sqrt{2gh(1-\cot\theta)}$$

The General Law of Energy Conservation

The basic forces in nature, gravity and electromagnetism are all conservative.

Where do nonconservative forces come from?

To understand this seeming paradox, we must broaden the concept of energy beyond just the kinetic and potential energies.

When the block slides down the inclined plane, mechanical energy $K+U$ is lost... the final mechanical energy is less than the initial mechanical energy by an amount = to the nonconservative work.

We observe, however, that during this motion the plane(and block) get hotter, their temperature rises.

The net mechanical energy lost = the amount of energy (called heat) that is necessary to raise the temperature(s).

Heat energy is actually the mechanical energy of the atoms in the bodies.

As the block slides down the plane it causes(electromagnetic forces) the atoms in the surface of the plane to vibrate around their equilibrium positions such that their average speed increases ... this increase in average speed is what we call a temperature rise and the increase in mechanical energy needed to cause it is called heat.

If we redefine total energy to include mechanical energy and heat (lost or gained) then once again the total energy is conserved.

In fact, if one takes into account all the forms of energy, then the conservation law for total energy is exact.

Stability

The result

$$F(x) = -\frac{dU(x)}{dx}$$

is also useful for understanding the stability of a system.

If the net force on a body is $= 0$, the body is in translational equilibrium, that is, if the body is at rest, it will remain at rest.

There are two kinds of equilibrium, however, namely **stable and unstable**.

In stable equilibrium, if a body is displaced slightly from the equilibrium configuration, then it will always return to the equilibrium configuration, whereas, in unstable equilibrium it will not.

To see what is happening we consider a system that we know has an equilibrium configuration, namely, a mass on a spring.

In this case the potential energy is

$$U(x) = \frac{1}{2}kx^2$$

If we plot this potential energy function it looks like figure 145.

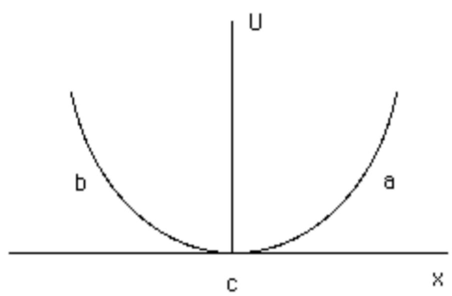

Figure 145:

At point c $(x = 0)$, $F(x) = -\frac{dU}{dx} = 0$ or the force = 0. This is an **equilibrium point**.

Is it stable or unstable?

Writing a vector equation we have

$$\vec{F} = F_x\hat{x} = F(x)\hat{x} = -\frac{dU}{dx}\hat{x}$$

Therefore, in region a ($x > 0$) where $F(x) = -\frac{dU}{dx} < 0$, the force points back (in negative x-direction) towards $x = 0$.

In region b ($x < 0$), $F(x) = -\frac{dU}{dx} > 0$ or the force points back (in positive x-direction) towards $x = 0$.

Thus, if we displace the mass from the equilibrium position, it will always return to the equilibrium position and hence this is a stable equilibrium position.

Can we generalize this result?

If we are at equilibrium then

$$F(x) = -\frac{dU}{dx} = 0$$

That means that the $U(x)$ curve can look like one of the following cases shown in figure 146:

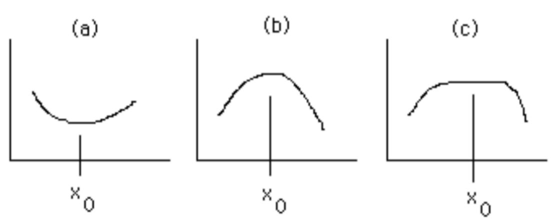

Figure 146:

where

(a) $\frac{d^2U}{dx^2} > 0 \rightarrow$ stable equilibrium

(b) $\frac{d^2U}{dx^2} < 0 \rightarrow$ unstable equilibrium

(c) $\frac{d^2U}{dx^2} = 0 \rightarrow$ neutral equilibrium

Now consider a pendulum as shown in figure 147

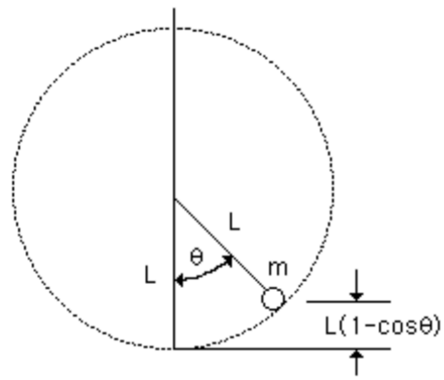

Figure 147:

In this case the potential energy is given (where is the reference level) by

$$U(\theta) = mgL(1 - \cos\theta)$$

which looks like figure 148

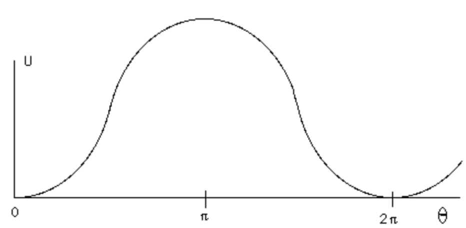

Figure 148:

It is clear that $\theta = 0, \pi$ are equilibrium points since

$$\frac{dU(\theta)}{d\theta} = mgL\sin\theta \text{ when } \theta = 0, \pi$$

but they are very different in nature. $\theta = 0$ is a stable equilibrium point since

$$\frac{d^2U(\theta)}{d\theta^2} = mgL\cos\theta = mgL > 0 \text{ near } \theta = 0$$

while $\theta = \pi$ is an unstable equilibrium point since

$$\frac{d^2U(\theta)}{d\theta^2} = mgL\cos\theta = -mgL < 0 \text{ near } \theta = \pi$$

So we can summarize our **equilibrium/stability criteria** as

$$\frac{dU(q)}{dq} = 0 \text{ and } \frac{d^2U(q)}{dq^2} > 0$$

A Complicated Example - The Teeter Toy

The teeter toy is shown in figure 149.

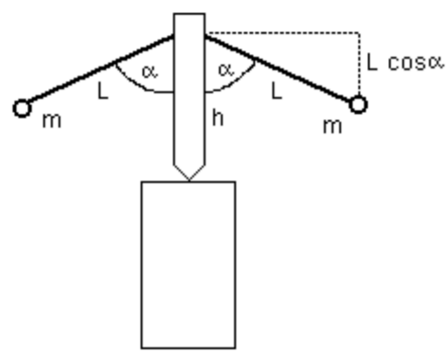

Figure 149:

It is an extremely stable device!

Let us look at its stability during rocking motion.

Suppose the teeter toy is tipped at angle θ as shown in figure 150.

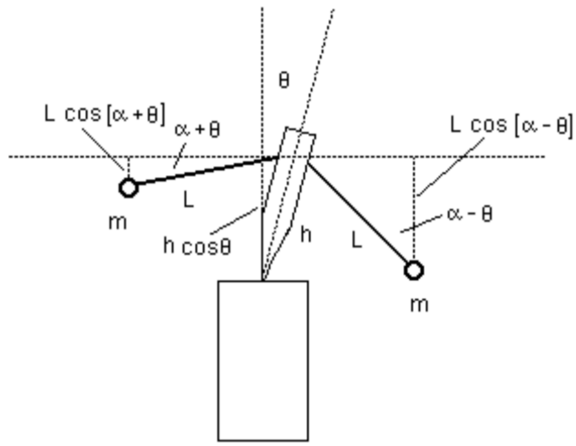

Figure 150:

Let the gravitational potential energy be = 0 at the pivot (remember it is arbitrary). The potential energy(massless center piece) is then

$$U(\theta) = mg(h\cos\theta - L\cos(\alpha + \theta) + mg(h\cos\theta - L\cos(\alpha - \theta)$$

which becomes

$$U(\theta) = 2mg\cos\theta(h - L\cos\alpha)$$

Equilibrium occurs when

$$\frac{dU(\theta)}{d\theta} = -2mg\sin\theta(h - L\cos\alpha) = 0$$

The solution, of course, is $\theta = 0$. What about stability?

$$\frac{d^2U(\theta)}{d\theta^2} = -2mg\cos\theta(h - L\cos\alpha)$$

This must be greater than zero at equilibrium ($\theta = 0$) for stability, which requires that

$$h - L\cos\alpha < 0 \rightarrow h < L\cos\alpha$$

This means that in order for the toy to be stable the weights must be **below** the pivot point.

Energy Diagrams

Potential energy diagrams enable us to understand, qualitatively, the possible motions of a system.

Consider the potential energy diagram in figure 151(for a mass attached to a spring):

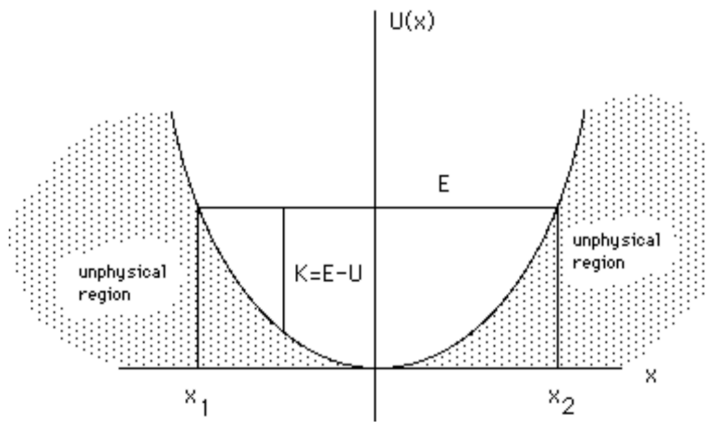

Figure 151:

The way we use these diagrams is as follows:

(1) draw a horizontal line representing the energy, which is constant during the motion

(2) the vertical distance between the energy line and the potential energy curve = the kinetic energy

(3) any region where we would have $K < 0$ is unphysical, which means the particle cannot be in that region

(4) any point where $E = U$ is called a turning point of the motion; $K = 0$ at these points

Class discussion of motion for $E > 0$ and $E = 0$.

(1) $E > 0$: bounded motion between two turning points - SHM

(2) $E = 0$: stable equilibrium - SHM for small oscillations

Now consider this potential energy diagram in figure 152:

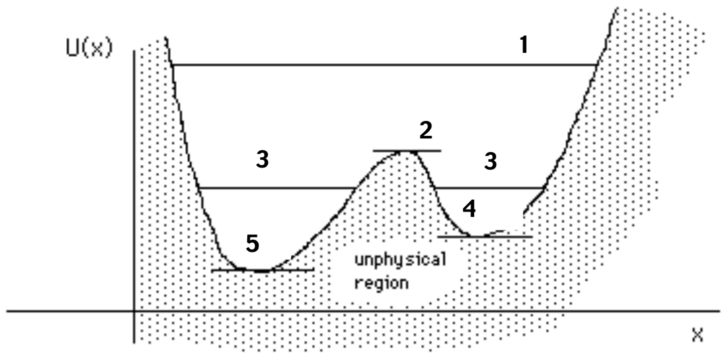

Figure 152:

Class discussion of motion for 5 indicated E-values:

(1) bounded motion between two turning points - not SHM

(2) unstable equilibrium

(3) bounded motion between two turning points - not SHM - 2 distinct regions

(4) stable equilibrium - SHM for small oscillations

(5) stable equilibrium - SHM for small oscillations

Now consider the potential energy diagram in figure 153:

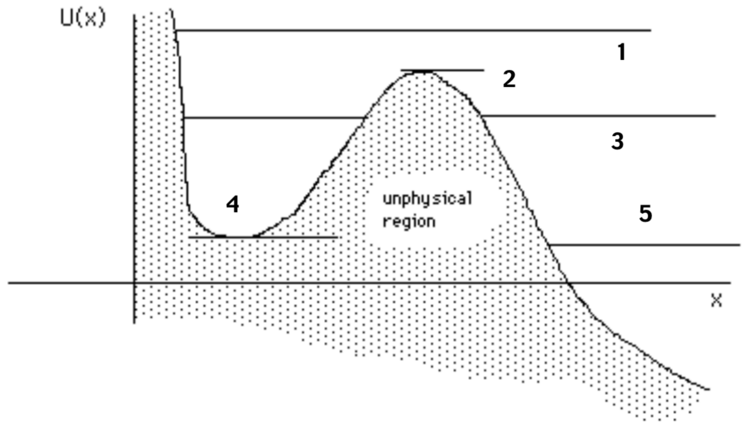

Figure 153:

Class discussion of motion for 5 indicated E-values:

(1) unbounded motion with one turning point

(2) unstable equilibrium

(3) bounded motion with two turning points - not SHM; unbounded motion with one turning point

(4) stable equilibrium - SHM for small oscillations; unbounded motion with one turning point

(5) unbounded motion with one turning point

Small Oscillations in a Bound System

A very interesting feature exists for **all** potential energy curves that have a minimum point (equilibrium point).

First, any well-behaved function $f(x)$ can be expanded in a series (called a Taylor series) about an arbitrary point x_0 in the form

$$f(x) = f(x_0) + (x - x_0) \left. \frac{df(x)}{dx} \right|_{x=x_0} + \frac{1}{2}(x - x_0)^2 \left. \frac{d^2 f(x)}{dx^2} \right|_{x=x_0} + \cdots \tag{6.38}$$

In particular if we are close to the equilibrium point $x = x_0$ potential energy function $U(x)$ we can write

$$U(x) = U(x_0) + (x - x_0)\,\frac{dU(x)}{dx}\bigg|_{x=x_0} + \frac{1}{2}(x - x_0)^2\,\frac{d^2U(x)}{dx^2}\bigg|_{x=x_0} + \cdots \qquad (6.39)$$

But since $\frac{dU(x)}{dx}\Big|_{x=x_0}$ by the definition of an equilibrium point and since the reference level is arbitrary allowing us to choose $U(x_0) = 0$, we always can write near equilibrium

$$U(x) = \frac{1}{2}(x - x_0)^2\,\frac{d^2U(x)}{dx^2}\bigg|_{x=x_0} = \frac{1}{2}k(x - x_0)^2 \qquad (6.40)$$

where

$$k = \frac{d^2U(x)}{dx^2}\bigg|_{x=x_0} \qquad (6.41)$$

This means that near an equilibrium point any potential energy function looks like the potential energy function of a simple harmonic oscillator (spring) and hence for small displacements from equilibrium any body will undergo SHM with frequency given by

$$\omega^2 = \frac{k}{m} = \frac{1}{m}\,\frac{d^2U(x)}{dx^2}\bigg|_{x=x_0} \qquad (6.42)$$

Example - Back to the teeter totter toy .....

When we displace the toy from equilibrium by angle θ we found earlier that the potential energy was given by

$$U(\theta) = -A\cos\theta \quad , \quad A = 2mg(L\cos\alpha - h)$$

Let us expand $U(\theta)$ about the equilibrium position $\theta_0 = 0$ We get

$$\frac{dU}{d\theta}\bigg|_0 = 0 \quad \text{(as we should)}$$

$$\frac{d^2U}{d\theta^2}\bigg|_0 = A$$

which gives

$$U(\theta) = \frac{1}{2}A\theta^2$$

and thus the equation of motion is

$$\theta t = C_1 \cos \omega t + C_2 \sin \omega t$$

where

$$\omega = \sqrt{\frac{A}{m}}$$

Example

Choose

$$U(x) = x - 2x^2 + x^3$$

Then

$$\frac{dU}{dx} = 1 - 4x + 3x^2 = 0 \to x = 1, \frac{1}{3}$$

$$\frac{d^U}{dx^2} = -4 + 6x = \begin{cases} -2 < 0 & \to \text{ unstable } (x = 1/3) \\ +2 > 0 & \to \text{ stable } (x = 1) \end{cases}$$

This looks like figure 154:

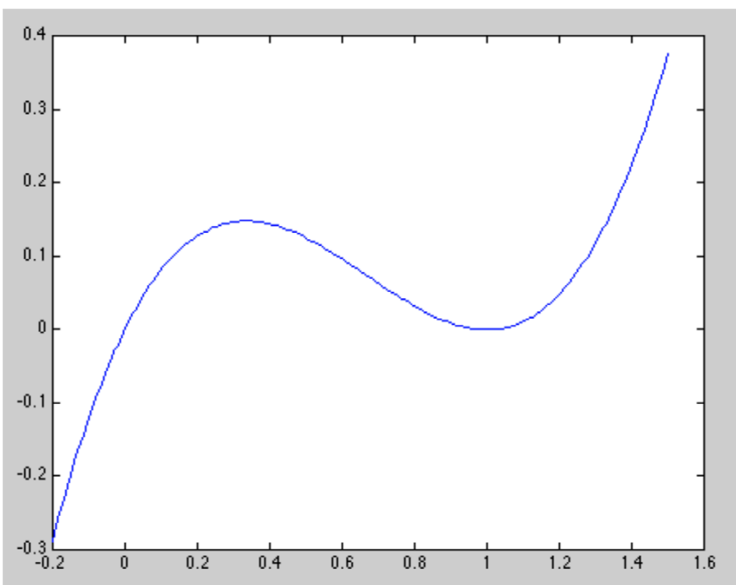

Figure 154:

Near $x = 1$ we can write

$$U(x) = \frac{1}{2}(x-1)^2 \left.\frac{d^2U(x)}{dx^2}\right|_{x=1} = \frac{1}{2}(x-1)^2(2) = (x-1)^2 = \frac{1}{2}k(x-1)^2$$

so $k = 2$ and we would have simple harmonic motion with frequency

$$\omega = \sqrt{\frac{k}{m}}$$

Example

Mass m whirls on a frictionless table, held to circular motion by a string which passes through a hole in the table. See figure 155

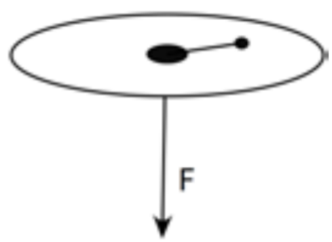

Figure 155:

The string is slowly pulled through the hole so that the radius of the circle changes from ℓ_1 to ℓ_2.

Show that the work done in pulling the string equals the increase in kinetic energy of the mass.

We have

$$\vec{F} = m(\ddot{r} - r\dot{\theta}^2)\hat{r} + m(2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} = -F\hat{r}$$
$$-F = m(\ddot{r} - r\dot{\theta}^2)$$
$$0 = m(2\dot{r}\dot{\theta} + r\ddot{\theta}$$

which gives

$$\frac{\ddot{\theta}}{\dot{\theta}} = \frac{\dot{\omega}}{\omega} = -2\frac{\dot{r}}{r} \to \frac{d\omega}{\omega} = -2\frac{dr}{r}$$
$$\log\omega = -2\log r + \log c$$
$$\omega = \dot{\theta} = \frac{\text{constant}}{r^2} = \frac{\ell_1^2\omega_1}{r^2}$$

where $r(t_1) = \ell_1$, $\dot{\theta}(t_1) = \omega - 1$. We then have

$$W = \int \vec{F}\cdot d\vec{r} = m\int_{t-1}^{t_2}\left(\ddot{r} - \frac{\ell_1^4\omega_1^2}{r^4}\right)dr$$
$$= \frac{1}{2}mv_r^2(t_2) - \frac{1}{2}mv_r^2(t_1) + m\ell_1^4\omega_1^2\left(\frac{1}{2\ell_2^2} - \frac{1}{2\ell_1^2}\right)$$

where we have used

$$\int_{t-1}^{t_2}\ddot{r}dr = \int_{t-1}^{t_2}\frac{d^2r}{dt^2}dr = \int_{t-1}^{t_2}\frac{d}{dt}\left(\frac{dr}{dt}\right)dr = \int_{t-1}^{t_2}\frac{dr}{dt}\frac{d}{dr}\left(\frac{dr}{dt}\right)dr = \int_{t-1}^{t_2}v_r dv_r$$

We then have

$$W = \left(\frac{1}{2}mv_r^2(t_2) + \frac{m\ell_1^4\omega_1^2}{2\ell_2^2}\right) - \left(\frac{1}{2}mv_r^2(t_1) + \frac{m\ell_1^4\omega_1^2}{2\ell_1^2}\right)$$

Now

$$\frac{m\ell_1^4\omega_1^2}{2\ell_2^2} = \frac{m\ell_1^4\omega_1^2}{2\ell_1^2}$$

so that

$$W = \left(\frac{1}{2}mv_r^2(t_2) + \frac{1}{2}m\ell_2^2\omega_2^2\right) - \left(\frac{1}{2}mv_r^2(t_1) + \frac{1}{2}m\ell_1^2\omega_1^2\right) = K(t_2) - K(t_1)$$

Conservation Laws and Particle Collisions

Particle scattering experiments of one sort or another have produced much of the knowledge we have about atoms, molecules, nuclei, solids and elementary particles. Determining the detailed dynamics of a scattering experiment depends on knowing the potential energy function (or force law) between the particle involved in the scattering.

It turns out, however, that we can learn a great deal about the kinematics of a scattering experiment just by using conservation of energy and momentum.

Suppose we consider the collision or scattering experiment shown in figure 156:

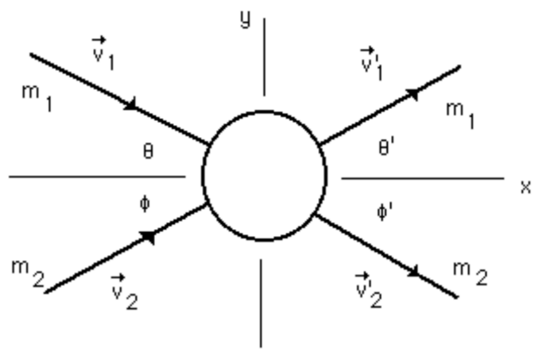

Figure 156:

Experimentally we will usually know the initial velocity vectors $\vec{v}_1$ and $\vec{v}_2$. Often one of the initial particles is at rest and then we usually choose $\theta = 0$.

External forces are usually $= 0$ (the dynamics comes from internal forces), This means that linear momentum is always conserved and we can write

$$\vec{P}_{initial} = \vec{P}_{final} \tag{6.43}$$

$$m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{v}_1' + m_2\vec{v}_2' \tag{6.44}$$

for a two-particle collision.

This represents 2 equations for a 2-dimensional collision (all particles remain in a single plane).

There are 4 unknowns, however, the x- and y-components of the final velocities.

Energy considerations usually provide another equation, so we are short one equation necessary for the solution of the problem.

In general, however, we will also know something about the final state, i.e., in which direction one of the final particles emerges.

This reduces the number of unknowns to 3 and allows us to solve the problem.

Elastic and Inelastic Collisions

We define two types of collisions:

elastic - total kinetic energy is conserved in the collision

inelastic - total kinetic energy is not conserved (heat generated by deformations, etc)

Both types of collisions can be covered by writing the kinetic energy conservation relation as

$$K_{initial} = K_{final} + Q \tag{6.45}$$

where Q = amount of kinetic energy converted into another form of energy.

Inelastic collisions are the result of nonconservative forces such as friction, deformation, etc and are signaled by a heat transfer occurring.

The conservation of kinetic energy relation is then

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2 + Q \tag{6.46}$$

We have

$Q > 0 \Rightarrow$ kinetic energy is lost in collision(as heat)

$Q < 0 \Rightarrow$ internal energy converted to kinetic energy

$Q = 0 \Rightarrow$ elastic collision

Collisions in 1-Dimension

In 1 dimensional collisions, particles are constrained to move along a line (say the x-axis).

Our equations in the laboratory frame become:

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \tag{6.47}$$

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2 + Q \tag{6.48}$$

and the configuration of particles before and after the collision is shown figure 157:

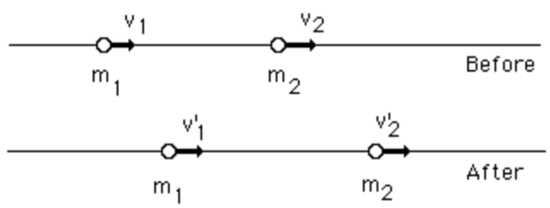

Figure 157:

We have 2 equations in 2 unknowns and they can be solved with **no extra information** about the final state.

Example

Suppose $m_2 = 3m_1$ and $v_1 = v = -v_2$ (they approaching each other before the collision).

Also suppose the collision is elastic.

We have the configurations shown in figure 158.

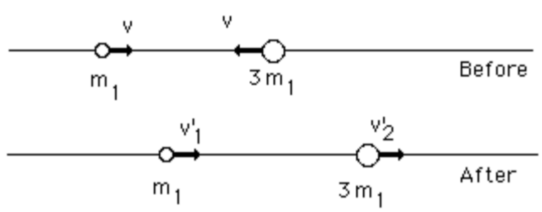

Figure 158:

We then have the equations

$$m_1 v - 3m_1 v = m_1 v_1' + 3m_1 v_2'$$
$$\frac{1}{2}m_1 v^2 + \frac{3}{2}m_1 v^2 = \frac{1}{2}m_1 v_1'^2 + \frac{3}{2}m_2 v_2'^2$$

We then obtain:

$$v_1' = -2v - 3v_2' \to 4v^2 = (-2v - 3v_2')^2 + 3v_2'^2 = 4v^2 + 12vv_2' + 12v_2'^2$$
$$0 = 12vv_2' + 12v_2'^2 = 12v_2'(v + v_2')$$

We have 2 solutions

#1 (nothing happens ... this solution will always be there)

$$v'1 = v \quad , \quad v'_2 = -v$$

which is **unphysical** and we ignore it.

#2

$$v'1 = -2v \quad , \quad v'2 = 0$$

After the collision, the smaller mass is moving to the left with twice its original speed and the larger mass is at rest.

Collisions as seen in the CM Frame

Let us redo the above problem in the CM frame.

In this 1 dimensional situation we have

$$V_{CM} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = \frac{mv - 3mv}{4m} = -\frac{v}{2}$$

$$v_{1,CM} = v_1 - V_{CM} = v - \left(-\frac{v}{2}\right) = \frac{3v}{2}$$

$$v_{2,CM} = v_2 - V_{CM} = -v - \left(-\frac{v}{2}\right) = -\frac{v}{2}$$

as shown in figure 159.

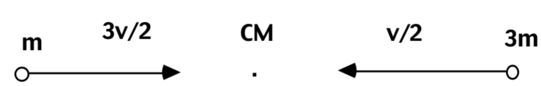

Figure 159:

Now the total momentum = 0 in the CM frame, hence

$$m_1 v'_{1,c} + m_2 v'_{2,c} = 0 \to v'_{2,c} = -\frac{1}{3} v'_{1,c}$$

That corresponds to momentum conservation.

Conservation of energy says

$$\frac{1}{2} m_1 \left(\frac{3v}{2}\right)^2 + \frac{1}{2} m_1 \left(-\frac{v}{2}\right)^2 = \frac{1}{2} m_1 v'^2_{1,c} + \frac{1}{2} m_1 \left(-\frac{1}{3} v'_{1,c}\right)^2 \to 3v^2 = \frac{4}{3} v'^2 1, c$$

$$v'_{1,c} = \frac{3}{2}v \quad , \quad v'_{2,c}\frac{1}{2}v$$

which gives

$$v'_{1,CM} + V_{CM} = v'_1 = -\frac{3v}{2} - \frac{v}{2} = -2v$$

$$v'_{2,CM} + V_{CM} = v'_2 = \frac{v}{2} - \frac{v}{2} = 0$$

as shown in figure 160.

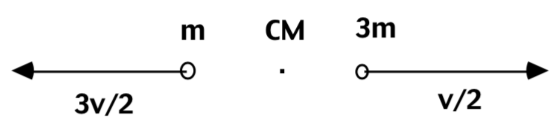

Figure 160:

Same result as earlier, but no complicated equations to solve!!!!

Note the very important result that we find ..... the particles **only reverse direction** in the CM and do not change their speed (magnitude of the velocities) in an elastic collision.

With that new piece of knowledge the solution is almost trivial ..... we do not need to use energy conservation at all in the CM frame.

$$V_{CM} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = \frac{mv - 3mv}{4m} = -\frac{v}{2}$$

$$v_{1,CM} = v_1 - V_{CM} = v - \left(-\frac{v}{2}\right) = \frac{3v}{2}$$

$$v_{2,CM} = v_2 - V_{CM} = -v - \left(-\frac{v}{2}\right) = -\frac{v}{2}$$

$$v'_{1,CM} = -v_{1,c} = -\frac{3v}{2}$$

$$v'_{2,CM} = -v_{2,c} = \frac{v}{2}$$

(the last step is equivalent to kinetic energy conservation)

$$v'_{1,CM} + V_{CM} = v'_1 = -\frac{3v}{2} - \frac{v}{2} = -2v$$

$$v'_{2,CM} + V_{CM} = v'_2 = \frac{v}{2} - \frac{v}{2} = 0$$

That is a very powerful result!!!!

Does this result generalize to more than 1 dimension. Yes!! Let us see how

$$m_1\vec{v}_{1,c} + m_2\vec{v}_{2,c} = m_1\vec{v}'_{1,c} + m_2\vec{v}'_{2,c} = 0 \quad \text{total momentum} = 0 \text{ in CM}$$

$$m_1\vec{v}_{1,c} = -m_2\vec{v}_{2,c} \quad , \quad m_1\vec{v}'_{1,c} = -m_2\vec{v}'_{2,c}$$

$$m_1 v_{1,c}^2 + m_2 v_{2,c}^2 = m_1 v'^2_{1,c} + m_2 v'^2_{2,c} \quad K \text{ conserved in elastic collision}$$

Combining these results we get

$$(m_1 + m_1^2/m_2)v_{1,c}^2 = (m_1 + m_1^2/m_2)v'^2_{1,c} \rightarrow v_{1,c} = -v'_{1,c}$$

$$(m_2 + m_1^2/m_1)v_{2,c}^2 = (m_2 + m_2^2/m_1)v'^2_{2,c} \rightarrow v_{2,c} = -v'_{2,c}$$

and the result clearly is **general**. This **does not determine the direction in the CM**. All we know is that they are **opposite** to each other.

A final example

A particle of mass m and initial velocity v_0 collides with a particle of unknown mass M coming from the opposite direction as shown in figure 161.

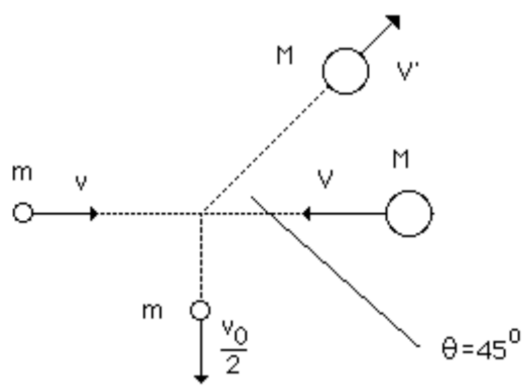

Figure 161:

After the collision, m has velocity $v_0/2$ at right angles to the incident(original) direction and M moves off in the direction shown in figure 161.

Find the ratio M/m.

Using the Laboratory Frame

$$x\text{-momentum: } mv_0 - MV = MV'\cos\theta = \frac{MV'}{\sqrt{2}}$$

$$y\text{-momentum: } 0 = MV'\sin\theta - m\frac{v_0}{2} = \frac{MV'}{\sqrt{2} - m\frac{v_0}{2}}$$

$$\text{Energy: } \frac{1}{2}mv_0^2 + \frac{1}{2}MV^2 = \frac{1}{2}m\left(\frac{v_0}{2}\right)^2 + \frac{1}{2}MV'^2$$

$$mv_0 - MV = m\frac{v_0}{2} \to V = \frac{1}{2}\frac{m}{M}v_0$$

$$V' = \frac{1}{\sqrt{2}}\frac{m}{M}v_0$$

$$v_0^2 + \frac{M}{m}\left(\frac{1}{2}\frac{m}{M}v_0\right)^2 = \left(\frac{v_0}{2}\right)^2 + \frac{M}{m}\left(\frac{1}{\sqrt{2}}\frac{m}{M}v_0\right)^2 \to \frac{M}{m} = \frac{1}{3}$$

7. An Alternative Path - The Lagrangian Method

Let us restrict our attention to one-dimensional systems. We then have

$$E = K + V \tag{7.1}$$

where

$$\begin{aligned} K &= \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2 = \text{ kinetic energy} \\ V &= V(x) = \text{ potential energy} \end{aligned} \tag{7.2}$$

Now Newton's second law takes the form

$$F_x = -\frac{\partial V(x)}{\partial x} = ma_x = m\ddot{x} \tag{7.3}$$

Some algebra gives

$$m\ddot{x} = \frac{d}{dt}(m\dot{x}) = \frac{d}{dt}\left(\frac{\partial}{\partial\dot{x}}\right)\left(\frac{1}{2}m\dot{x}^2\right) = \frac{d}{dt}\left(\frac{\partial K}{\partial\dot{x}}\right) \tag{7.4}$$

Combining everything we then have

$$-\frac{\partial V}{\partial x} = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}}\right) \tag{7.5}$$

Assuming that

$$\begin{aligned} K = K(\dot{x}) &\to \frac{\partial K}{\partial x} = 0 \\ V = V(x) &\to \frac{\partial V}{\partial \dot{x}} = 0 \end{aligned} \tag{7.6}$$

we have

$$\frac{\partial(K-V)}{\partial x} = \frac{d}{dt}\left(\frac{\partial(K-V)}{\partial \dot{x}}\right) \tag{7.7}$$

or

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} \tag{7.8}$$

where we define

$$L = K - V = \text{ Lagrangian} \tag{7.9}$$

and equation (7.8) is called Lagrange's equation. It is important to realize that I have not introduced any new physical content into the theory at this point.

I have only found an alternative way to write Newton's second law.

Examples

(1) For an object falling in a gravitational field, we have

$$\begin{aligned} K &= \frac{1}{2}m\dot{y}^2 \\ V &= mgy \\ L = K - V &= \frac{1}{2}m\dot{y}^2 - mgy \end{aligned} \tag{7.10}$$

We then have

$$\begin{aligned} \frac{\partial L}{\partial \dot{y}} &= m\dot{y} \to \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = m\ddot{y} \\ \frac{\partial L}{\partial y} &= -mg \end{aligned} \tag{7.11}$$

so that the equation of motion is

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \\
&m\ddot{y} = -mg \rightarrow \ddot{y} = -g
\end{aligned}
\tag{7.12}
$$

which agrees with our earlier result.

(2) For an object attached to a spring, we have

$$
\begin{aligned}
K &= \frac{1}{2}m\dot{x}^2 \\
V &= \frac{1}{2}kx^2 \\
L &= K - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\dot{x}^2
\end{aligned}
\tag{7.13}
$$

We then have

$$
\begin{aligned}
&\frac{\partial L}{\partial \dot{x}} = m\dot{x} \rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{yx}}\right) = m\ddot{x} \\
&\frac{\partial L}{\partial x} = -kx
\end{aligned}
\tag{7.14}
$$

so that the equation of motion is

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \\
&m\ddot{x} = -kx \rightarrow \ddot{x} = -\frac{k}{m}x = -\omega_0^2 x
\end{aligned}
\tag{7.15}
$$

which agrees with our earlier result.

(3) For an object attached to a string (simple pendulum) (see figure 162)

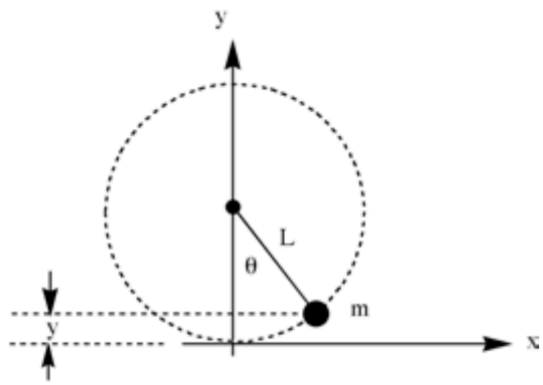

Figure 162:

we have

$$\begin{aligned} K &= \frac{1}{2} m L^2 \dot{\theta}^2 \\ V &= mgy = mgL(1 - \cos\theta) \\ L &= K - V = \frac{1}{2} m L^2 \dot{\theta}^2 - mgL(1 - \cos\theta) \end{aligned} \tag{7.16}$$

We then have

$$\begin{aligned} \frac{\partial L}{\partial \dot{\theta}} &= mL^2 \dot{\theta} \to \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mL^2 \ddot{\theta} \\ \frac{\partial L}{\partial \theta} &= -mgL \sin\theta \end{aligned} \tag{7.17}$$

so that the equation of motion is

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= 0 \\ mL^2 \ddot{\theta} = -mgL \sin\theta &\to \ddot{\theta} + \frac{g}{L} \sin\theta = 0 \end{aligned} \tag{7.18}$$

For small angles we have $\sin\theta \approx \theta$, so that (7.17) reduces to

$$\ddot{\theta} + \frac{g}{L}\theta = \ddot{\theta} + \omega_0^2 \theta = 0 \tag{7.19}$$

which agrees with our earlier result.

(4) Now consider the sliding blocks shown in figure 163:

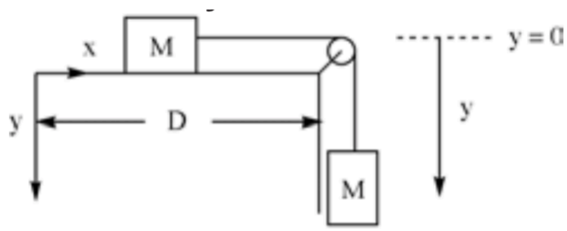

Figure 163:

We have the following constraint:

$$\begin{aligned} \ell &= \text{ length of string } = (D - x) + y \\ \rightarrow \dot{\ell} &= 0 = -\dot{x} + \dot{y} \rightarrow \dot{x} = \dot{y} \end{aligned} \tag{7.20}$$

If we ignore the string mass (first approximation) we have

$$K = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}M\dot{y}^2 = M\dot{y}^2$$

$$V = -Mgy \tag{7.21}$$

$$L = K - V = M\dot{y}^2 + Mgy \tag{7.22}$$

We then have

$$\begin{aligned} \frac{\partial L}{\partial \dot{y}} &= 2M\dot{y} \rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 2M\ddot{y} \\ \frac{\partial L}{\partial y} &= Mg \end{aligned} \tag{7.23}$$

so that the equation of motion is

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} &= 0 \\ 2M\ddot{y} = Mg \rightarrow \ddot{y} &= \frac{g}{2} \end{aligned} \tag{7.24}$$

which has the solution

$$y(t) = \frac{g}{4}t^2 \tag{7.25}$$

which agrees with an earlier result if we remember that y is increasing downwards.

Now what happens if the string has a mass m. We then have

$$\begin{aligned} K &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}M\dot{y}^2 + \frac{1}{2}m\dot{y}^2 = \left(M + \frac{1}{2}m\right)\dot{y}^2 \\ V &= -Mgy + U_{string} == Mgy - g\int_0^y ydm \\ &= -Mgy - \int_0^y y\frac{m}{\ell}dy = -Mgy - \frac{mg}{2\ell}y^2 \\ L &= K - V = \left(M + \frac{1}{2}m\right)\dot{y}^2 + Mgy + \frac{mg}{2\ell}y^2 \end{aligned} \tag{7.26}$$

We then have

$$\begin{aligned} &\frac{\partial L}{\partial \dot{y}} = (2M + m)\dot{y} \to \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = (2M + m)\ddot{y} \\ &\frac{\partial L}{\partial y} = Mg + \frac{mg}{\ell}y \end{aligned} \tag{7.27}$$

so that the equation of motion now becomes

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \\ &(2M + m)\ddot{y} = Mg + \frac{mg}{\ell}y \to \ddot{y} = \frac{M}{2M + m}g + \frac{mg}{(2M + m)\ell}y \end{aligned} \tag{7.28}$$

which has the solution

$$y(t) = \frac{M}{m}\ell(1 - \cosh{(\gamma t)}) \quad , \quad \gamma = \sqrt{\frac{mg}{(2M + m)\ell}} \tag{7.29}$$

Is this the same result as earlier?

Consider the limit as $m \to 0$. We have

$$\begin{aligned} y(t) &= \lim_{m\to 0}\frac{M}{m}\ell\left(1 - \cosh\sqrt{\frac{mg}{(2M + m)\ell}}t\right) \\ &= \lim_{m\to 0}\frac{M}{m}\ell\left(1 - \frac{1}{2}\left(e^{\sqrt{\frac{mg}{(2M+m)\ell}}t} + e^{-\sqrt{\frac{mg}{(2M+m)\ell}}t}\right)\right) \\ &= \lim_{m\to 0}\frac{M}{m}\ell\left(1 - \frac{1}{2}\left(1 + \sqrt{\frac{mg}{(2M + m)\ell}}t + \frac{1}{2}\frac{mg}{(2M + m)\ell}t^2\right.\right. \\ &\quad \left.\left. +1 - \sqrt{\frac{mg}{(2M + m)\ell}}t + \frac{1}{2}\frac{mg}{(2M + m)\ell}t^2\right)\right) \end{aligned}$$

or

$$y(t) = \lim_{m\to 0} \frac{M}{m}\ell\frac{mg}{2(2M_m)\ell}t^2 = \frac{g}{4}t^2 \tag{7.30}$$

which is the same as earlier in (7.25).

(5) Now let us modify this pendulum problem and turn the string into a spring of spring constant$k$ and unstretched length b. See figure 164.

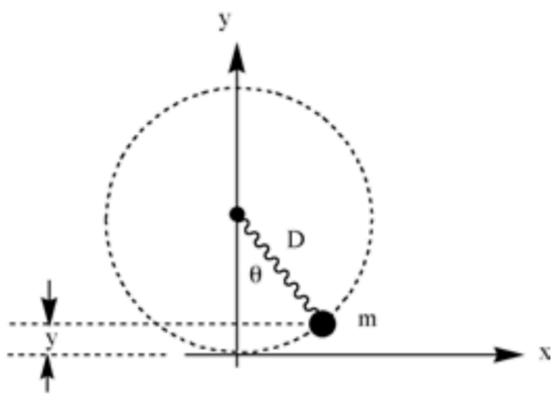

Figure 164:

Now L is the variable length of the spring.

We have

$$\begin{aligned} K &= \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{1}{2}m(\dot{D}^2 + D^2\dot{\theta}^2) \\ V &= mgy = mgD(1-\cos\theta) + \frac{1}{2}k(D-b)^2 \\ L &= K - V = \frac{1}{2}m(\dot{D}^2 + D^2\dot{\theta}^2) - mgD(1-\cos\theta) - \frac{1}{2}k(D-b)^2 \end{aligned} \tag{7.31}$$

We then have one equation for each variable(the two-dimensional problem generates two equations).

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}} &= mL^2\dot{\theta} \rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mL^2\ddot{\theta} \\
\frac{\partial L}{\partial \theta} &= -mgL\sin\theta \\
\frac{\partial L}{\partial \dot{D}} &= m\dot{D} \rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{D}}\right) = m\ddot{D} \\
\frac{\partial L}{\partial D} &= -mg(1-\cos\theta) - k(D-b)
\end{aligned}
\tag{7.32}
$$

so that the equations of motion are

$$
\begin{aligned}
&= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \\
&= mL^2\ddot{\theta} = -mgL\sin\theta \rightarrow \ddot{\theta} + \frac{g}{L}\sin\theta = 0 \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{D}}\right) - \frac{\partial L}{\partial D} = 0 \\
&= m\ddot{D} = -mg(1-\cos\theta) - k(D-b) + mD\dot{\theta}^2 \\
&= \rightarrow \ddot{D} + \frac{k}{m}(D-b) - D\dot{\theta}^2 + g(1-\cos\theta) = 0
\end{aligned}
\tag{7.33}
$$

Clearly this is a much more complicated problem (further study left for more advanced courses).

In both of the last examples, you could have solved the problems using force diagrams without the new methods.

What happens, however, in the case shown in figure 165?

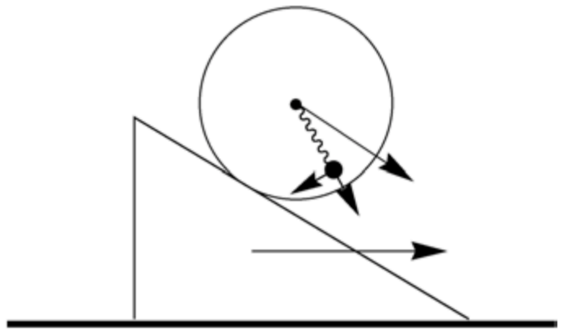

Figure 165:

where we have a cylinder with an attached springy pendulum rolling down an inclined plane that can slide along the table with all the motions shown allowed! You will do this one in more advanced courses.

Doing an old problem: See figure 166.

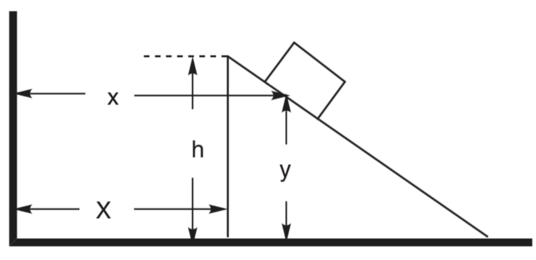

Figure 166:

Contraint equation

$$x - X = (h - y)\cot\theta \to \dot{x} - \dot{X} = -\dot{y}\cot\theta \to \ddot{x} - \ddot{X} = -\ddot{y}\cot\theta$$

Energies and Lagrangian

$$K = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$V = mgy$$

$$L = K - V = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

$$\dot{X} = \dot{x} + \dot{y}\cot\theta$$

Lagrangian rewritten

$$L = \frac{1}{2}(m+M)\dot{x}^2 + M\cot\theta\,\dot{x}\dot{y} + \frac{1}{2}(m + M\cot\theta)\dot{y}^2 - mgy$$

Equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 = (m+M)\ddot{x} + M\cot\theta\,\ddot{y}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{xy}}\right) - \frac{\partial L}{\partial y} = 0 = (m + M\cot\theta)\ddot{y} + M\cot\theta\,\ddot{x} + mg$$

so that

$$(m + M\cot\theta)\left(-\frac{m+M}{M}\tan\theta\,\ddot{x}\right) + M\cot\theta\,\ddot{x} = -mg$$

Letting

$$\theta = 45^\circ \quad , \quad M = 5m$$

we get

$$\ddot{x} = \frac{5}{11}g \to \ddot{y} = -\frac{6}{11}g \to \ddot{X} = -\frac{1}{11}g$$

8. Oscillations

Part 1

Simple Harmonic Motion

If any system characterized by a physical variable Q can be described by an equation of motion

$$\ddot{Q} + \omega^2 Q = 0 \tag{8.1}$$

then the system is said to exhibit SHM and the solution is found using complex exponential substitution.

Complex Exponential Substitution

Digression: Second-Order diffEQs

Although in various special cases we will be able to convert many of the 2nd order diffEQs that arise from the application of Newton's 2nd law to 1st order diffEQs and thus use simpler solution methods, in many cases we will need to solve the 2nd order equation directly.

The solution technique we will use in most cases is called **exponential substitution**.

Exponential Substitution

This method is applicable to all 2nd-order differential equations of the form

$$A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = 0 \tag{8.2}$$

where A, B, and C are constants. The SHM equation has this form

$$M\frac{dv}{dt} = -kx \rightarrow M\frac{d}{dt}\left(\frac{dx}{dt}\right) = -kx \rightarrow M\frac{d^2x}{dt^2} + kx = 0 \tag{8.3}$$

so that $A = M$, $C = k$ and $B = 0$.

The Method: Consider a typical equation of the form

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0 \tag{8.4}$$

We make the exponential substitution

$$y = e^{\alpha t} \tag{8.5}$$

into the diffEQ. This will convert the diffEQ into an **algebraic equation** for α.

We have

$$\frac{d^2y}{dt^2} = \frac{d^2e^{\alpha t}}{dt^2} = \alpha\frac{de^{\alpha t}}{dt} = \alpha^2 e^{\alpha t} \tag{8.6}$$

$$\frac{dy}{dt} = \frac{de^{\alpha t}}{dt} = \alpha e^{\alpha t} \tag{8.7}$$

$$y = e^{\alpha t} \tag{8.8}$$

which, upon substitution, gives the result

$$(\alpha^2 + 3\alpha + 2)e^{\alpha t} = 0 \rightarrow \alpha^2 + 3\alpha + 2 = 0 \tag{8.9}$$

since $e^{\alpha t} \neq 0$.

The solutions of this algebraic equation tell us the **allowed** values of α that give valid solutions to the diffEQ.

In particular, in this case we get

$$\alpha = -1, -2 \tag{8.10}$$

as solutions to the quadratic equation.

This result means that $y = e^{-t}$ and $y = e^{-2t}$ satisfy the original diffEQ.

If there is more than one allowed value of α (as in this case), then the most general solution will be a linear combination of all possible solutions (because this is a linear diffEQ, that is, all derivative and functions enter in the equation to the first-power).

Thus, the most general solution of the diffEQ is

$$y(t) = ae^{-t} + be^{-2t} \tag{8.11}$$

where a and b are constants to be determined by the **initial conditions**.

The number of arbitrary constants that need to be determined by the initial conditions is equal to the order (highest derivative → 2 in this case) of the diffEQ.

Suppose the initial conditions are $y = 0$ and $dy/dt = 1$ at $t = 0$. Then we have

$$\begin{aligned} y(t) &= ae^{-t} + be^{-2t} \\ y(0) &= 0 = a + b \\ \frac{dy}{dt} &= -ae^{-t} - 2be^{-2t} \\ \frac{dy}{dt}(0) &= -a - 2b - 1 \end{aligned}$$

which gives $a = -b = 1$ and

$$y(t) = e^{-t} - e^{-2t} \tag{8.12}$$

Substitute this solution into the original equation and see that it works and has the correct initial conditions!!

Although this method is very powerful as described, we can make it even more powerful by defining a new mathematical quantity called the **complex exponential**.

This will allow us to use the method for the SHM case.

Complex Exponentials - Alternative Very Powerful Method

Remember our discussion earlier in course about power series expansions of a function around some point

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \tag{8.13}$$

the point being expanded about in this case is $x = 0$ in this case.

If we apply this to the exponential function we get

$$f(x) = e^{\alpha x}$$

$$f^{(0)}(0) = f(0) = 1$$

$$f^{(1)}(0) = \left.\frac{df}{dx}\right|_{x=0} = \left.\alpha e^{\alpha x}\right|_{x=0} = \alpha$$

$$f^{(2)}(0) = \left.\frac{d^2 f}{dx^2}\right|_{x=0} = \left.\alpha^2 e^{\alpha x}\right|_{x=0} = \alpha^2$$

and so on ....

Therefore, we get

$$e^{\alpha x} 1 + \alpha x + \frac{\alpha^2}{2!} x^2 + \frac{\alpha^3}{3!} x^3 + \frac{\alpha^4}{4!} x^4 + \cdots = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} x^n \tag{8.14}$$

If we apply this to the sine and cosine functions in the same mannerwe get (you should do this for at least one of these functions)

$$\sin \alpha x = \sum_{n=0}^{\infty} (-1)^n \frac{\alpha^{2n+1}}{(2n+1)!} x^{2n+1} = \alpha x - \frac{\alpha^3}{3!} x^3 + \frac{\alpha^5}{5!} x^5 + \cdots \tag{8.15}$$

$$\cos\alpha x = \sum_{n=0}^{\infty}(-1)^n \frac{\alpha^{2n}}{(2n)!}x^{2n} = 1 - \frac{\alpha^2}{2!}x^2 + \frac{\alpha^4}{4!}x^4 + \cdots \tag{8.16}$$

We can then show the neat result that

$$e^{\pm i\alpha t} = \cos\alpha t \pm i\sin\alpha t \tag{8.17}$$

which will be very useful throughout the course. It is called Euler's formula.

Proof:

$$\begin{aligned} e^{i\alpha t} &= 1 + i\alpha t + \frac{(i\alpha t)^2}{2!} + \frac{(i\alpha t)^3}{3!} + \frac{(i\alpha t)^4}{4!} + \cdots \\ &= \left(1 - \frac{(\alpha t)^2}{2!} + \frac{(\alpha t)^4}{4!} - \cdots\right) + i\left(\alpha t - \frac{(\alpha t)^3}{3!} + \frac{(\alpha t)^5}{5!} - cdots\right) \\ &= \cos\alpha t + i\sin\alpha t \end{aligned}$$

and similarly

$$e^{-i\alpha t} = \cos\alpha t - i\sin\alpha t$$

From these results we can also derive the relations

$$\frac{e^{i\alpha t} - e^{-i\alpha t}}{2i} = \frac{\cos\alpha t + i\sin\alpha t - \cos\alpha t + i\sin\alpha t}{2i} = \sin\alpha t$$

$$\frac{e^{i\alpha t} + e^{-i\alpha t}}{2} = \frac{\cos\alpha t + i\sin\alpha t + \cos\alpha t - i\sin\alpha t}{2} = \cos\alpha t$$

Finally, we use these results to solve the SHM equation.

$$M\frac{d^2y}{dt^2} + ky = 0 \to \frac{d^2y}{dt^2} + \omega^2 y = 0 \quad , \quad \omega^2 = \frac{k}{m}$$

using the exponential substitution method.

$$\frac{d^2y}{dt^2} + \omega^2 y = 0$$

Substituting $y = e^{\alpha t}$ we get the algebraic equation

$$\alpha^2 + \omega^2 = 0$$

which has solutions (allowed values of α) of

$$\alpha = \pm i\omega$$

so that the most general solution takes the form

$$y = Ae^{i\omega t} + Be^{-i\omega t}$$

Suppose now that the initial conditions are $y = y_0$ and $dy/dt = 0$ at $t = 0$.

Then we have

$$y(t) = Ae^{i\omega t} + Be^{-i\omega t}$$
$$y(0) = y_0 = A + B$$
$$\frac{dy}{dt} = i\omega Ae^{i\omega t} - i\omega Be^{-i\omega t}$$
$$\frac{dy}{dt}(0) = i\omega A - i\omega B = 0 \rightarrow A - B = 0$$

or

$$A = B = \frac{y_0}{2}$$

and

$$y(t) = y_0 \frac{e^{i\omega t} + e^{-i\omega t}}{2} = y_0 \cos \omega t$$

as we found earlier.

Alternatively, if the initial conditions are $y = 0$ and $dy/dt = v_0$ at $t = 0$ then we have

$$y(t) = Ae^{i\omega t} + Be^{-i\omega t}$$
$$y(0) = 0 = A + B$$
$$\frac{dy}{dt} = i\omega Ae^{i\omega t} - i\omega Be^{-i\omega t}$$
$$\frac{dy}{dt}(0) = i\omega A - i\omega B = v_0 \rightarrow A - B = \frac{v_0}{i\omega}$$

or

$$A = -B = \frac{v_0}{2i\omega}$$

and

$$y(t) = \frac{v_0}{\omega} \frac{e^{i\omega t} - e^{-i\omega t}}{2i} = \frac{v_0}{\omega} \sin \omega t$$

and in general we have for the initial conditions $y = y_0$ and $dy/dt = v_0$ at $t = 0$,

$$y(t) = Ae^{i\omega t} + Be^{-i\omega t}$$

$$y(0) = y_0 = A + B$$

$$\frac{dy}{dt} = i\omega A e^{i\omega t} - i\omega B e^{-i\omega t}$$

$$\frac{dy}{dt}(0) = i\omega A - i\omega B = v_0 \to A - B = \frac{v_0}{i\omega}$$

or

$$A = \frac{1}{2}\left(y_0 + \frac{v_0}{i\omega}\right) \quad , \quad B = \frac{1}{2}\left(y_0 - \frac{v_0}{i\omega}\right)$$

and

$$\begin{aligned} y(t) &= \frac{1}{2}\left(y_0 + \frac{v_0}{i\omega}\right)\frac{e^{i\omega t} + \frac{1}{2}\left(y_0 - \frac{v_0}{i\omega}\right)e^{-i\omega t}}{2i} \\ &= y_0 \frac{e^{i\omega t} + e^{-i\omega t}}{2} + \frac{v_0}{\omega}\frac{e^{i\omega t} - e^{-i\omega t}}{2i} \\ &= y_0 \cos\omega t + \frac{v_0}{\omega}\sin\omega t \end{aligned}$$

As stated above, the most general solution is

$$Q(t) = Ge^{i\alpha t} + He^{-i\alpha t} \tag{8.18}$$

where G and H are constants determined by the initial conditions.

Now we have:

$$e^{\pm i\alpha t} = \cos(\alpha t) \pm i\sin(\alpha t) \tag{8.19}$$

and

$$\sin(\alpha t) = \frac{e^{i\alpha t} - e^{-i\alpha t}}{2i} \quad , \quad \cos(\alpha t) = \frac{i\alpha t + e^{-i\alpha t}}{2} \tag{8.20}$$

With some algebra (see above) we can show that

$$\begin{aligned} Q(t) &= Ge^{i\alpha t} + He^{-i\alpha t} = G(\cos\omega t + i\sin\omega t) + H(\cos\omega t - i\sin\omega t) \\ &= C\cos\omega t + D\sin\omega t \end{aligned}$$

where $C = G + iH$ and $D = G - iH$.

Alternatively we can write this as

$$C = A\cos\phi \quad , \quad D = -A\sin\phi \tag{8.21}$$

or

$$A^2 = C^2 + D^2 \quad , \quad \tan\phi = -\frac{D}{C} \tag{8.22}$$

which gives

$$Q(t) = A\cos\omega t\cos\phi - AD\sin\omega t\sin\phi = A\cos(\omega t + \phi) \tag{8.23}$$

All these forms of the solution are **equivalent**.

Meaning of the Constants

$$A = \text{ amplitude measured from equilibrium } (Q = 0)$$

$$\begin{aligned}\omega &= \text{ angular frequency of the motion} \\ &= 2\pi\nu \quad , \quad \text{where } \nu = \text{ frequency} = \text{oscillations / second} \\ &= \frac{2\pi}{T} \quad , \quad \text{where } T = \text{ period} = \text{time between identical } Q \text{ values}\end{aligned}$$

$$\phi = \text{ phase angle}$$

The initial conditions $Q(0), \dot{Q}(0)$ determine the unknown constants

$$(G, H) \text{ or } (C, D) \text{ or } (A, \phi)$$

Special Case of Undamped, Unforced Spring

For a spring we have

$$m\frac{d^2x}{dt^2} = -kx \rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$\rightarrow x(t) = B\sin\omega_0 t + C\cos\omega_0 t \ , \ \omega_0 = \sqrt{\frac{k}{m}} = 2\pi\nu_0 = \frac{2\pi}{T_0}$$

or

$$x(t) = A\cos(\omega_0 t + \phi)$$

Suppose at time $t = 0$, we have

$$x(0) = x_0 \quad , \quad \dot{x}(0) = v_0$$

We then have

$$x(0) = x_0 = C \quad , \quad \dot{x}(0) = v_0 = \omega B$$

or

$$x(t) = x_0 \cos\omega t + \frac{v_0}{\omega}\sin\omega t$$

Alternatively, we have

$$x(0) = x_0 = A\cos\phi \quad , \quad \dot{x}(0) = v_0 = -\omega_0 A\sin\phi$$

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2} \quad , \quad \tan\phi = -\frac{v_0}{x_0\omega}$$

$$x(t) = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}\cos\left(\omega t + \tan^{-1}\left(= -\frac{v_0}{x_0\omega}\right)\right)$$

The different solution forms are appropriate for particular problems.

Energy Considerations

If we choose the potential energy $U(x) = 0$ at $x = 0$, we have

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t + \phi) \tag{8.24}$$

The kinetic energy is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2\sin^2(\omega t + \phi) = \frac{1}{2}kA^2\sin^2(\omega t + \phi) \tag{8.25}$$

Thus, the total energy is

$$E = K + U = \frac{1}{2}kA^2\cos^2(\omega t + \phi) + \frac{1}{2}kA^2\sin^2(\omega t + \phi) = \frac{1}{2}kA^2 \tag{8.26}$$

Note that the energy = constant since this is a conservative force.

Time Averages

The time average value $\langle f\rangle$ of a function $f(t)$ over a given time interval is defined as

$$\langle f\rangle = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} f(t)dt \tag{8.27}$$

In particular, over one period

$$\langle \sin\omega t\rangle = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\sin\omega t dt = 0 = \langle\cos\omega t\rangle = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\cos\omega t dt \tag{8.28}$$

and

$$\langle \sin^2\omega t\rangle = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\sin^2\omega t dt = \frac{1}{2} = \langle\cos^2\omega t\rangle = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\cos^2\omega t dt \tag{8.29}$$

This allows us to calculate the time average values of various physical quantities associated with SHM.

$$\langle U\rangle = \frac{1}{4}kA^2 = \langle K\rangle \tag{8.30}$$

$$\langle E\rangle = \langle U\rangle + \langle K\rangle = \frac{1}{2}kA^2 \tag{8.31}$$

The Damped Harmonic Oscillator

Suppose that we have a spring with a mass attached and that the mass is experiencing an extra force due to frictional effects.

Let us assume that

$$F - m\ddot{x} = -kx - b\dot{x} \tag{8.32}$$

where the term $-b\dot{x}$ represents the frictional effects.

This is the typical form for such forces.

Our equation of motion then becomes:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \quad ,\gamma = \frac{b}{m} \;\;,\;\; \omega_0^2 = \frac{k}{m} \tag{8.33}$$

We solve this equation using the complex exponential method.

We substitute

$$x(t) = e^{i\alpha t} \tag{8.34}$$

into the diffEQ.

As before, this will convert the diffEQ into an algebraic equation for α.

$$(\alpha^2 + \gamma\alpha + \omega_0^2)e^{i\alpha t} = 0 \tag{8.35}$$

or

$$\alpha^2 + \gamma\alpha + \omega_0^2 = 0 \tag{8.36}$$

The solutions of this equation tell us the allowed values of α that give solutions to the diffEQ.

If there is more than one allowed value of α , then the most general solution will be a linear combination of all possible solutions.

In this case, the allowed values of α are

$$\alpha = i\frac{\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \tag{8.37}$$

Therefore the most general solution is

$$x(t) = e^{-\frac{\gamma}{2}t}(Be^{i\omega_1 t} + Ce{-}i\omega_1 t) \quad , \quad \omega_1 = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \tag{8.38}$$

or

$$x(t) = Ae^{-\frac{\gamma}{2}t}\cos(\omega_1 t + \phi) \tag{8.39}$$

where the constants are determined by the initial conditions.

The solution shown is valid when

$$\omega_0^2 > \frac{\gamma^2}{4} \tag{8.40}$$

It corresponds to **UNDERDAMPED** oscillatory motion. It looks like:

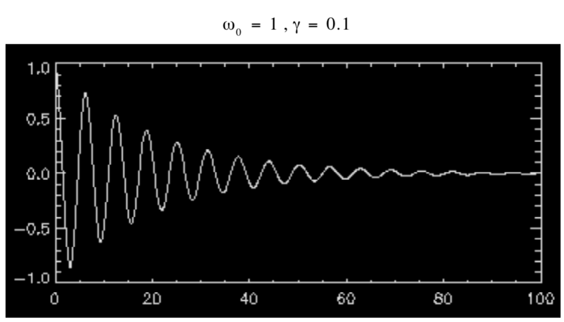

Figure 167:

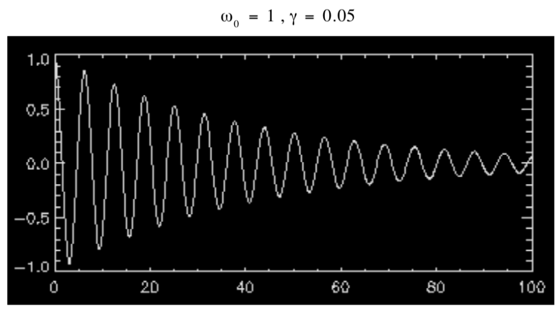

Figure 168:

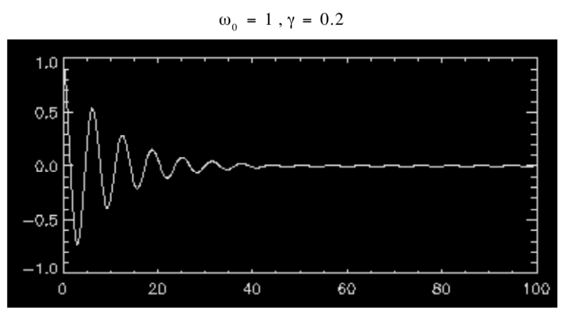

Figure 169:

All these are examples of **underdamped harmonic motion**.

It is SHM where the amplitude A is modulated by a decaying exponential function.

$$x(t) = A(t)\cos(\omega_1 t + \phi) \quad , \quad A(t) = Ae^{-\frac{\gamma}{2}t} \tag{8.41}$$

We have oscillatory motion where the amplitude decreases to zero (friction eventually removes all the energy from the system).

The damped frequency ω_1 is smaller than the undamped frequency ω_0.

The zero crossings are separated by equal time intervals equal to

$$\frac{2\pi}{\omega_1} \tag{8.42}$$

but the peaks do not lie half way between the zero crossings(as in the case of SHM).

For weak damping $\omega_1 \approx \omega_0$ and for heavy damping ω_1 is significantly smaller than ω_0.

Energy

Using the work-energy theorem we have

$$E(t) = E(0) + W_{friction} \tag{8.43}$$

$$E(t) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 - K(t) + U(t) \tag{8.44}$$

$$W_{friction} = \text{ work done by friction between times } t_1 \text{ and } t_2 \tag{8.45}$$

with

$$f = -bv = \text{ frictional force} \tag{8.46}$$

We then have

$$W_{friction} == \int_{x(0)}^{x(t)} f dx = \int_0^t f v dt = -\int_0^t bv^2 dt < 0 \tag{8.47}$$

which says that $E(t)$ decreases with time because the friction force continually dissipates energy.

One can shown explicitly that

$$E(t) = E(0)e^{-\gamma t} \tag{8.48}$$

The energy decreases exponentially in time.

The parameter γ determines the nature of the exponential behavior.

The energy decays to $E(0)/e$ in a time $\tau = 1/\gamma =$ = **damping time** = **time constant** = **characteristic time of the system**.

The Q of an Oscillator

The degree of damping is determined by a dimensionless number defined by

$$Q = \frac{\text{energy stored in the oscillator}}{\text{energy dissipated per radian}} \tag{8.49}$$

The energy dissipated per radian = energy lost when ωt changes by 1 or when the oscillator oscillates through 1 radian.

This corresponds to:

$$T = \text{ period } = \frac{2\pi}{\omega_1} = \text{ time to oscillate through } 2\pi \text{ radians}$$

$$\frac{T}{2\pi} = \text{ time to oscillate through 1 radian } = \frac{1}{\omega_1}$$

In the lightly damped case we have:

$$E(t) = E(0)e^{-\gamma t} \rightarrow \frac{dE}{dt} = -\gamma E(0)e^{-\gamma t} = \gamma E(t)$$

$$\Delta E = \text{ energy dissipated in time } \Delta t = \left|\frac{dE}{dt}\right| \Delta t = \gamma E \Delta t$$

$$\Delta E(1 \text{ rad}) = \gamma E \frac{1}{\omega_1}$$

$$Q = \frac{E}{\gamma E}\frac{1}{\omega_1} = \frac{\omega_1}{\gamma} \approx \frac{\omega_0}{\gamma}$$

Some typical numbers are:

Typical damped spring ? $Q = 1 - 10$

Tuning fork ? $Q \approx 1000$

Laser- $Q \approx 10^{14}$

Other Solutions

Now if we have

$$\omega_0^2 < \frac{\gamma^2}{4} \tag{8.50}$$

the solution looks like

$$x(t) = e^{-\frac{\gamma}{2}}\left(Be^{\beta_1 t} + Ce^{-\beta_1 t}\right) \quad , \quad \beta_1 = \sqrt{\frac{\gamma^2}{4} - \omega_0^2} \tag{8.51}$$

It corresponds to **OVERDAMPED** oscillatory motion.

It looks like:

$\omega_0 = 0.9$, $\gamma = 4.0$

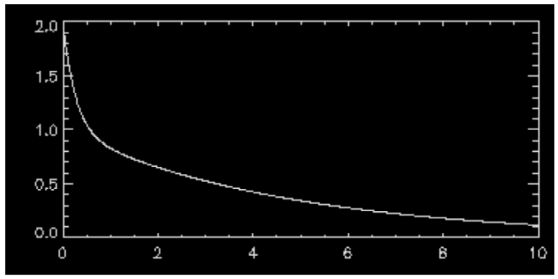

Figure 170:

There are no oscillations(it does not cross the equilibrium line).

There is a special case (you will learn about it in detail in an advanced Physics class) corresponding to

$$\omega_0^2 = \frac{\gamma^2}{4} \tag{8.52}$$

where the solution is

$$x(t) = e^{-\omega_0 t}(A + Bt) \tag{8.53}$$

This corresponds CRITICALLY DAMPED motion and is the fastest decay to equilibrium.

Phase Space

Until now we have been plotting solutions as time series, that is, $x(t)$ vs t or $v(t)$ vs. t.

A very powerful informational and plotting tool is called **phase space** where we plot $x(t)$ vs $v(t)$.

This technique will be especially powerful when we study the chaotic motion of the nonlinear oscillator later.

Let us look at some examples below.

The different curved correspond to different energy values.

As we will see later, the closed curved in phase space is a signal of periodic motion of some kind.

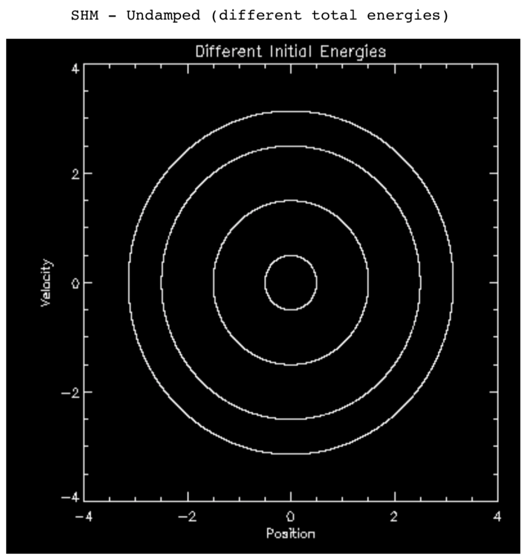

Figure 171:

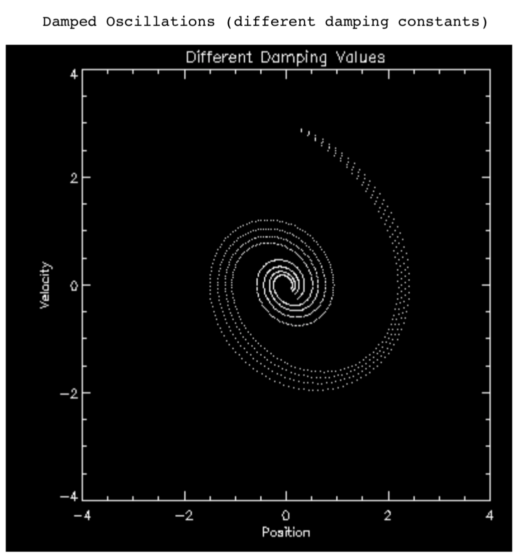

Figure 172:

We will say more about phase space shortly when we look at chaos.

Part 2

The Forced Damped Harmonic Oscillator

Suppose that we have a spring with a mass attached and that the mass is experiencing an extra force due to frictional effects.

Let us assume that the equation takes the form (as earlier)

$$F = m\ddot{x} = -kx - b\dot{x} \tag{8.54}$$

In addition, let us assume that there is an external time-dependent driving force.

In particular, a sinusoidal function with frequency ω.

This is the typical form for such forces.

Our equation of motion then becomes:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = F_0 \cos\omega t \quad , \gamma = \frac{b}{m} \ , \ \omega_0^2 = \frac{k}{m} \tag{8.55}$$

The solution of this equation is the sum of two solutions

$$x(t) = x_h(t) + x_p(t) \tag{8.56}$$

where

$$\ddot{x}_h + \gamma\dot{x}_h + \omega_0^2 x_h = 0 \tag{8.57}$$

and

$$\ddot{x}_p + \gamma\dot{x}_p + \omega_0^2 x_p = F_0 \cos\omega t \tag{8.58}$$

$x_h(t)$ is called the transient solution since it quickly dies out.

$x_p(t)$ is called the steady-state solution because it is the final steady motion of the system.

We already know the solution for $x_h(t)$ from our earlier discussions of the damped oscillator.

If one observes this type of system, then its time behavior looks like

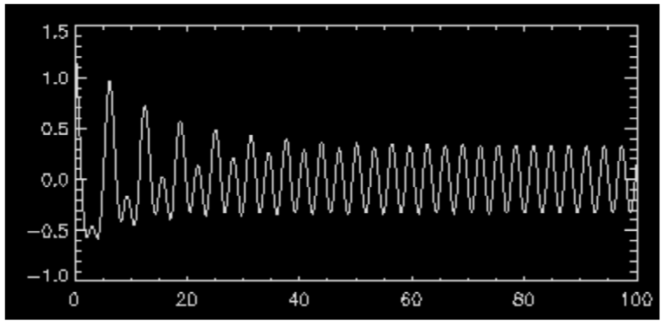

Figure 173:

We initially observe erratic behavior when both solutions are present.

As the transient solution decays away, the system settles into a steady-state solution as shown with frequency ω, the same as the frequency of the driving force.. This suggests that we assume a steady-state solution of the form

$$x_p(t) = R\cos(\omega t + \phi) \tag{8.59}$$

for $x_p(t)$.

This is standard technique used by physicists to solve differential equations – observe a system experimentally and then use that information to guess a solution.

Then check to see if it is correct.

If it is, then it is the solution to the problem because the solutions of these equations are **unique**.

Substitution of the assumed solution shows that it will work(is a valid solution) if we choose R and ϕ as derived below:

$$x_p(t) = R\cos(\omega t + \phi)\ ,\ \dot{x}_p(t) = -\omega R\sin(\omega t + \phi)\ ,\ \ddot{x}_p(t) = -\omega^2 R\cos(\omega t + \phi) \tag{8.60}$$

Substituting we get

$$-\omega^2 R\cos(\omega t + \phi) - \gamma\omega R\sin(\omega t + \phi) + \omega_0^2 R\cos(\omega t + \phi) = F_0\cos(\omega t) \tag{8.61}$$

Expanding the terms $\cos\omega t+\phi$ and $\sin\omega t+\phi$ we get

$$-\omega^2 R(\cos(\omega t)\cos(\phi)-\sin(\omega t)\sin(\phi))-\gamma\omega R(\sin(\omega t)\cos(\phi)+\cos(\omega t)\sin(\phi))$$

$$+\omega_0^2 R(\cos(\omega t)\cos(\phi)-\sin(\omega t)\sin(\phi))=F_0\cos(\omega t) \tag{8.62}$$

Collecting terms and rearranging we get

$$\begin{aligned}&\cos\omega t(-\omega^2 R\cos(\phi)-\gamma\omega R\sin(\phi)+\omega_0^2 R\cos(\phi)-F_0)\\&+\sin\omega t(\omega^2 R\sin(\phi)-\gamma\omega R\cos(\phi)-\omega_0^2 R\sin(\phi))=0\end{aligned} \tag{8.63}$$

which implies that

$$(-\omega^2 R\cos(\phi)-\gamma\omega R\sin(\phi)+\omega_0^2 R\cos(\phi)-F_0)=0 \tag{8.64}$$
$$(\omega^2 R\sin(\phi)-\gamma\omega R\cos(\phi)-\omega_0^2 R\sin(\phi))=0 \tag{8.65}$$

separately since $\cos\omega t$ and $\sin\omega t$ cannot both be zero at the same time. Solving these two equations gives

$$\tan\phi=\frac{\gamma\omega}{\omega^2-\omega_0^2} \quad , \quad R=\frac{F_0}{(\omega^2-\omega_0^2)\cos(\phi)+\gamma\omega\sin(\phi)} \tag{8.66}$$

Finishing off we get

$$\cos^2\phi=\frac{1}{1+\tan^2\phi}=\frac{(\omega^2-\omega_0^2)^2}{(\omega^2-\omega_0^2)^2+(\gamma\omega)^2} \tag{8.67}$$

$$\sin^2\phi=\frac{\tan^2\phi}{1+\tan^2\phi}=\frac{(\gamma\omega)^2}{(\omega^2-\omega_0^2)^2+(\gamma\omega)^2} \tag{8.68}$$

$$R=\frac{F_0}{[(\omega^2-\omega_0^2)^2+(\gamma\omega)^2]^{1/2}} \tag{8.69}$$

$$x_p(t)=R\cos(\omega t+\phi) \tag{8.70}$$

Let us first look at the behavior of R and ϕ as functions of ω and the parameter ratio γ/ω_0:

Case #1 - light damping - $\gamma=0.15$, $\omega_0=1.0$

Plotting R versus ω we get

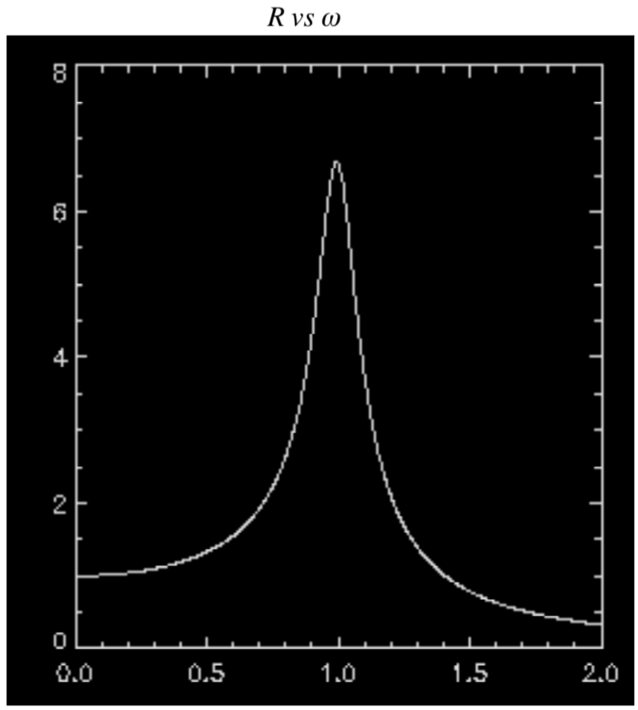

Figure 174:

For small(light) damping R is a maximum for $\omega = \omega_0$ and the amplitude at maximum (or resonance) is

$$R(\omega_0) = \frac{F_0}{\gamma\omega_0} \tag{8.71}$$

which can get very large for small damping.

In general there is very little final amplitude for these system unless the driving frequency is near to the natural frequency.

The size of the amplitude as a function of frequency is an indication of how the system would absorb or emit energy at that frequency.

ϕ represents the phase difference between the driving force $F_0 \cos \omega t$ and the response to the driving force

$$x_p(t) = R\cos(\omega t + \phi) \tag{8.72}$$

A typical plot of ϕ versus ω looks like:

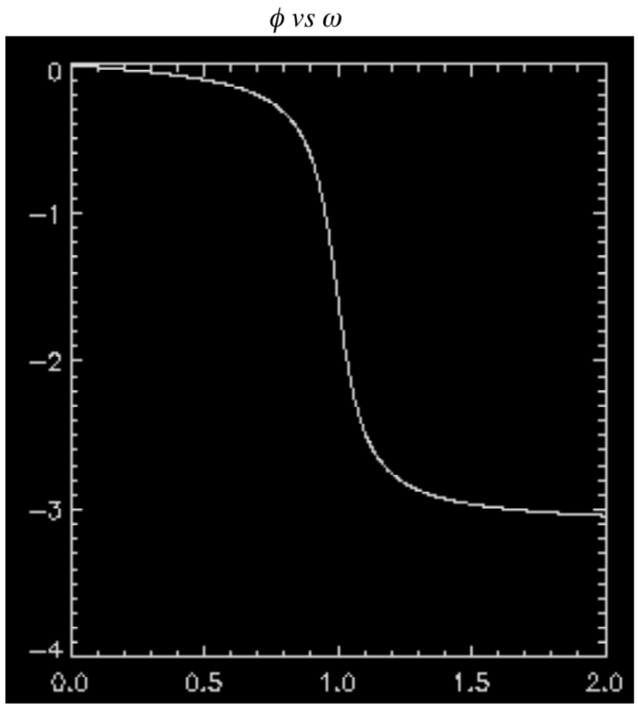

Figure 175:

If we plot the driving force and the response function together they look like (the horizontal separation corresponds to the phase difference):

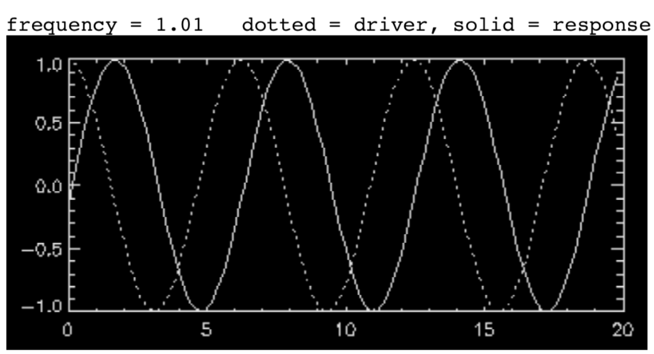

Figure 176:

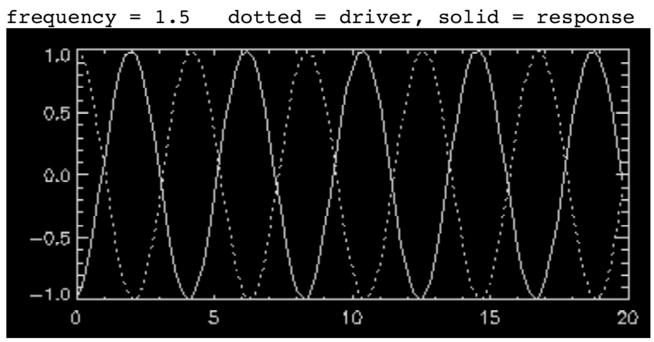

Figure 177:

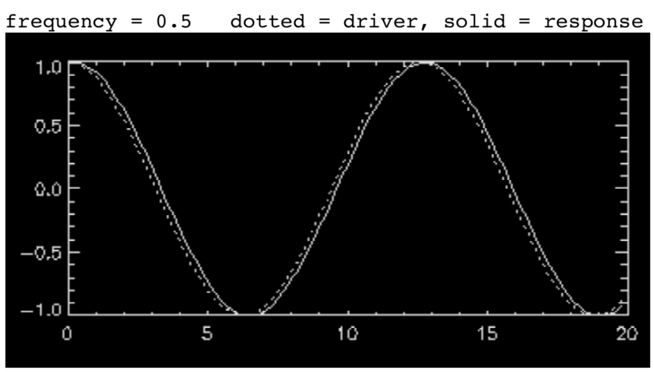

Figure 178:

For the driving frequency less than the natural frequency, the driver and the response are **in phase**. For the driving frequency greater than the natural frequency, the driver and the response are **out of phase**.

For highly damped systems the behavior is dramatically different as can be seen below.

Case #2 - heavy damping - γ = 0.9 , ω_0 = 1.0

R vs ω

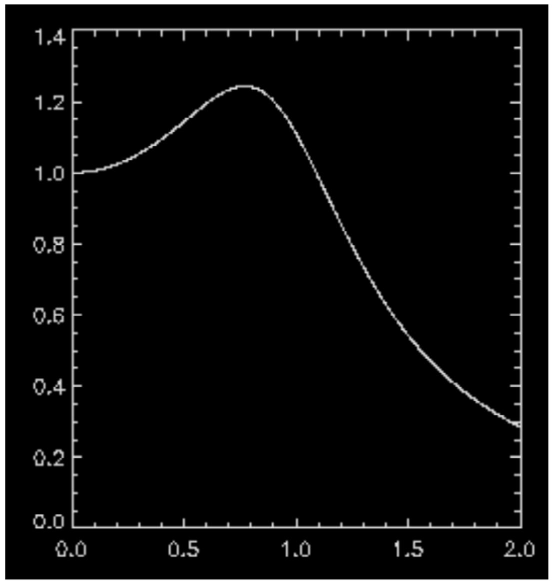

Figure 179:

φ vs ω

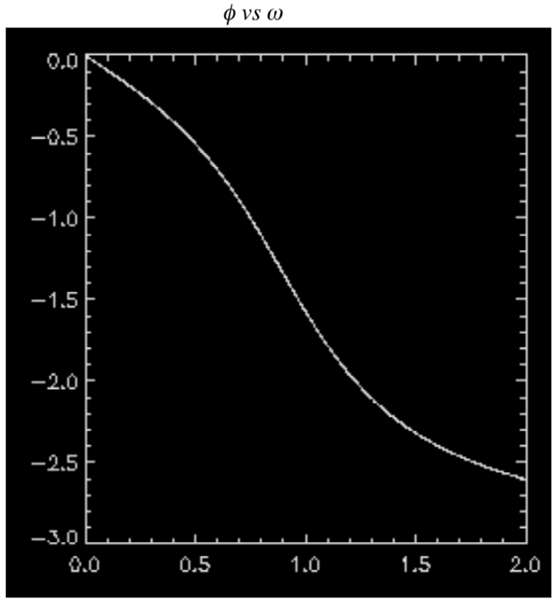

Figure 180:

Properties of Resonance in a Lightly Damped System: The Quality factor Q

For small damping the resonance amplitude is large at resonance and sharp (small width or small spread of frequencies where the amplitude is large).

As the damping gets larger the resonance amplitude gets smaller and the curve spreads out.

These features are very important in determining the system response to external oscillatory forces.

For the steady-state motion, the amplitude is constant in time (after transients have died out).

We have

$$x = R\cos(\omega t + \phi) \quad , \quad v = -\omega R \sin(\omega t + \phi) \tag{8.73}$$

$$K(t) = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 R^2 \sin^2(\omega t + \phi) \tag{8.74}$$

$$U(t) = \frac{1}{2}kx^2 = \frac{1}{2}kR^2\cos^2(\omega t + \phi) \tag{8.75}$$

$$E(t) = K(t) + U(t) = \frac{1}{2}R^2[m\omega^2 \sin^2(\omega t + \phi) + k\cos^2(\omega t + \phi)] \tag{8.76}$$

The energy is time-dependent ($k \neq m\omega^2$).

We are pumping energy into the system and it is continually being dissipated.

We focus our attention on time-average quantities

$$\langle K \rangle = \frac{1}{4}m\omega^2 R^2 \quad , \quad \langle U \rangle = \frac{1}{4}kR^2 \tag{8.77}$$

$$\langle E \rangle = \frac{1}{4}R^2[m\omega^2 + k] = \frac{1}{4}mR^2[\omega^2 + \omega_0^2] \tag{8.78}$$

As a function of the driving frequency we then have:

$$\langle E(\omega) \rangle = \frac{F_0^2}{4}\frac{\omega^2 + \omega_0^2}{(\omega^2 - \omega_0^2)^2 + (\gamma\omega)^2} \tag{8.79}$$

Let us consider this expression in the case of small damping, where $\gamma \ll \omega_0$.

If we plot $E(\omega)$ as a function of ω we see that for light damping $E(\omega)$ is effectively zero (solid curve) except near resonance. Other curves are larger damping values.

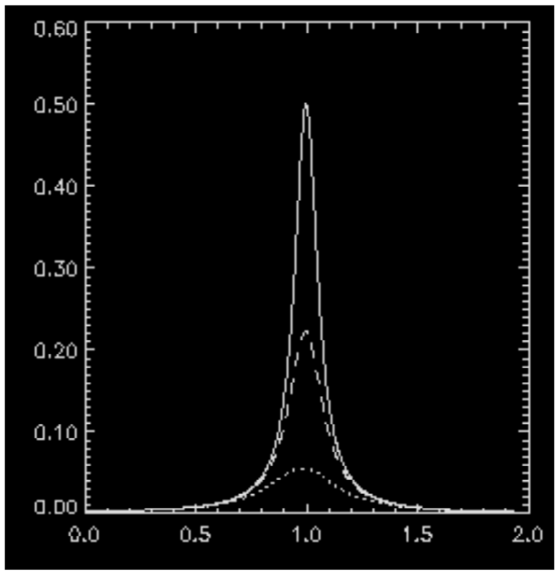

Figure 181:

Hence we can replace $\omega \approx \omega_0$ except in the term involving $\omega - \omega_0$.

We obtain

$$\begin{aligned} \langle E(\omega) \rangle &= \frac{F_0^2}{4} \frac{2\omega_0^2}{(\omega - \omega_0)^2(\omega + \omega_0)^2 + (\gamma\omega)^2} \\ &= \frac{F_0^2}{2} \frac{\omega_0^2}{(\omega - \omega_0)^2(2\omega_0)^2 + (\gamma\omega)^2} \\ &= \frac{F_0^2}{8} \frac{1}{(\omega - \omega_0)^2 + \left(\frac{\gamma}{2}\right)^2} \end{aligned} \tag{8.80}$$

This is called a **resonance curve** or **Lorentzian**. Its properties are:

maximum height = $\frac{4}{\gamma^2}$

$\frac{1}{2}$-maximum height when $(\omega - \omega_0)^2 = \left(\frac{\gamma}{2}\right)^2$ or when $\omega - \omega_0 = \pm\frac{\gamma}{2} = \omega_\pm$

$$\text{full width at } \tfrac{1}{2} \text{ maximum} = \text{resonance width} = \omega_+ - \omega_- = \gamma = \Delta\omega$$

As γ decreases the curve becomes higher and narrower.

The range of frequencies over which the system responds becomes smaller and the oscillator becomes increasingly selective in frequency.

This frequency selective property is characterized by the Q-value.

Remember

$$Q = \frac{\text{energy stored in the oscillator}}{\text{energy dissipated per radian}} \tag{8.81}$$

which for a lightly damped system gives (after much algebra)

$$Q = \frac{\omega_0}{\gamma} \tag{8.82}$$

This same damped oscillator, when driven has a resonance curve with frequency width $\Delta\omega = \gamma$.

Thus,

$$Q = \frac{\omega_0}{\gamma} = \frac{\omega_0}{\Delta\omega} = \frac{\text{resonance frequency}}{\text{frequency width of resonance curve}} \tag{8.83}$$

A very sharp resonance curve means that a system will not respond unless driven very near its resonance frequency.

Coupled Oscillations

Let us now consider the system of **coupled** oscillators shown below:

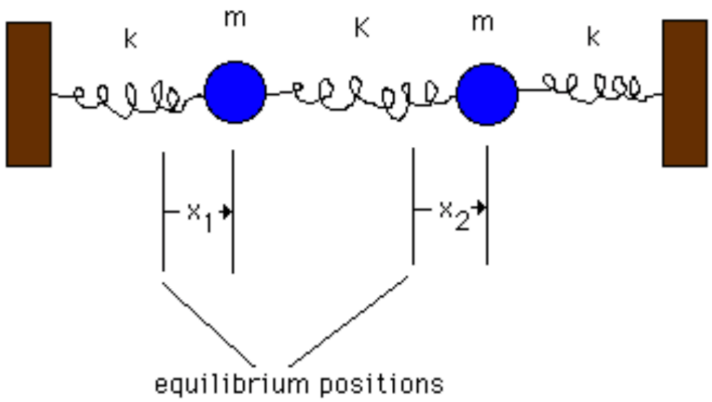

Figure 182:

We can write down Newton's 2nd law equations for each of the masses:

$$m\ddot{x}_1 = -kx_1 + K(x_2 - x_1) \tag{8.84}$$

$$m\ddot{x}_2 = -kx_2 - K(x_2 - x_1) \tag{8.85}$$

These are **coupled differential equations**.

The general solution can be found using a couple of tricks.

Rearrange the equations as follows:

$$(\ddot{x}_1 + \ddot{x}_2) + \frac{k}{m}(x_1 + x_2) = 0 \tag{8.86}$$

$$(\ddot{x}_1 - \ddot{x}_2) + \frac{k + 2K}{m}(x_1 - x_2) = 0 \tag{8.87}$$

These are SHM equations (variable $x_1 \pm x_2$).

Therefore we have

$$x_1 + x_2 = A\cos\omega_1 t + B\sin\omega_1 t \quad , \quad \omega_1 = \sqrt{\frac{k}{m}} \tag{8.88}$$

$$x_1 - x_2 = C\cos\omega_2 t + D\sin\omega_2 t \quad , \quad \omega_2 = \sqrt{\frac{k + 2K}{m}} \tag{8.89}$$

Let us choose some special cases (choice of initial conditions) so we can see what is happening:

Case 1: Symmetric case

$$x_1(0) = d = x_2(0) \quad , \quad \dot{x}_1(0) = 0 = \dot{x}_2(0)$$
$$\rightarrow 2d = A \ , \ 0 = C \ , \ 0 = \omega_1 B \ , \ 0 = \omega_2 D$$
$$\rightarrow x_1 + x_2 = 2d\cos\omega_1 t$$
$$\rightarrow x_1 - x_2 = 0$$
$$\rightarrow x_1 = x_2 = d\cos\omega_1 t$$

→ the masses oscillate in same direction (in phase) with frequency $\omega_1 = \sqrt{\dfrac{k}{m}}$

Case 2: Antisymmetric case

$$x_1(0) = d = -x_2(0) \quad , \quad \dot{x}_1(0) = 0 = \dot{x}_2(0)$$
$$\rightarrow 0 = A \ , \ 2d = C \ , \ 0 = \omega_1 B \ , \ 0 = \omega_2 D$$
$$\rightarrow x_1 - x_2 = 2d\cos\omega_2 t$$
$$\rightarrow x_1 + x_2 = 0$$
$$\rightarrow x_1 = -x_2 = d\cos\omega_2 t$$

→ oscillate in opposite directions (out of phase) with frequency $\omega_2 = \sqrt{\dfrac{k+2K}{m}}$

These first two cases represent situations where the system as a whole oscillates with a single frequency.

They are called **NORMAL MODES** and the frequencies are called **characteristic** or **eigenfrequencies**.

Case 3: General case

The initial conditions

$$x_1(0) = d \quad , \quad x_2(0) = 0 \quad , \quad \dot{x}_1(0) = 0 = \dot{x}_2(0)$$

give the results

$$d = A \ \ , \ \ d = C \ \ , \ \ 0 = \omega_1 B \ \ , \ \ 0 = \omega_2 D$$

$$x_1 + x_2 = d \cos \omega_1 t$$

$$x_1 - x_2 = d \cos \omega_2 t$$

which lead to

$$x_1 = \frac{d}{2}(\cos \omega_1 t + \cos \omega_2 t) = d \cos \left(\frac{1}{2}(\omega_1 + \omega_2) \right) \cos \left(\frac{1}{2}(\omega_1 - \omega_2) \right)$$

$$x_2 = \frac{d}{2}(\cos \omega_1 t - \cos \omega_2 t) = d \sin \left(\frac{1}{2}(\omega_1 + \omega_2) \right) \sin \left(\frac{1}{2}(\omega_1 - \omega_2) \right)$$

These correspond to a low frequency $\frac{1}{2}(\omega_1 - \omega_2)$ **envelope** modulating a high frequency $\frac{1}{2}(\omega_1 + \omega_2)$ **wiggle**.

What does this look like (choosing some numerical values):

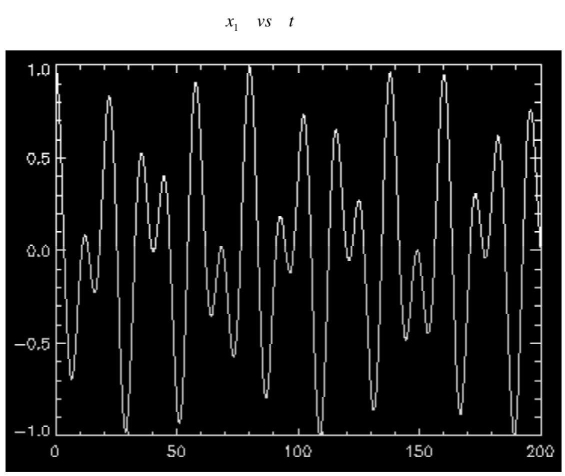

Figure 183:

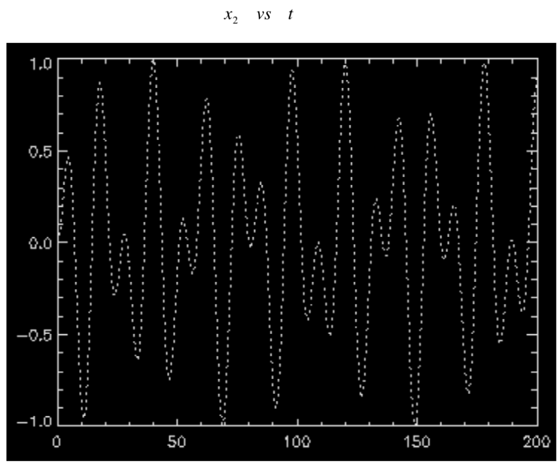

Figure 184:

Example:

The pendulum of a grandfather's clock activates an escapement mechanism every time it passes through the vertical.

The escapement is under tension (provided by a hanging weight) and gives the pendulum a small impulse a distance ℓ from the pivot.

The energy transferred by this impulse compensates for the energy dissipated by friction, so that the pendulum swings with constant amplitude.

(a) What is the impulse needed to sustain the motion of a pendulum of length L and mass m, with an amplitude of swing θ_0 and quality factor Q? Let the pendulum have speed v_0 as it starts to swing up (from equilibrium position) and speed v_1 as it returns (to the equilibrium position).

The loss in energy is then

$$\Delta E = \frac{1}{2}m(v_0^2 - v_1^2)$$

The motion takes half of a cycle (π rad).

By definition, Q = fraction of energy lost per radian.

Therefore,

$$\Delta E = \frac{\pi E}{Q} = \frac{1}{2}\frac{\pi m v_0^2}{Q}$$

Therefore,

$$\frac{1}{2}\frac{\pi m v_0^2}{Q} = \frac{1}{2}m(v_0^2 - v_1^2) \to (v_0 - v_1)(v_0 + v_1) = \frac{\pi v_0^2}{Q}$$

Since $v_1 \approx v_0$, we have

$$(\Delta v)(2v_0) = \frac{\pi v_0^2}{Q} \to \Delta v = v_0 - v_1 \approx \frac{\pi v_0}{2Q}$$

The required impulse is

$$I = m\Delta v = \frac{m\pi v_0}{2Q}$$

If the motion is given by $\theta(t) = \theta_0 \sin\omega t$, where $\omega = \sqrt{g/L}$, then

$$v_0 = L\dot{\theta} = \omega\theta_0 L = \sqrt{gL}\theta_0\sqrt{gL}\theta_0$$

and

$$I = \frac{\pi\theta_0}{2Q}m\sqrt{gL}$$

(b) Why is it desirable for the pendulum to engage the escapement as it passes vertical rather than at some other point of the cycle?

The change in velocity for a given impulse is $\Delta v = I/m$.

However, the change in energy is

$$\Delta E = \frac{1}{2}m(v + \Delta v)^2 - \frac{1}{2}mv^2 = m\Delta v + \frac{1}{2}m(\Delta v)^2$$

The point in the cycle where the impulse occurs can vary due to mechanical imperfections such as play in the mechanism or wear.

To minimize changes in the energy transferred, the impulse should occur when v is constant, that is, at the bottom of the swing.

Proof: The energy equation is

$$\frac{1}{2}mv^2 + gL(1 - \cos\theta) \approx \frac{1}{2}mv^2 + \frac{1}{2}g\ell\theta^2$$

This gives

$$vmdv + g\ell\theta d\theta = 0 \rightarrow \frac{dv}{d\theta} = \frac{g\ell}{m}\frac{\theta}{v}$$

so that

$$\frac{dv}{d\theta} = 0 \quad \text{when} \quad \theta = 0$$

Example:

A small cuckoo clock has a pendulum 25 cm long with a mass of 10 g and a period of 1 s.

The clock is powered by a 200 g weight which falls 2 m between daily windings.

The amplitude of the swing is 0.2 rad.

What is the Q of the clock?

How long would the clock run if it were powered by a battery with 1 J capacity?

The power to the clock from the descending weight equals the power dissipated by friction.

If the weight descends distance L in time T, then $\bar{p} = MgL/T$.

The energy lost per radian is $\overline{\Delta E} = \bar{p}/\omega$.

The average stored energy is

$$\bar{E} = \frac{1}{2}m\overline{v^2} + \frac{1}{2}mg\ell\overline{\theta^2}$$

The kinetic and potential energies are equal, on the average, and

$$\bar{E} = \frac{1}{2}mg\ell\overline{\theta_0^2}$$

where θ_0 is the angular amplitude. We then have

$$Q = \frac{\bar{E}}{\overline{\Delta E}} = \frac{\frac{1}{2}mg\ell\overline{\theta_0^2}}{\bar{p}}\omega = \frac{\frac{1}{2}mg\ell\overline{\theta_0^2}}{MgL}\omega T = \frac{1}{2}\frac{m}{M}\frac{\ell}{L}\theta_0^2\omega T$$

We have numerical values

$$\ell = 25\,cm,\ L = 2\,m, m = 10^{-2}\,kg, M = 0.2\,kg, \theta_0 = 0.2$$

$$\omega = 2\pi, rad/s, T = 8.64\times 10^4\,s$$

Therefore,

$$Q = 68$$

The energy to drive the clock one day is $E = MgL = 4$ joules.

Therefore, the clock would run only 6 hours on a 1 joule battery.

9. Non-Linear Mechanics : Approach to Chaos

Introduction

There are two major reasons for studying non-linear mechanics.

The first and most basic is that the equations of motion of almost all real systems are non-linear.

The second reason is that even a relatively simple system which obeys a non-linear equation of motion can exhibit unusual and surprisingly complex behavior for certain ranges of the system parameters.

In a wide variety of dramatically different non-linear systems identical features show up.

Much of the existing knowledge of non-linear behavior has been obtained from numerical solutions.

The traditional methods of mechanics, which lead to analytic (explicit equations) expressions for the motion, fail for most problems in non-linear mechanics.

Numerical integration of the equations of motion is usually necessary.

From the time of Newton until the 20th century, physicists and philosophers viewed the universe as a sort of enormous clock which, once wound up, behaves in a predictable manner.

This idea was dramatically shaken by the discovery of quantum mechanics and the Heisenberg uncertainty principle, but physicists still thought that the motion of classical systems (macroscopic) that obey Newton's equations of motion would exhibit predictable or deterministic behavior.

It turns out, however, that even macroscopic systems obeying Newton's equations can exhibit so-called **chaotic** motion or motion that seems very difficult to predict (or is even unpredictable).

The main difference between a chaotic system and a non-chaotic system is the degree of predictability of the motion given the initial conditions to some level of accuracy (note the new idea being introduced that we might not be able to specify the initial conditions exactly).

In addition, we will find an extraordinary sensitivity to initial conditions.

Let us look at a system we considered earlier, namely, the linear, damped oscillator with sinusoidal driving force.

This system satisfies the differential equation

$$\frac{d^x}{dt^2} + b\frac{dx}{dt} + cx = a\cos\omega t \tag{9.1}$$

with $a = 0.9$, $b = 1.0$, $c = 0.5$, $\omega = 0.66666$. If this system is started off with different initial conditions (values of x, dx/dt at $t = 0$) we observe the following behavior:

> **after a time long enough for the transient motion to die out, the different oscillatory systems will end up with the same motion**

Thus, the final motion is independent of the initial conditions.

In the Matlab simulation below, we plot the motion of two oscillators with different initial conditions on the same diagram (in **phase space** where we plot velocity(y-axis) versus position(x-axis) and different initial conditions correspond to different starting points in phase space) with each oscillator represented by a different color.

The final steady-state motion in both cases is clearly an elliptical path in phase space.

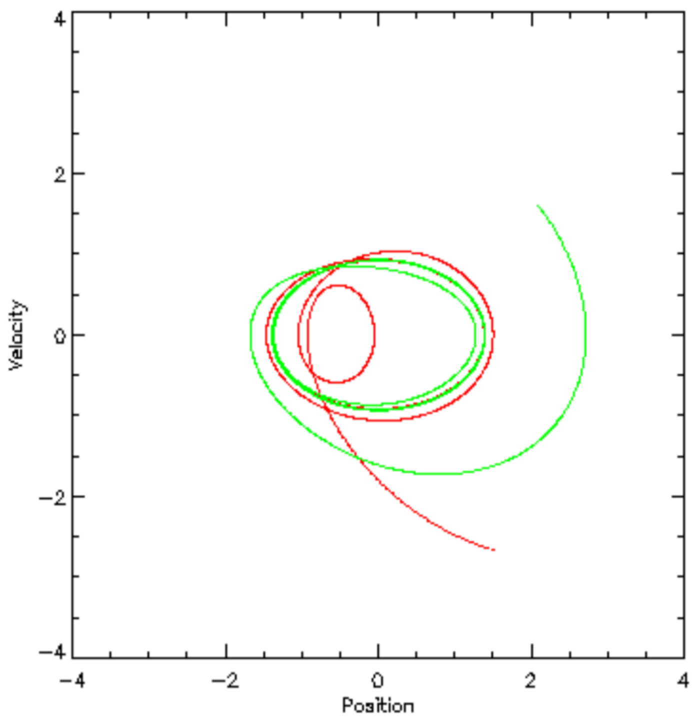

Figure 185:

Analytically, the final **steady-state motion** is given by

$$\begin{aligned} x &= A\sin\omega t \quad , \quad \frac{dx}{dt} = v = A\omega\cos\omega t \\ &= \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1 \rightarrow \text{ an ellipse} \end{aligned} \tag{9.2}$$

in agreement with the simulated motion.

The simulation illustrates the independence of initial conditions rather dramatically.

Later we shall investigate this system when the linear approximation is not valid, that is, the correct differential equation is

$$\frac{d^x}{dt^2} + b\frac{dx}{dt} + c\sin x = a\cos\omega t \tag{9.3}$$

and (9.1) is only valid when we can make the approximation

$$\sin x \approx x \tag{9.4}$$

If we simulate with the non-linear version of the equation, then the system exhibits a dramatic sensitivity to initial conditions, i.e., the solution for different(even slightly different) initial conditions bear no resemblance to each other as shown below

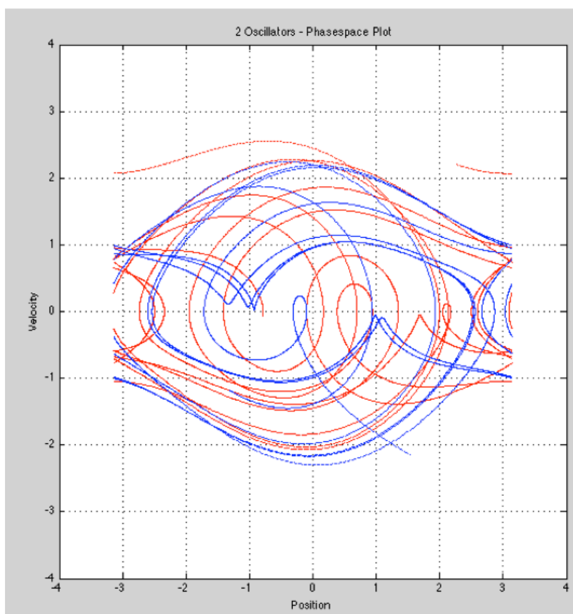

Figure 186:

One of the best know examples of poor predictability is the weather.

At one time it was believed that with a large number of atmospheric measurements and powerful computers to integrate the fluid mechanics equations it would be possible to make long term weather predictions.

It is now realized that this was a naive hope and that the weather equations are extremely non-linear and the solutions of the weather equations are exponentially sensitive to the initial conditions; this means that an infinitesimal difference in initial conditions will eventually produce a completely different solution to the equations, that is, no two solutions will have any relation to each other.

A simple weather model are the three-dimensional Lorenz equations which generate solutions that fall on a curve in three dimensions called a **strange attractor** (looks like butterfly wings). This led to the famous **butterfly effect** where one could imagine that the perturbation in the weather system due to a butterfly flapping its wings in Africa would grow exponentially into a great weather front in North America. This is illustrated dramatically by the simulation of the Lorenz equations (lorentz0.m and lorentz4n.m) video(in class). Again we observe extreme sensitivity to initial conditions.

Even though the motion of a complex system cannot be precisely predicted certain features can often still be relied upon.

For example, the exact path of a given molecule of water that come out of a faucet is certainly not predictable.

We can say, however, with high probability that the molecule will fall vertically downward within a well-defined cylindrical surface.

It is a real challenge to deduce such **"robust"** features of the solutions of nonlinear equations.

Toward an Understanding of Chaos

In general, non-linear differential equations are difficult to solve either analytically or numerically.

Before investigating in detail the properties of the non-linear damped, driven oscillator, we will look at a simple system that can be easily solved numerically but still exhibits all the important properties of a chaotic system.

There exists a class of elementary model systems that can give insight into the mechanisms leading to chaotic behavior.

These are stated in the form of **difference equations**, rather than differential equations.

A typical difference equation is of the form

$$x_{n+1} = f(\mu, x_n) \tag{9.5}$$

where x_n refers to the n^{th} value of x, which is always a real number on the unit interval [0,1], and μ is a parameter.

The way to think of this system is the following.

Think of nT as a time, where T is a basic time interval.

Starting from some initial value of x, x_0, we can generate a sequence of x values, $x_1, x_2, ...$ using (9.5).

The function f is called a **map** of the interval [0,1] onto itself, since it generates x_{n+1} from x_n.

The function f can be nonlinear in its argument x_n.

Difference equations are readily solved by iteration, and their numerical solution is much less time consuming than is the case for nonlinear differential equations.

The Quadratic or Logistic Map

The map is

$$x_{n+1} = \lambda x_n (1 - x_n) \tag{9.6}$$

If $0 \leq \lambda \leq 4$, then $0 \leq x_n \leq 1$ implies that that $0 \leq x_{n+1} \leq 1$, so that we can always assume that x_n is in the interval [0,1].

A **fixed point** of a mapping is a point that maps into itself.

If there are points which, after more and more iterations of the mapping approach closer and closer to a fixed point, then the fixed point is called an **attractor**.

If $\lambda < 1$, then we can see from (9.6) that $x_{n+1} < x_n$ for all x_n.

This implies that the ultimate result of repeated iterations is inevitably $x = 0$.

Thus, when $\lambda < 1$ the mapping has one fixed point, which is an attractor.

The fixed points are found for any λ by using the fixed point condition

$$x = \lambda x(1 - x) \tag{9.7}$$

which has solutions

$$x = 0 \quad , \quad x = 1 - \frac{1}{\lambda} \tag{9.8}$$

Geometrically, the fixed point is the intersection of the quadratic map function with the line $x_{n+1} = x_n$. Two examples are shown in the figures below:

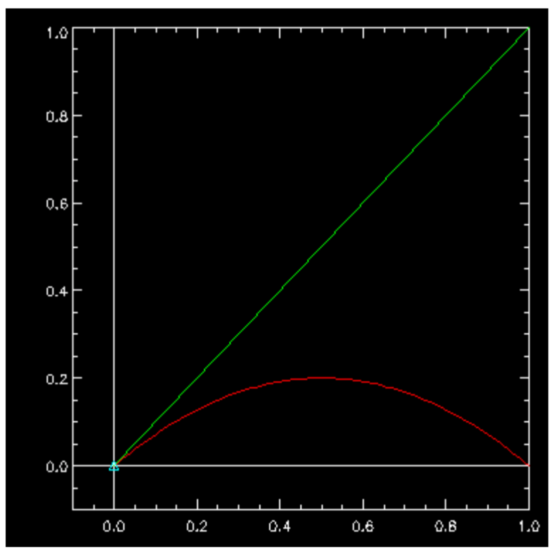

Figure 187:

The case above corresponds to $\lambda = 0.8$ and fixed point is $x = 0$ (cyan triangle).

As stated earlier, the fixed point will always be $x = 0$ when $\lambda \leq 1$.
The case above corresponds to $\lambda = 2.8$ and fixed point is $x = 0.643$ (cyan triangle).

Since $\lambda > 1$ in this case, the fixed point satisfies

$$x = 1 - \frac{1}{\lambda} = 1 - \frac{1}{2.8} = 1 - 0.357 = 0.643 \text{ as stated above.}$$

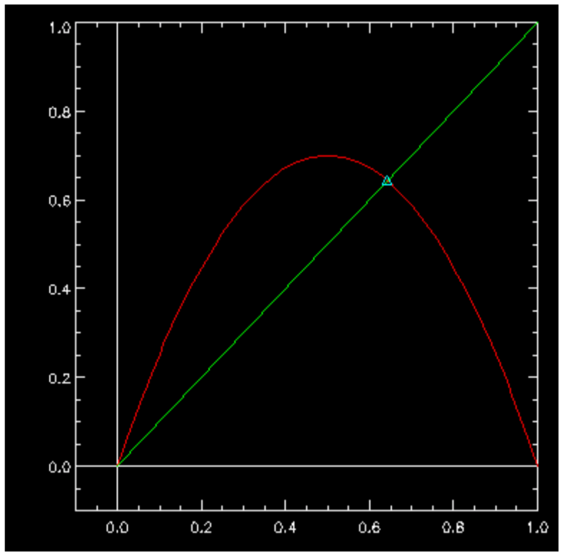

Figure 188:

The next question is whether the fixed points of (9.8) are stable, that is, they are attractors.

To settle this question we start with a point near a fixed point and see if the result of repeated mapping converges to the fixed point.

We write x_n as

$$x_n = \bar{x} + \delta_n \tag{9.9}$$

where $\bar{x}$ is a fixed point and δ_n is (at least initially) small in magnitude.

Substituting (9.9) into the map equation (9.6) and retaining only terms linear in δ_n (since δ_n is small), we find that

$$\begin{aligned} x_{n+1} &= \bar{x} + \delta_{n+1} = \lambda(\bar{x} + \delta_n)(1 - (\bar{x} + \delta_n)) \\ \bar{x} + \delta_{n+1} &= \lambda\bar{x}(1 - \bar{x}) + \lambda\delta_n - 2\lambda\bar{x}\delta_n - \lambda(\delta_n)^2 \\ \delta_{n+1} &= \lambda\delta_n - 2\lambda\bar{x}\delta_n \rightarrow \frac{\delta_{n+1}}{\delta_n} = \lambda(1 - 2\bar{x}) \end{aligned} \tag{9.10}$$

where we have used the definition of the fixed point $\bar{x} = \lambda\bar{x}(1-\bar{x})$ and dropped the small $\lambda(\delta_n)^2$ term. If $|\delta_{n+1}| < |\delta_n|$, then with repeated mappings the point

(9.9) moves closer and closer to $\bar{x}$ with increasing n and so this fixed point is called **stable or attracting**. On the other hand, if $|\delta_{n+1}| > |\delta_n|$, then the point moves away from $\bar{x}$ and the fixed point is called **unstable or repelling**.

For the logistic map, setting $x_{n+1} = \lambda x_n(1 - x_n) = F(x_n)$, we have

$$\left(\frac{dF}{dx}\right)_{x=\bar{x}} = \lambda(1 - 2\bar{x}) = \frac{\delta_{n+1}}{\delta_n} \tag{9.11}$$

so that the criterion for stability becomes

$$\left(\frac{dF}{dx}\right)_{x=\bar{x}} < 1 \tag{9.12}$$

We note that this result is general for all maps of the form $x_{n+1} = F(x_n)$.

Also note that $\left(\frac{dF}{dx}\right)_{x=\bar{x}}$ is the slope of the mapping function in the neighborhood of the fixed point.

A simple but informative geometrical construction of the iteration process near the fixed point is shown in the figures below for the two stable fixed point cases previously discussed.

In the construction we plot two curves, the magenta curve is $f(x) = x$ and the yellow curve is $g(x) = \lambda x(1 - x)$.

The cyan triangle is a fixed point where $f(x) = g(x)$.

The axes are x_{n+1} versus x_n.

The iteration starts from a point(near the fixed point) on the horizontal axis and proceeds as follows.

$$\begin{aligned}(x_0, 0) &\to (x_0, g(x_0)) \to (g(x_0), g(x_0)) \to (g(x_0), g(g(x_0))) \\ &\to (g(g(x_0)), g(g(x_0))) \to (g(g(x_0)), g(g(g(x_0)))) \to \text{ etc}\end{aligned}$$

The plot is called a **return map**.

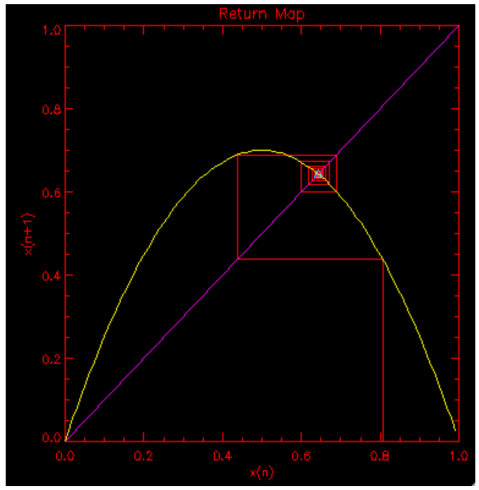

Figure 189:

This figure above shows the stable fixed point ($\bar{x} = 0.643$) for $\lambda = 2.8$.

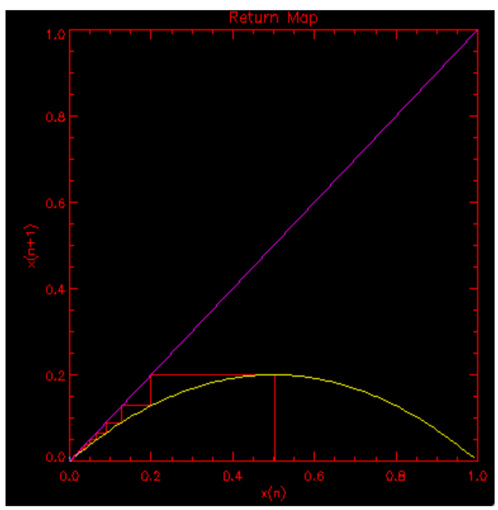

Figure 190:

This figure above shows the stable fixed point ($\bar{x} = 0$) for $\lambda = 0.8$.

According to (9.11) the fixed point $\bar{x} = 0$ is stable (it is an attractor) when $0 < \lambda < 1$, while the fixed point $\bar{x} = 1 - \frac{1}{\lambda}$ is stable for $1 < \lambda < 3$. Now that we have established the stability of the fixed points when $0 < \lambda < 3$, we venture into the region $3 < \lambda < 4$.

According to (9.11) there are no stable fixed points (simple attractors) for $\lambda > 3$.

We then look for points with higher periodicity points which return to their original value after some number of mappings.

For instance, period-2 points satisfy $x_{n+2} = x_n$.

They are fixed points of the once iterated mapping

$$x_{n+2} = \lambda x_{n+1}(1 - x_{n+1}) - \lambda^2 x_n(1 - x_n) - \lambda^3 x_n^2(1 - x_n)^2 \tag{9.13}$$

Before seeing how to make a plot which uses this equation to find periodic points, we show below the return map plot for $\lambda = 3.3$, which should be an unstable fixed point.

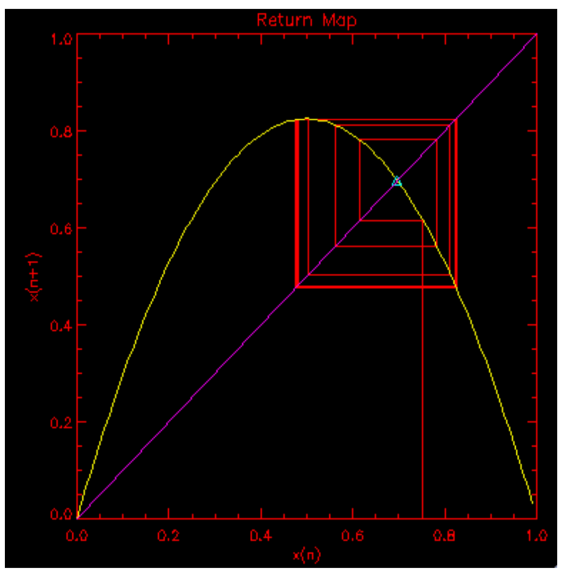

Figure 191:

Clearly, we do not have a period-1 fixed point.

We might have some periodic points indicated by the plot repeating the same path (large rectangle) but it is hard to determine what is happening.

Now using (9.13), called the double map function, we plot, the double map function, the single map function (the original map function) and the line $x_{n+1} = x_n$.

For $\lambda = 2.8$ all three curves should intersect in the same point (the fixed point at 0.643) since we already found a period-1 fixed point in this case. This is shown below.

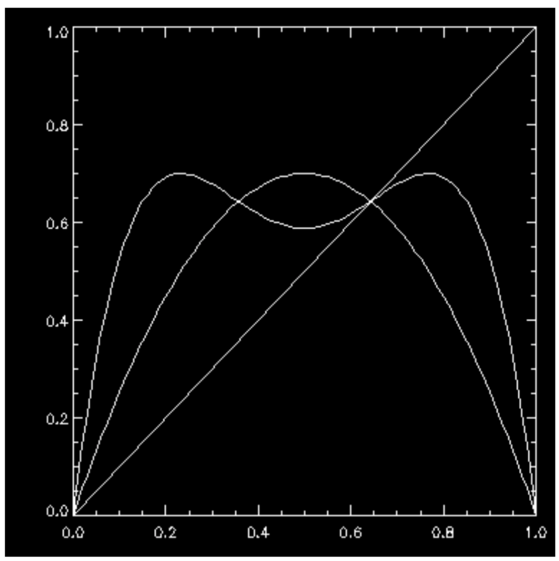

Figure 192:

Now looking in the region $\lambda > 3$, for $\lambda = 3.2$ we get the plot below

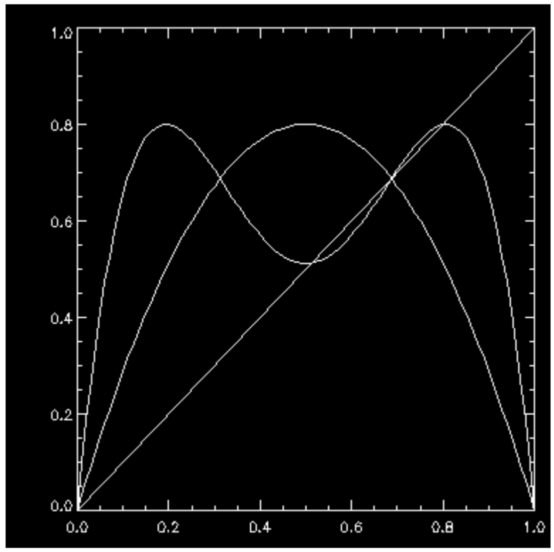

Figure 193:

There are three fixed points in this case.

The middle one is the unstable fixed point of the period-1 or single mapping at $x = 1 - (3.2)^{-1} = 0.6875$.

The two remaining fixed points of the double mapping are stable in the range $3 < \lambda < 3.449...$

Note that these two points are a single pair of period-2 points; calling them x_A and x_B, the mapping takes one into the other: $x_B = F(x_A)$ and $x_A = F(x_B)$.

This transition, as the value of λ is raised past a critical value (3 in this case), from one stable fixed point to a pair of stable period-2 points, is known as a **bifurcation** or **period doubling**.

We can see this in another way by plotting a **time series** as shown below.

The first plot is for $\lambda = 0.8$ and we clearly see the map iterate to the stable fixed point at $x = 0$.

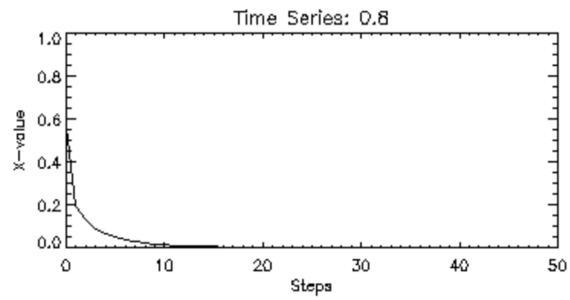

Figure 194:

The second plot is for $\lambda = 1.8$ and we clearly see the map iterate to the stable fixed point at $x = 0.44$.

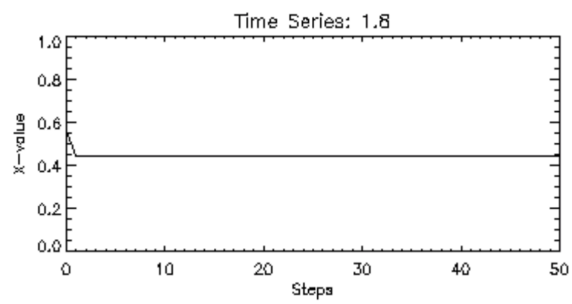

Figure 195:

The third plot is for $\lambda = 2.8$ and we clearly see the map iterate to the stable fixed point at $x = 0.64$ in agreement with the earlier result.

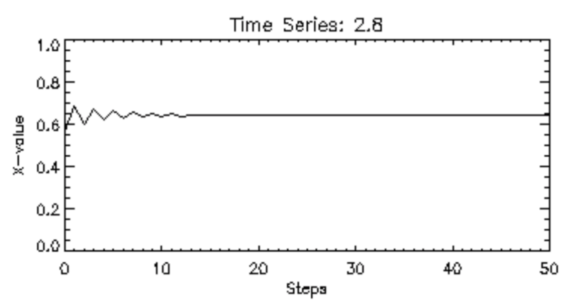

Figure 196:

The fourth plot is for $\lambda = 3.2$ and we clearly see the map iterate to two stable fixed points at $x = 0.52$ and $x = 0.80$ in agreement with the earlier result. These are period-2 points.

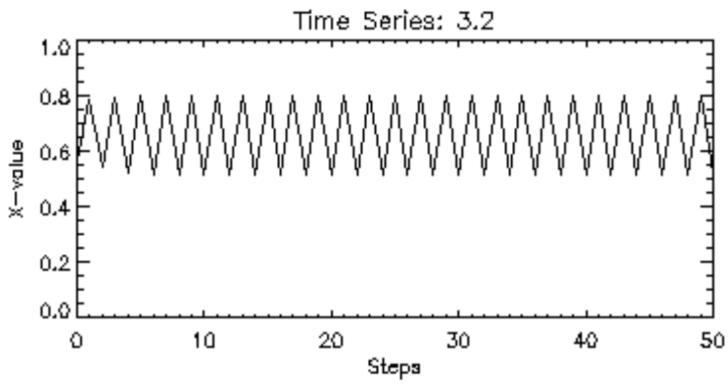

Figure 197:

The fifth plot is for $\lambda = 3.5$ and we clearly see the map iterate to four stable fixed points. These are period-4 points.

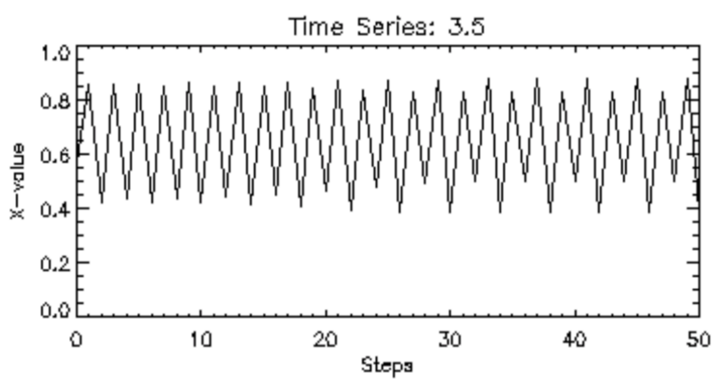

Figure 198:

The sixth plot is in a chaotic regime with no periodicity and no fixed points.

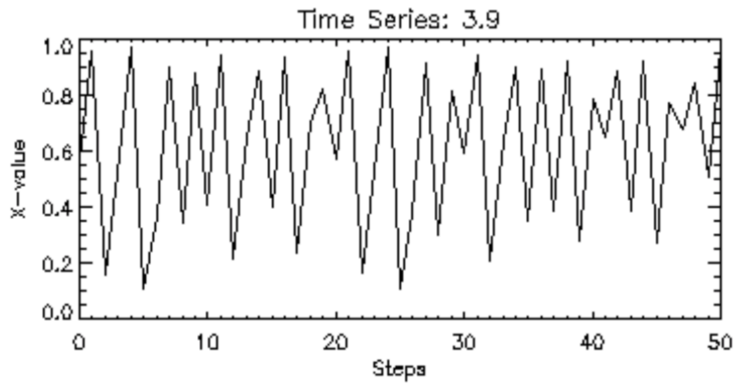

Figure 199:

As λ is raised above 3.499 a second bifurcation occurs (see period-4 points for λ = 3.5 above), that is, the pair of stable period-2 point turns into a quartet of period-4 points.

Such bifurcations occur faster and faster until an infinite number of bifurcations occur at λ = 3.56994...........

We can see the entire structure of the logistic map in the plot of fixed points below

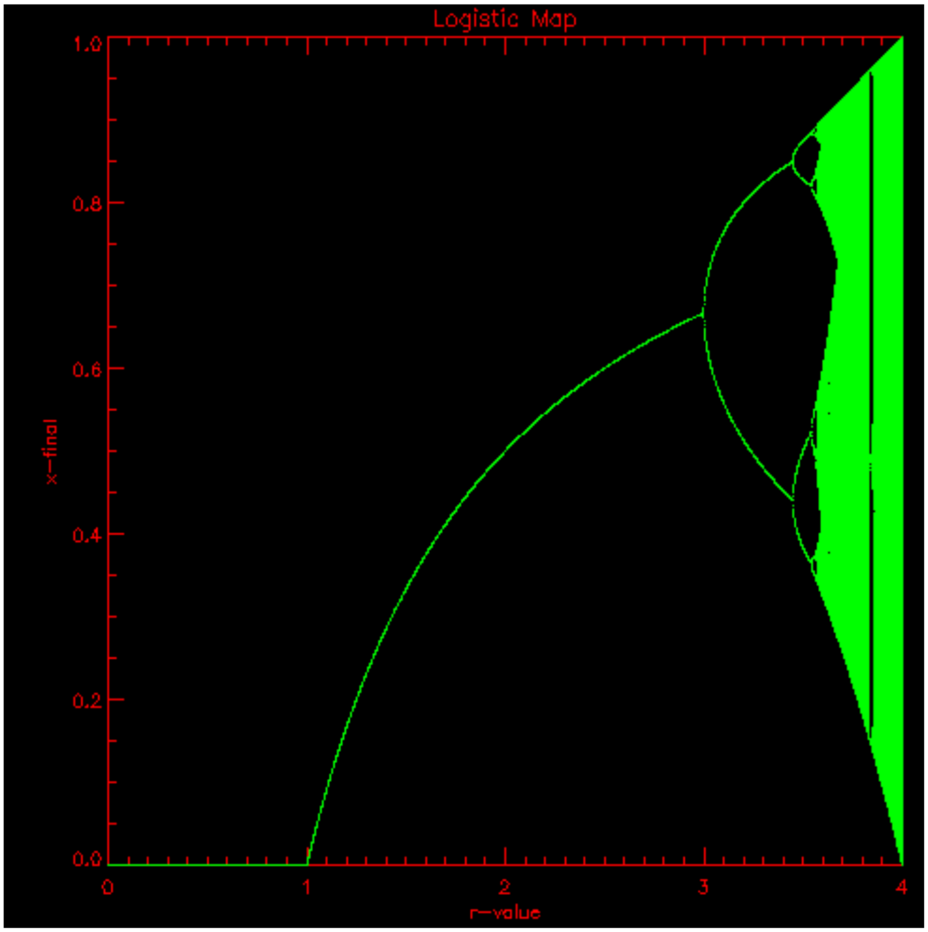

Figure 200:

Clearly, we can see the period-1, period-2, period-4, etc regions, the bifurcations or period doublings, the chaotic regions and so on.

If we blowup the region from 3.545 to 3.575 we have

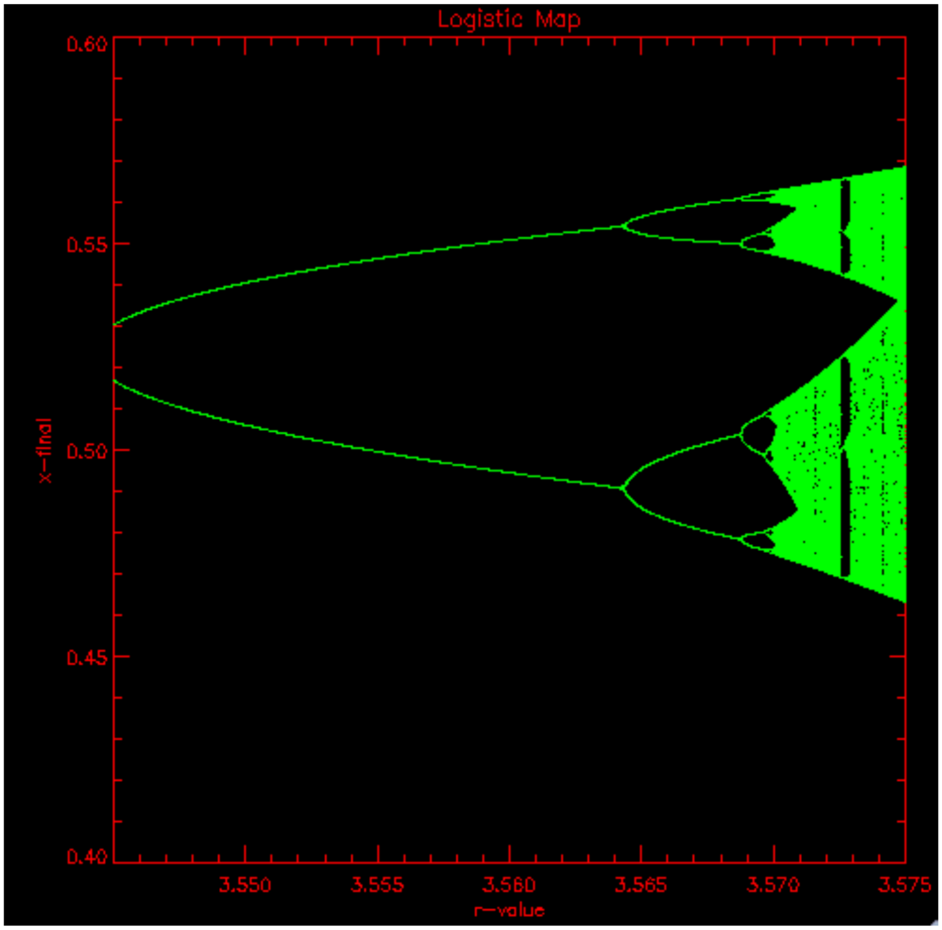

Figure 201:

Finally if we blowup the region 3.5680 to 3.5710 we have

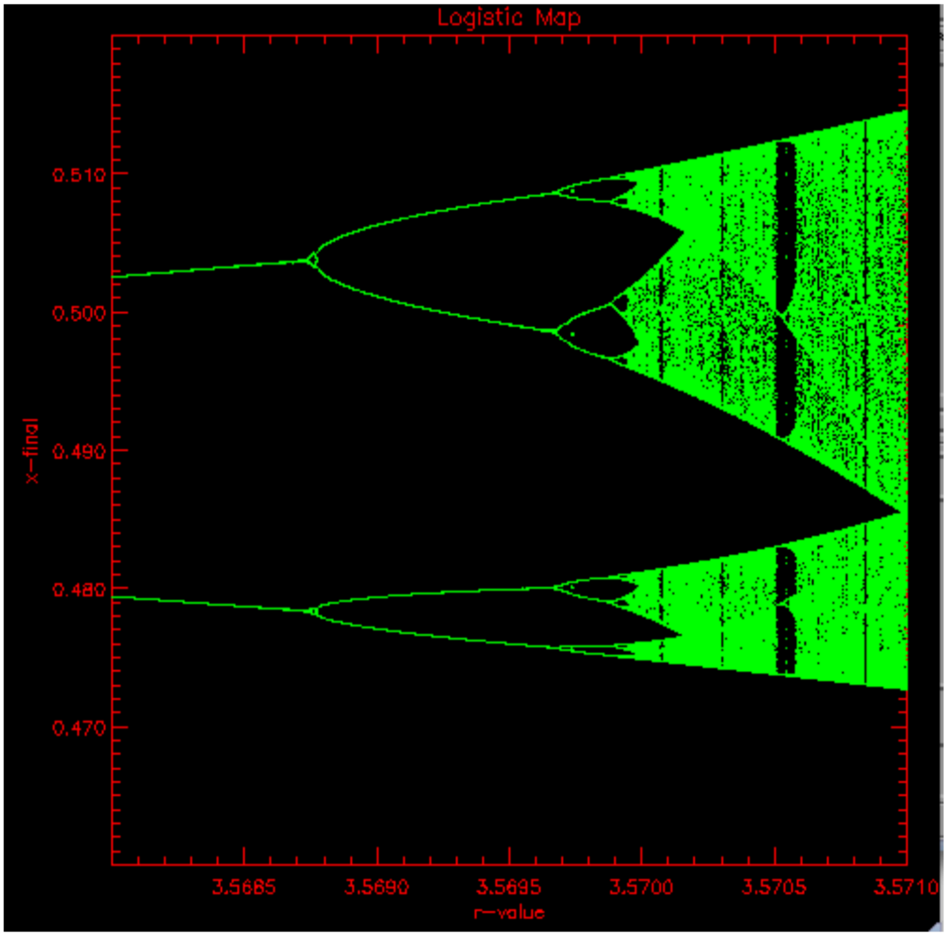

Figure 202:

We note at this time that the plots show that the entire original plot seems to repeat itself as we magnify the image.

There are similar structures within similar structures and so on. More about this later.

Denoting by λ_k the critical value of λ at which the bifurcation from a stable period-k set of points to a stable period-$(k+1)$ set occurs, it is found that

$$\lim_{k\to\infty} \frac{\lambda_k - \lambda_{k-1}}{\lambda_{k+1} - \lambda_k} = 4.669201.... \tag{9.14}$$

known as the **Feigenbaum number**.

This ratio turns out to be universal for any map with a quadratic maximum and is seen in a wide range of physical problems.

One of the conclusions one can draw from the existence of the Feigenbaum number is that each bifurcation looks similar up to a magnification factor.

This scale invariance or self- similarity plays an important role in the **transition to or onset of chaos** and in the structure of the **strange attractor** that we will discuss shortly.

We note from the pictures that above $\lambda_c = 3.56994....$ the attractor set for many (but not all) values of λ shows no periodicity at all.

For these values of λ the quadratic map exhibits **chaos** and is a strange attractor.

In the region $\lambda_c < \lambda < 4$ there are **"windows"** where attractors of small period reappear.

An important property of chaotic motion is extreme sensitivity to initial conditions (as we mentioned earlier).

To express the sensitivity quantitatively we introduce the **Lyapunov exponent**.

Consider two points in phase space separated by distance d_0 at time $t = 0$. If the motion is regular (non-chaotic) these two points will remain relatively close, separating at most according to a power of time.

In chaotic motion the two points separate exponentially with time according to

$$d(t) = d_0 e^{\lambda_L t} \tag{9.15}$$

Remember the Lorenz map video!

The parameter λ_L is the Lyapunov exponent.

If λ_L is positive the motion is chaotic.

A zero or negative coefficient indicates non-chaotic motion.

There are as many Lyapunov exponents for a particular system as there are variables.

Thus, for the logistic map there is one Lyapunov exponent.

For the logistic map a semi-logplot of the separation of two initially nearby time series trajectories in a chaotic region $\lambda = 3.64$ gives an approximate straight line with positive slope indicating exponential separation with a positive Lyapunov exponent. See plot below.

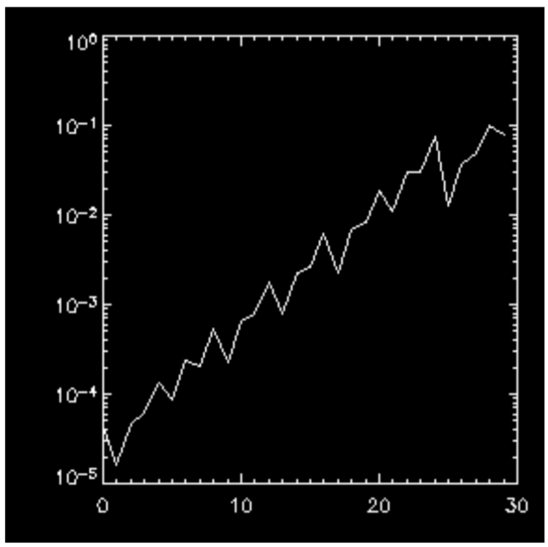

Figure 203:

Let us try to be more quantitative.

Consider a system with two initial states differing by a small amount; we call the initial states x_0 and $x_0 + \epsilon$.

We want to investigate the eventual values of x_n after n iterations from the two initial values.

The Lyapunov exponent λ_L represents the coefficient of the average exponential growth per unit time between the two states(if it exists).

After n iterations, the difference d_n between the two x_n values is, as stated above, approximately

$$d(t) = \epsilon e^{\lambda t} \tag{9.16}$$

From this equation, we can see that if λ_L is **negative**, the two orbits will eventually converge, but if it is **positive**, the nearby trajectories will diverge and chaos results.

Let us look specifically at a one-dimensional map described by $x_{n+1} = f(x_n)$.

The initial difference between the states $d_0 = \epsilon$, and after one iteration, the difference d_1 is

$$d_1 = f(x_0 + \epsilon) - f(x_0) \approx \epsilon \left.\frac{df}{dx}\right|_{x_0} \tag{9.17}$$

where the last result on the right occurs because ϵ is very small. After n iterations, the difference d_n between two initially nearby states is given by

$$d_n = f^n(x_0 + \epsilon) - f^n(x_0) = \epsilon e^{n\lambda_L} \tag{9.18}$$

where we have indicated the n^{th} iterate of the map $f(x)$ by the superscript n.

If we divide by ϵ and take the logarithm of both sides, we have

$$\ln\left(\frac{f^n(x_0 + \epsilon) - f^n(x_0)}{\epsilon}\right) = \ln\left(e^{n\lambda_L} = n\lambda_L \right. \tag{9.19}$$

Now, because ϵ is very small, we have for λ_L

$$\lambda_L = \frac{1}{n}\ln\left(\frac{f^n(x_0 + \epsilon) - f^n(x_0)}{\epsilon}\right) = \frac{1}{n}\ln\left|\left.\frac{df^n(x)}{dx}\right|_{x_0}\right| \tag{9.20}$$

The value of $f^n(x_0)$ is obtained by iterating the function $f(x_0)$ n times.

$$f^n(x_0) = f(f(.....(f(x_0))....)) \tag{9.21}$$

We use the derivative chain rule of the n^{th} iterate to obtain

$$\left.\frac{df^n(x)}{dx}\right|_{x_0} = \left.\frac{df}{dx}\right|_{x_{n-1}} \left.\frac{df}{dx}\right|_{x_{n-2}} \cdots \left.\frac{df}{dx}\right|_{x_0} \tag{9.22}$$

Taking the limit as $n \to \infty$ we finally obtain

$$\lambda_L = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| \frac{df(x_i)}{dx} \right| \tag{9.23}$$

where the sum arises from the logarithm of a product.

We plot the Lyapunov exponent as a function of r (the parameter r replaces the logistic parameter λ on the graph), the logistic parameter, below.

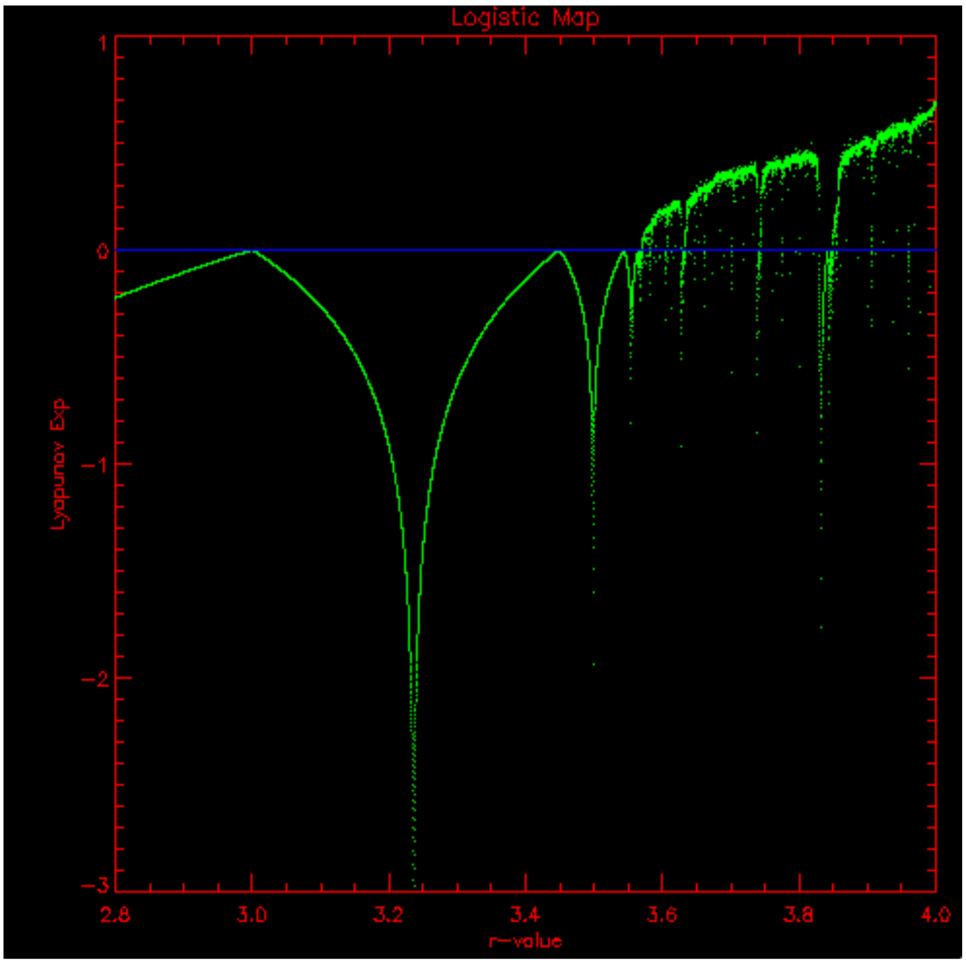

Figure 204:

If we put the Lyapunov exponent plot and the logistic map plot of fixed points on the same graph we get

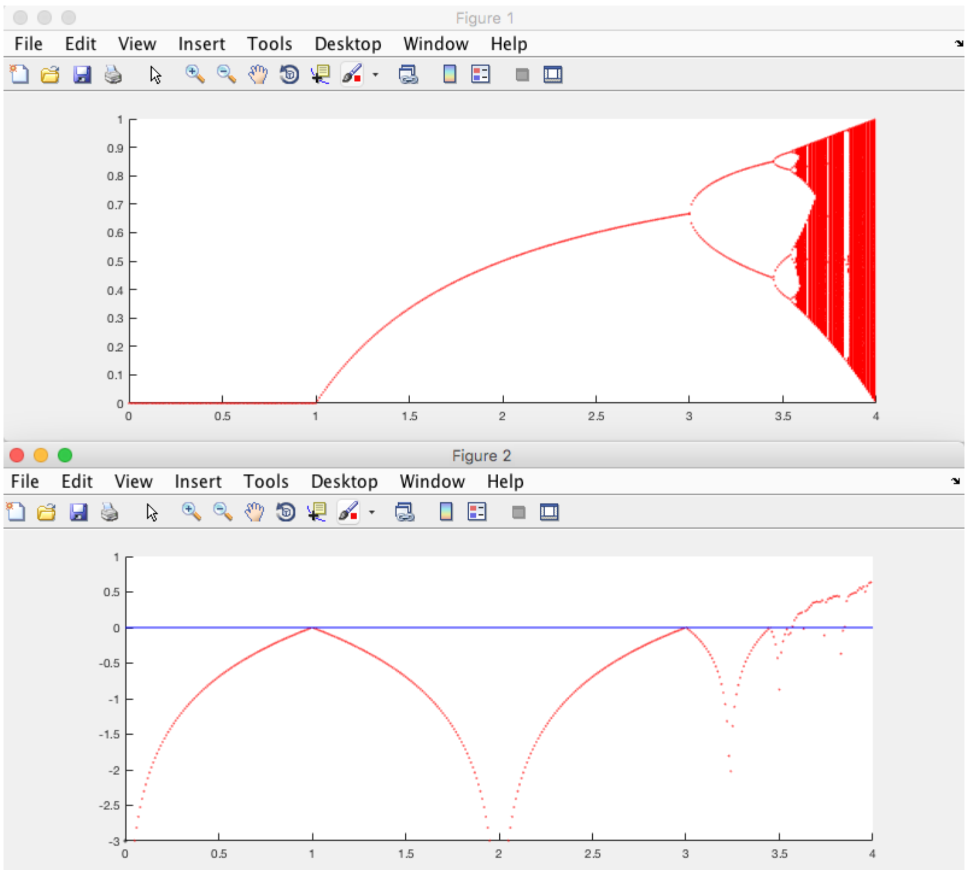

Figure 205:

Clearly, the Lyapunov exponent is negative whenever the map is stable and positive whenever the map is chaotic.

The value of λ is zero when bifurcation occurs because $|df/dx| = 1$ and the solution becomes unstable.

A superstable point occurs where $df/dx = 0$ and this implies that $\lambda_L = -\infty$.

We can see clearly from the plot that when λ_L goes above zero, there are windows of stability where λ_L goes negative for a while and period orbits occur amid the chaotic behavior.

The relatively wide window just above $\lambda = 3.8$ is apparent. Having introduced a wide variety of ideas and concepts related to the behavior of nonlinear system that exhibit chaotic motion, let us now return to a discussion of the nonlinear, damped driven oscillator, which is a real physical system.

The Nonlinear Damped Driven Oscillator

First we discuss the Simple Pendulum

The simple pendulum is shown below.

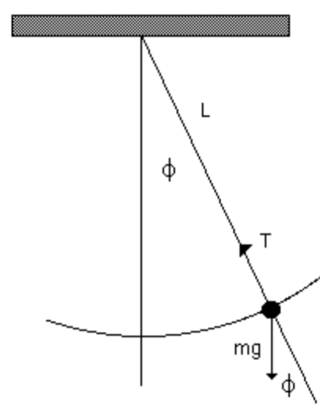

Figure 206:

Using Newton's laws, we have these equations of motion:

$$
\begin{aligned}
m\vec{a} &= ma_r\hat{r} + ma_\phi\hat{\phi} = m(\ddot{r} - r\dot{\phi}^2)\hat{r} + m(2\dot{r}\dot{\phi} + r\ddot{\phi})\hat{\phi} \\
&= (mg\cos\phi - T)\hat{r} - mg\sin\phi\hat{\phi} \qquad (9.24)
\end{aligned}
$$

or

$$mg\cos\phi - T = m(\ddot{r} - r\dot{\phi}^2) \qquad (9.25)$$

$$-mg\sin\phi = m(2\dot{r}\dot{\phi} + r\ddot{\phi}) \qquad (9.26)$$

Now $\dot{r} = \ddot{r} = 0$, $r = L$ so that we have

$$T = g\cos\phi - L\dot{\phi}^2 \tag{9.27}$$

$$p\ddot{h}i + \frac{g}{L}\sin\phi = 0 \tag{9.28}$$

The first equation simply determines the tension T.

The 2nd equation is the equation of motion of the pendulum.

If we add damping and a periodic driving force we have the equation of motion

$$m\frac{d^2\theta}{dt^2} + \alpha\frac{d\theta}{dt} + \beta\sin\theta = \gamma\cos(\omega t) \tag{9.29}$$

We rewrite this as

$$\frac{d^2\theta}{dt^2} = a\cos(\omega t) - c\frac{d\theta}{dt} - b\sin\theta \tag{9.30}$$

First we investigate the motion of this physical system in phase space.

We fixed some of the parameters

$$b = 1.0 = \frac{\beta}{m} = \frac{1}{m}\sqrt{\frac{g}{\ell}} \;\;,\;\; \ell = \text{ pendulum length}$$

$$c = 0.5 = \frac{\alpha}{m} \;\;,\;\; \alpha = \text{ damping parameter}$$

$$\omega = 0.66666666 = \text{ driving frequency}$$

We will use as a variable parameter (like λ in the logistic map) the constant a where $a = \frac{\gamma}{m}$, $\gamma =$ driving amplitude.

In particular, we will look at

a=0.90 periodic motion

a=1.07 periodic doubling

a=1.15 chaotic motion

a=1.35 periodic motion

a=1.45 periodic doubling

a=1.47 periodic doubling

a=1.50 chaotic motion

The plots below are then generated.

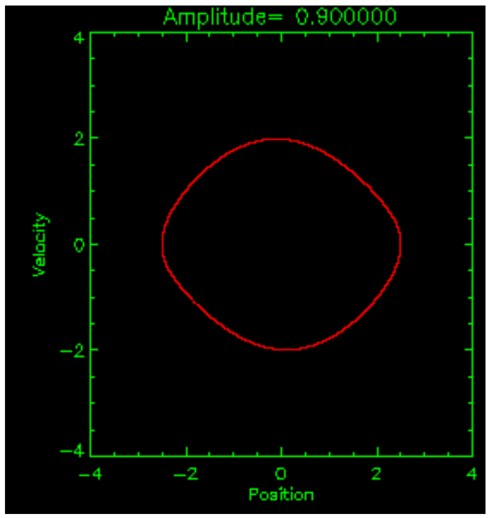

Figure 207: a=0.90 periodic motion

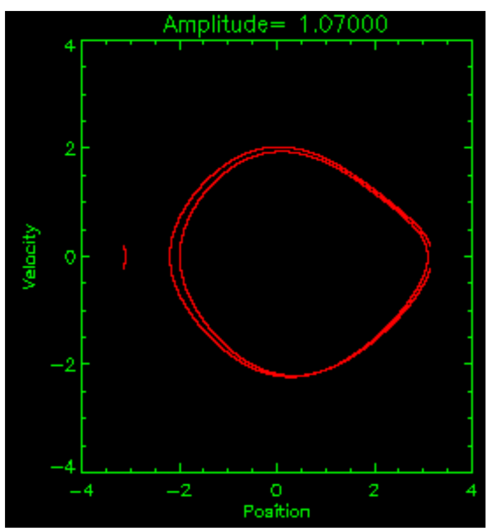

Figure 208: a=1.07 periodic doubling

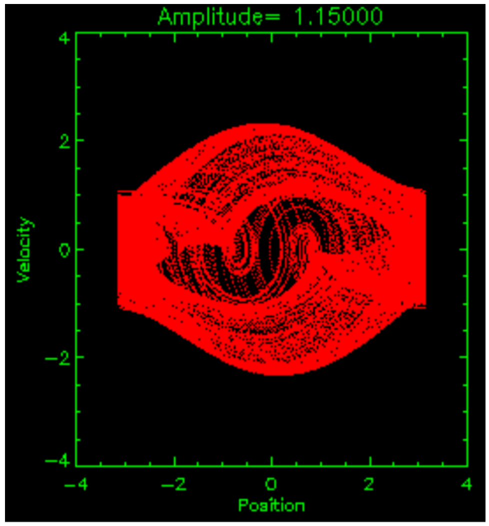

Figure 209: a=1.15 chaotic motion

If the simulation were run longer the area shown would be solid red since the oscillator never repeats the same point in phase space.

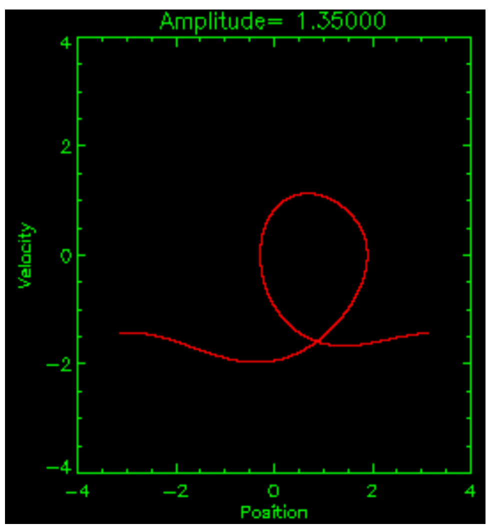

Figure 210: a=1.35 periodic motion

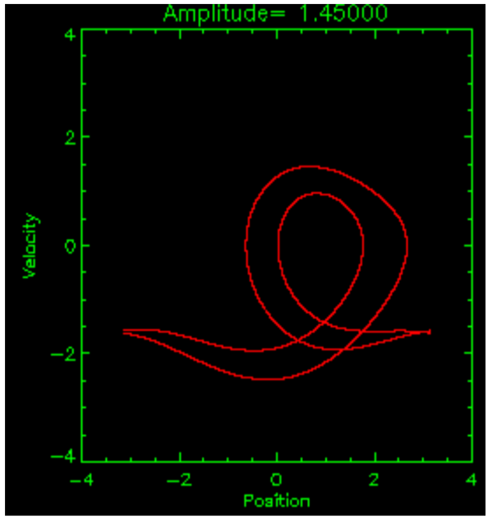

Figure 211: a=1.45 periodic doubling

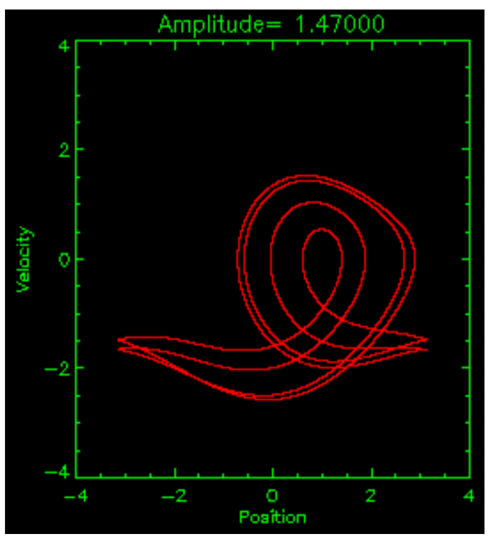

Figure 212: a=1.47 periodic doubling

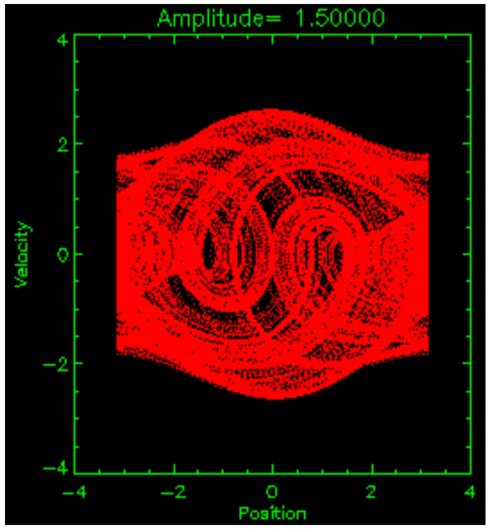

Figure 213: a=1.50 chaotic motion

Clearly, the dependence of the system motion on the amplitude is very complex and sensitive to value.

Another way to visualize the behavior of these systems is via Poincare Plots.

The Poincare plot is the same as using a stroboscope on the motion.

In this case we flash the strobe once every cycle of the driving force (frequency = ω).

We thus obtain one point per cycle in the plot.

We generate the plots shown below:

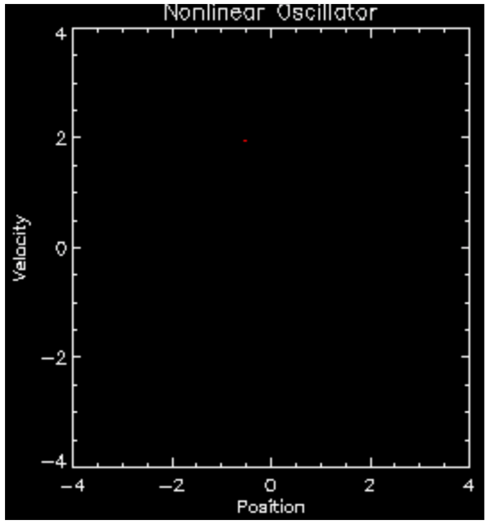

Figure 214: a=0.90 periodic motion

Since periodic only one point appears per cycle and it is the same each time.

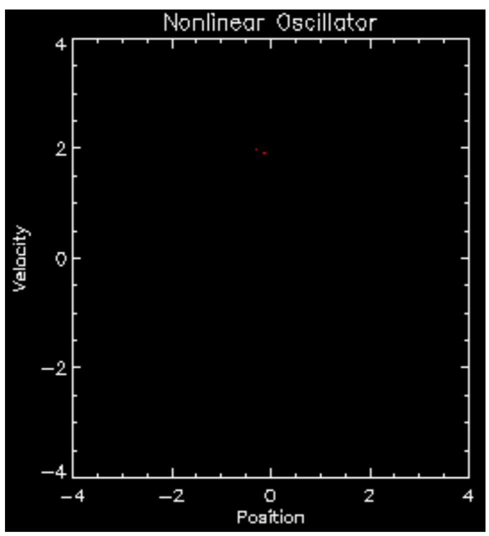

Figure 215: a=1.07 periodic doubling

Since period doubling occurred two points appear per cycle and they are the same each time.

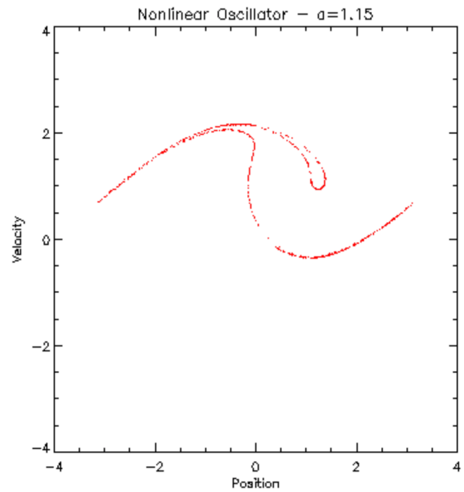

Figure 216: a=1.15 chaotic motion

The point never repeats in phase space.

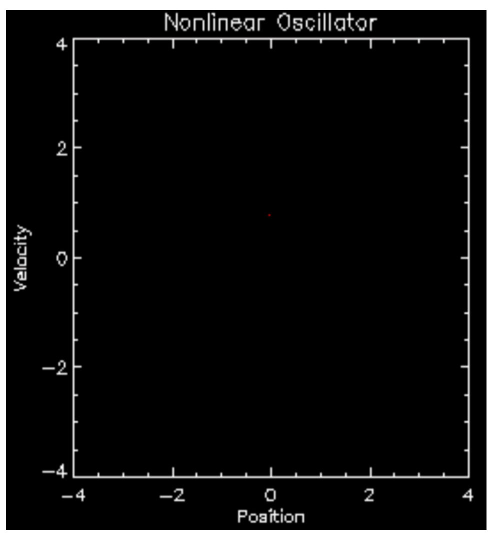

Figure 217: 1.35 periodic motion

Since periodic only one point appears per cycle and it is the same each time.

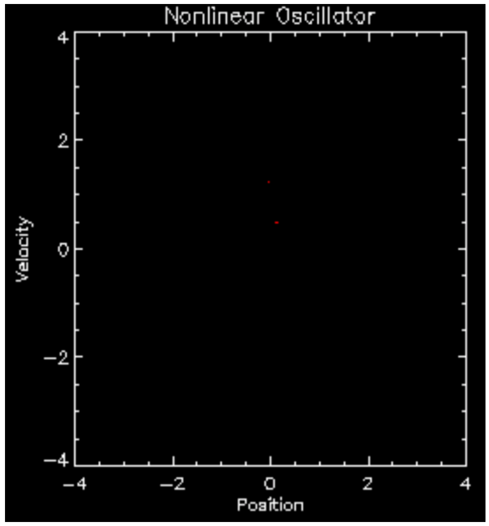

Figure 218: a=1.45 periodic doubling

Since period doubling occurred two points appear per cycle and they are the same each time.

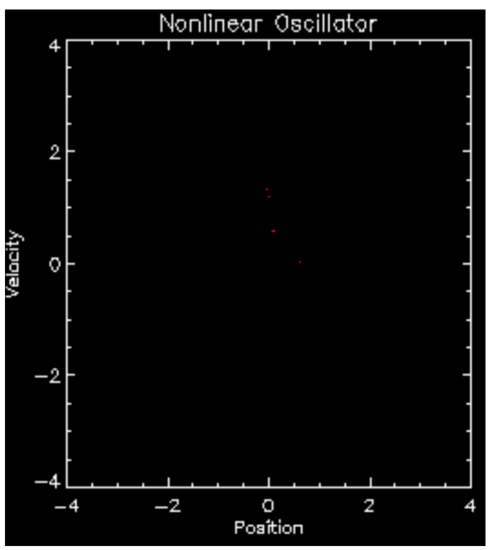

Figure 219: a=1.47 periodic doubling

Since another period doubling occurred four points appear per cycle and they are the same each time.

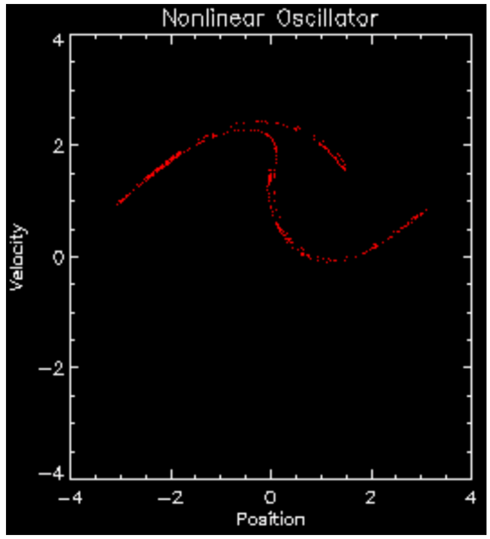

Figure 220: a=1.50 chaotic motion

The point never repeats in phase space.

In the two chaotic cases, the Poincare plot is an **attractor** with an infinite number of points. It is a **fractal curve** (more about this later) with **non-integer dimension**.

The steady state motion of the oscillator in these cases is not periodic at all; the motion is chaotic.

An attractor of this sort is known as a **strange attractor**.

Its infinity of points are arranged in a strange self-similar (fractal) manner.

Finally we can make a bifurcation plot of fixed points for the driven oscillator, where we plot the strobe values (from the Poincare plot) versus the driving amplitude.

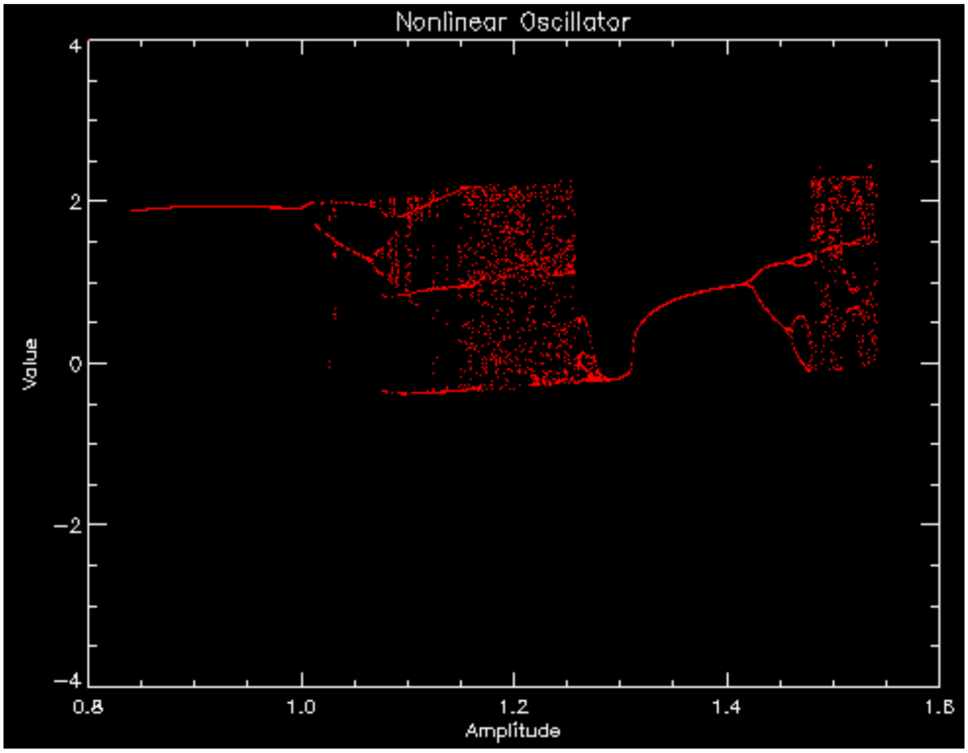

Figure 221:

We see the same structures as in the logistic map bifurcation plot.

The various periodic, period-doubling and chaotic regions are clear.

The critical points are also clear.

Thus, two systems, which really do not resemble each other in any way except that they are both nonlinear systems, exhibits very similar behaviors.

Show movie of calculation in **oscpoinbif.mpg**.

Zooming In

Let us zoom in on a strange attractor.

We consider the Poincare plot for the driven oscillator when $a = 1.50$.

The strange attractor looks like:

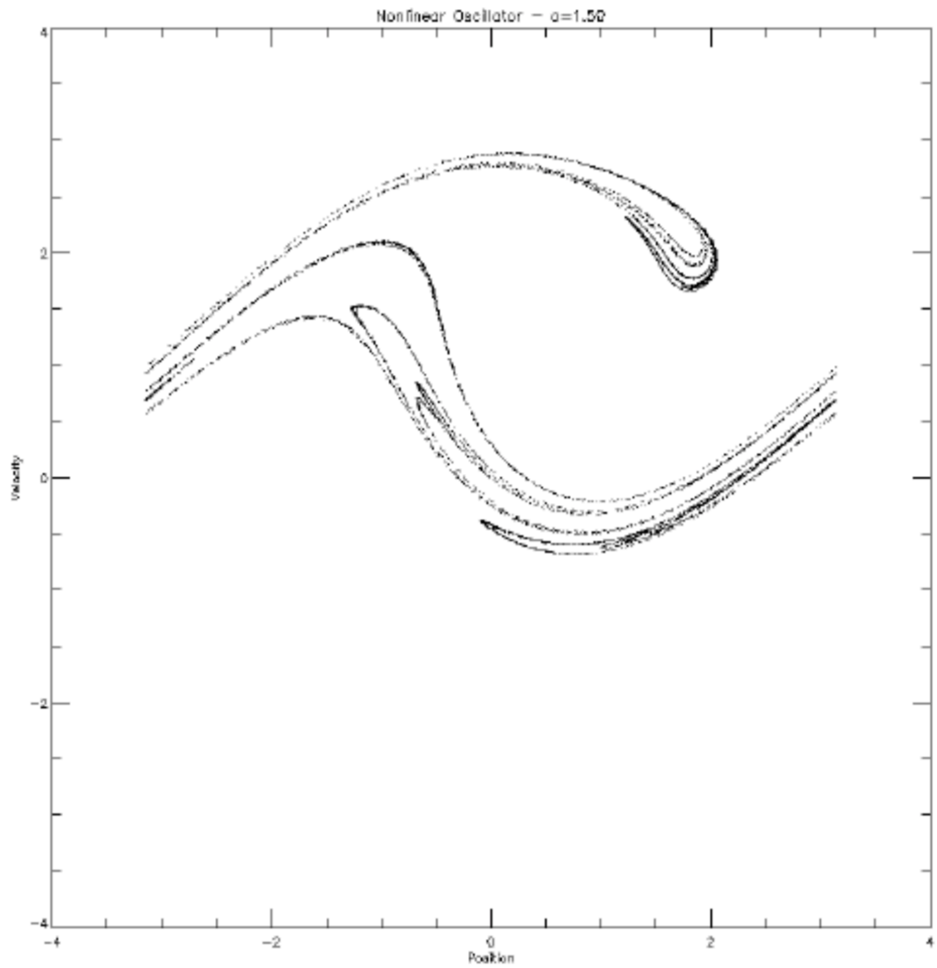

Figure 222:

We then start zooming in.....

ZOOM #1 below shows the detail present in the attractor.

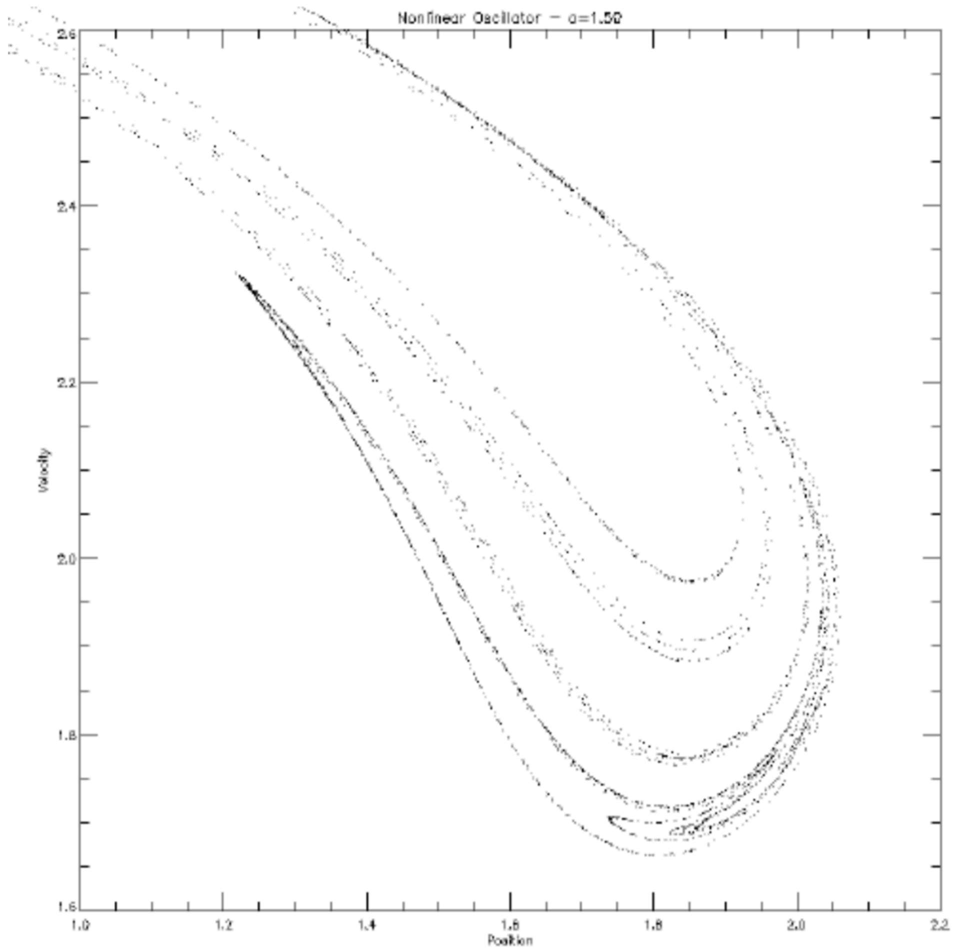

Figure 223:

ZOOM #2 below shows further detail present in the attractor.

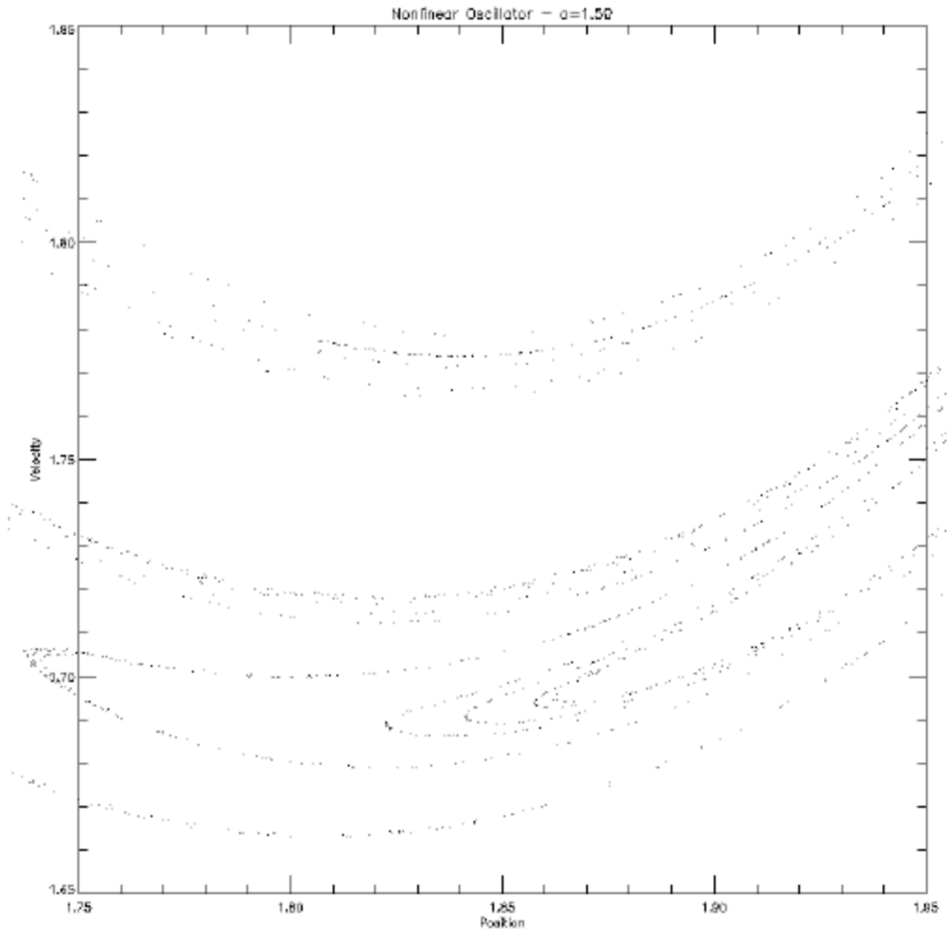

Figure 224:

These different magnifications clearly reveal the self-similar structure caused by the folding and stretching of the phase volume.

The stretching and folding processes lead to a cascade of scales: the attractor consists of an infinite number of layers.

The fine structure resembles the gross structure.

ZOOM #3 Pick another place.......

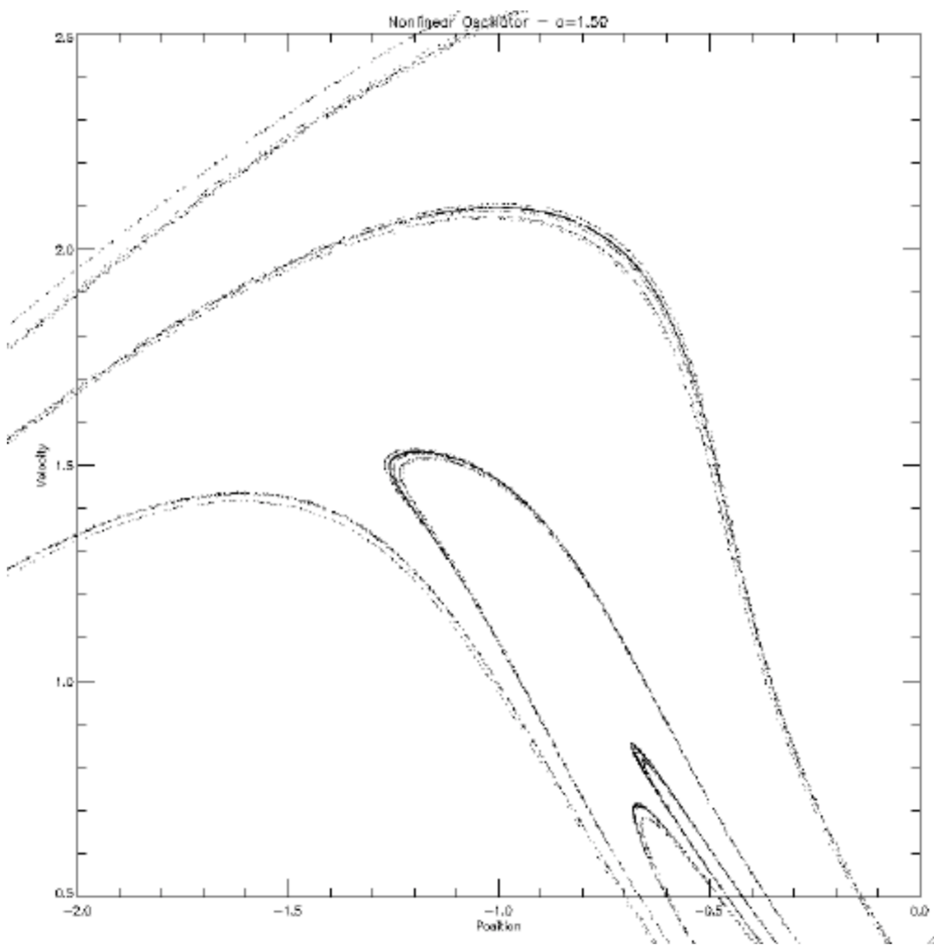

Figure 225:

and zoom again .....

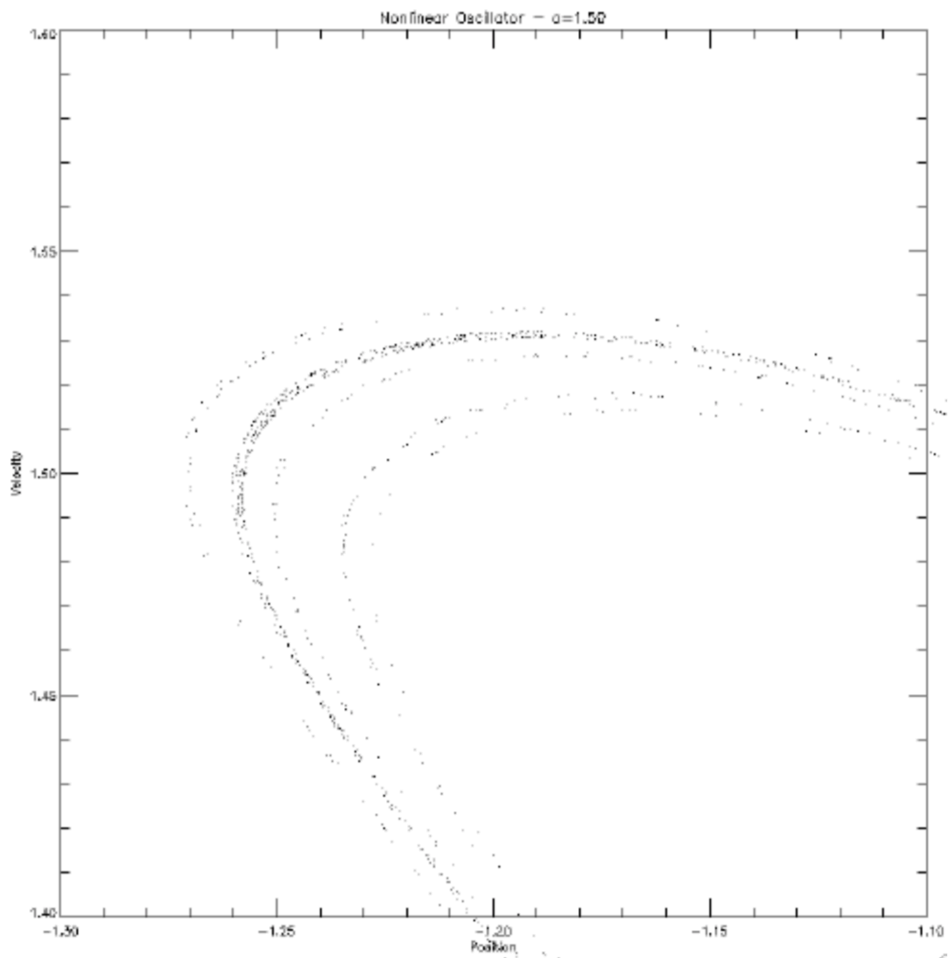

Figure 226:

Here we are only limited by the resolution of the screen and the accuracy of the calculation.

It is clear that there is complex structure in a strange attractor or fractal at all levels.

The dimension(show meaning later) of this strange attractor is D = 1.4954.......

The simulations of the chaotic attractor and its Poincare sections reveal a hierarchical structure that is uncharacteristic of ordinary compact geometrical objects.

The chaotic attractor as represented by the Poincare sections are fractals or mathematical sets of noninteger dimension(strange attractors).

Properties of a strange attractor

1. The trajectory of a strange attractor cannot intersect with itself
2. Nearby points on the attractor diverge exponentially.
3. The attractor is bounded in the phase space.
4. Even though it has an infinite number of different points the trajectory does not fill the phase space; it has zero area

A **strange attractor** is a **fractal**, and its fractal dimension is less than the dimension of its phase space.

Self-Similarity

An important (defining) property of a fractal is **self-similarity**, which refers to an infinite nesting of structure on all scales.

Strict self-similarity refers to a characteristic of a form exhibited when a substructure resembles a superstructure in the same form.

Fractals

Fractals are mathematical point sets with fractional dimension.

Fractals are not simple curves in Euclidean space.

A Euclidean object has integral dimension equal to the dimension of the space the object is being drawn in.

If a Euclidean line connects 2 points in 3-dimensions and also stays within a finite volume, then the length of the line is finite.

A fractal is a line that can stay within a finite volume, but still have an infinite length.

This implies that it has a complex structure at all levels of magnification.

In comparison, a Euclidean line will eventually look like a straight line at some magnification level.

To describe this property of a fractal, we need to generalize the usual concept of dimension.

The dimensionality D of a space is usually defined as the number of coordinates needed to determine a unique point in that space.

When defined in this way, the only allowed values for D are the non-negative integers 0,1,2,.....

There are several ways that the concept of dimension can be redefined so that it still takes on non-negative integer values when considering the systems described above, but can also take on non-negative real number values.

We will adopt a simplified version of the **"Hausdorff dimension"** called the **"box counting"** or **"capacity"** dimension.

In the box-counting scheme, the dimension of an object is determined by asking how many **"boxes"** are needed to cover the object.

Here the appropriate **"boxes"** for coverage are lines, squares, cubes, etc.

The size of the boxes is repeatedly decreased and the dimension of the object is determined by how the number of covering boxes scales with the length of the side of the box.

In one dimension, we consider a line of length ℓ.

We need

1 box(a line) of length ℓ

2 boxes of length $\ell/2$

4 boxes of length $\ell/4$

....

2^m boxes of length $\ell/2^m$

....

If we define δ_m = length of the m^{th} box, then $\delta_m = \ell/2^m$.

Thus, the number of boxes $N(\delta_m)$ scales as $N(\delta_m) = \frac{\ell}{\delta_m}$.

Note that in 1 dimension, the power of $\frac{\ell}{\delta_m}$ is 1.

In two dimensions, we consider a square of side ℓ.

We need

1 box(a square) of area ℓ^2

4 boxes of area $(\ell/2)^2$

....

2^m boxes of area $(\ell/2^m)^2$

....

Thus, the number of boxes $N(\delta_m)$ scales as $N(\delta_m) = \left(\frac{\ell}{\delta_m}\right)^2$.

Note that in 2 dimensions, the power of $\frac{\ell}{\delta_m}$ is 2.

Generalizing to D integer dimensions, we have

$$N(\delta_m) = \left(\frac{\ell}{\delta_m}\right)^D \tag{9.31}$$

Some algebra yields

$$\log\left(N(\delta_m)\right) = \log\left(\frac{\ell}{\delta_m}\right)^D = D\log\left(\frac{\ell}{\delta_m}\right) = D(\log\left(\ell\right) - \log\left(\delta_m\right))$$

$$D = \frac{\log\left(N(\delta_m)\right)}{\log\left(\ell\right) - \log\left(\delta_m\right)}$$

We then define the dimension D by

$$D = \lim_{m\to\infty} \frac{\log(N(\delta_m))}{\log(\ell) - \log(\delta_m)}$$

As $m \to \infty$, the length of the system $\log(\ell)$ becomes negligible and we have

$$D = \lim_{m\to\infty} \frac{\log(N(\delta_m))}{\log(\frac{1}{\delta_m})}$$

or letting $\delta_m = \epsilon$

$$D = \lim_{\epsilon\to 0} \frac{\log(N(\epsilon))}{\log\left(\frac{1}{\epsilon}\right)}$$

where $N(\epsilon)$ = the number of p-dimensional cubes of side ϵ needed to completely cover the set.

This is the Hausdorff or fractal dimension D of a set of points in a p-dimensional space.

Examples

(1) A **single point**: only one cube is required.

This means that $N(\epsilon) = 1$ or $\log(N(\epsilon)) = 0$ or $D = 0$(as expected).

(2) **A line of length ℓ in a plane:**

The number of cubes of side ϵ required = number of line segments of length ϵ.

Therefore, $N(\epsilon) = \frac{\ell}{\epsilon}$ and

$$D = \lim_{\epsilon\to 0} \frac{\log\left(\frac{\ell}{\epsilon}\right)}{\log\left(\frac{1}{\epsilon}\right)} = \lim_{\epsilon\to 0} \frac{\log\left(\frac{1}{\epsilon}\right) + \log(\ell)}{\log\left(\frac{1}{\epsilon}\right)} = \lim_{\epsilon\to 0}\left(1 + \frac{\log(\ell)}{\log\left(\frac{1}{\epsilon}\right)}\right) = 1$$

as expected.

(3) **A square of area ℓ^2:**

The number of cubes of side ϵ required $=$ number of squares of side ϵ.

Therefore, $N(\epsilon) = \frac{\ell^2}{\epsilon^2}$ and

$$D = \lim_{\epsilon \to 0} \frac{\log\left(\frac{\ell^2}{\epsilon^2}\right)}{\log\left(\frac{1}{\epsilon}\right)} = \lim_{\epsilon \to 0} \frac{2\log\left(\frac{1}{\epsilon}\right) + \log\left(\ell^2\right)}{\log\left(\frac{1}{\epsilon}\right)} = \lim_{\epsilon \to 0} \left(2 + \frac{\log\left(\ell\right)}{\log\left(\frac{1}{\epsilon}\right)}\right) = 2$$

as expected.

So this definition of dimension works for Euclidean objects and clearly makes sense.

(4) **The Koch Snowflake:** The Koch curve is constructed by recursion as exhibited in the first two figures below.

Figure 227: L = 1 , #(L) = 1 , total length = 1

Figure 228: L = 1/3 , #(L) = 4 , total length = (4/3)

At each step the middle-third of each segment is replaced with a "V" shaped bulge.

Continuing to iterate:

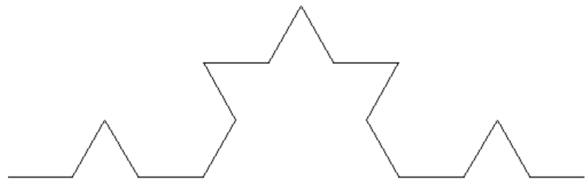

Figure 229: L=$(1/3)^2$, #(L)=4^2, total length = $(4/3)^2$

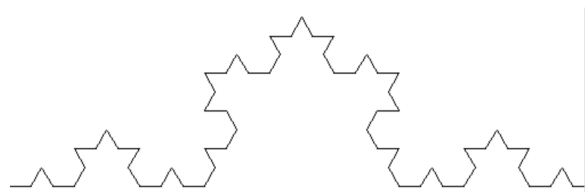

Figure 230: L =$(1/3)^3$, #(L)= 4^3 , total length = $(4/3)^3$

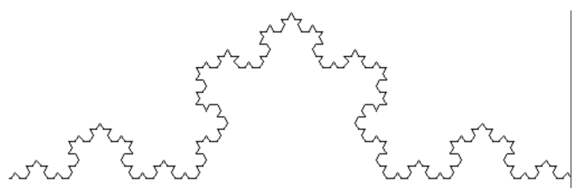

Figure 231: L =$(1/3)^4$, #(L)= 4^4 , total length = $(4/3)^4$

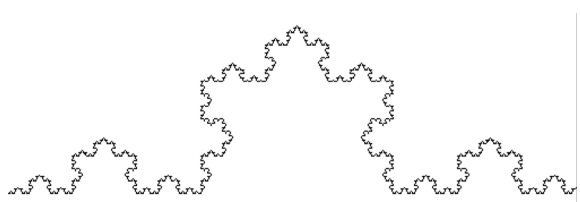

Figure 232: L =$(1/3)^5$, #(L)= 4^5 , total length = $(4/3)^5$

This curve turns out to have infinite length(see below), while enclosing (together with its natural base: the original line segment) a finite area.

The dimension is given by

$$D = \lim_{n\to\infty} \frac{\log(4^n)}{\left(\left(\frac{1}{3}\right)^n\right)} = \lim_{n\to\infty} \frac{n\log(4)}{n\log(3)} = \frac{\log(4)}{\log(3)} = 1.26.......$$

Clearly, the final length of the fractal line,

$$\lim_{n\to\infty}\left(\frac{4}{3}\right)^n \to \infty$$

is infinite even though it stays within a finite area of the 2-dimensional Euclidean plane.

A strange attractor is such a fractal curve.

And now for a digression.....

Fractal Dimension Program

We now illustrate a program that carries out the box dimension calculations on complex fractal structures.

The ideas of covering a fractal with boxes is shown below.

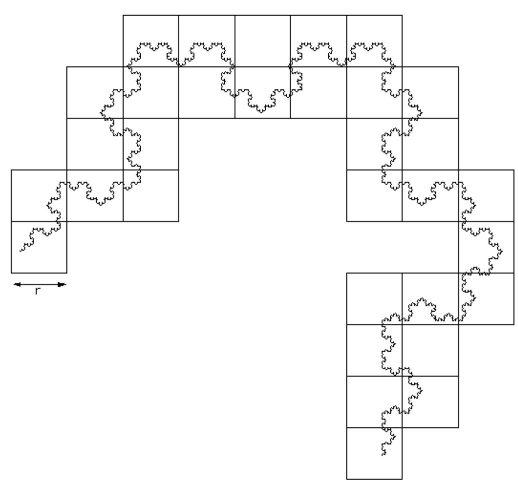

Figure 233: Koch curve covered by boxes of size r

The program simply chooses a "box" size, covers the fractal and counts the number of boxes.

It then reduces the box size and repeats until the box is very small.

At that point it can recognize the limit of plot $\log N(r)$ versus $\log r$ and determine the slope which corresponds to the dimension.

If we apply the program to the iterated Koch snowflake which looks like

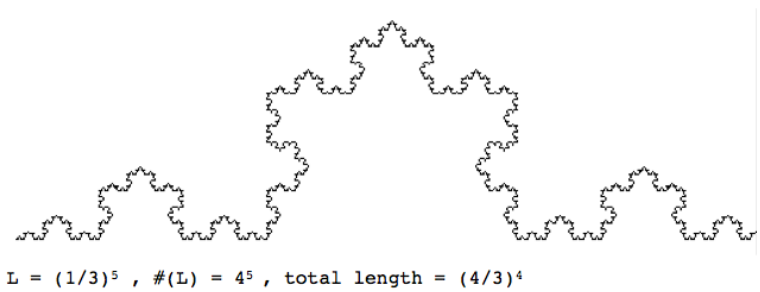

Figure 234: Koch snowflake curve

we get the result

```
reality:fdc_mac boccio$ ./fdc  -i snowflake.tga -b1 200 -b2 8 -n 1 -bf 1.2 -p
FDC - Fractal Dimension Calculator
Initialising GLUT
Creating windows
Creating menus (Right mouse button)
Reading file "snowflake.tga"
   Image dimensions: 1188 x 695 x 24
Creating screen version
   Full screen size: 2560 x 1440
   Display size:     1188 x 695
   Scale factor:     1
Processing starting
   Box size:  200 [  1], Total count = 15
   Box size:  166 [  2], Total count = 19
   Box size:  138 [  3], Total count = 22
   Box size:  114 [  4], Total count = 30
   Box size:   95 [  5], Total count = 31
```

```
   Box size:   79 [  6], Total count = 45
   Box size:   65 [  7], Total count = 58
   Box size:   54 [  8], Total count = 70
   Box size:   45 [  9], Total count = 82
   Box size:   37 [ 10], Total count = 109
   Box size:   30 [ 11], Total count = 115
   Box size:   24 [ 12], Total count = 183
   Box size:   20 [ 13], Total count = 225
   Box size:   16 [ 14], Total count = 308
   Box size:   13 [ 15], Total count = 424
   Box size:   10 [ 16], Total count = 547
   Box size:    8 [ 17], Total count = 747
Processing finished
Estimated fractal dimension:  1.211
```

The box process is illustrated below:

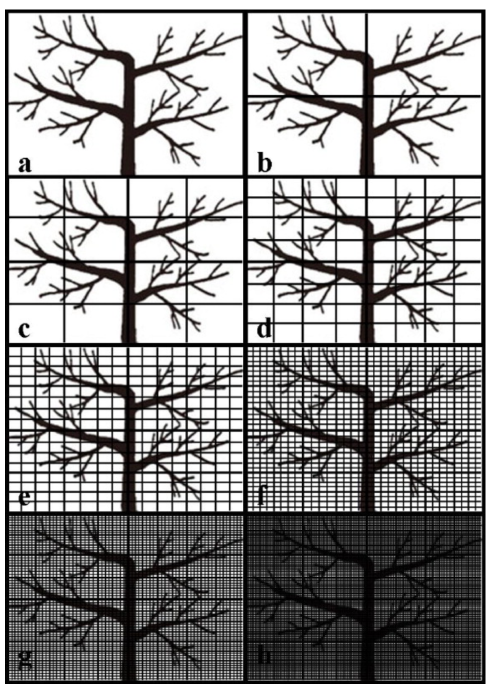

Figure 235: Boxing illustrated

Even more dramatic we apply the program to a fractal tree(created on the computer) as shown below

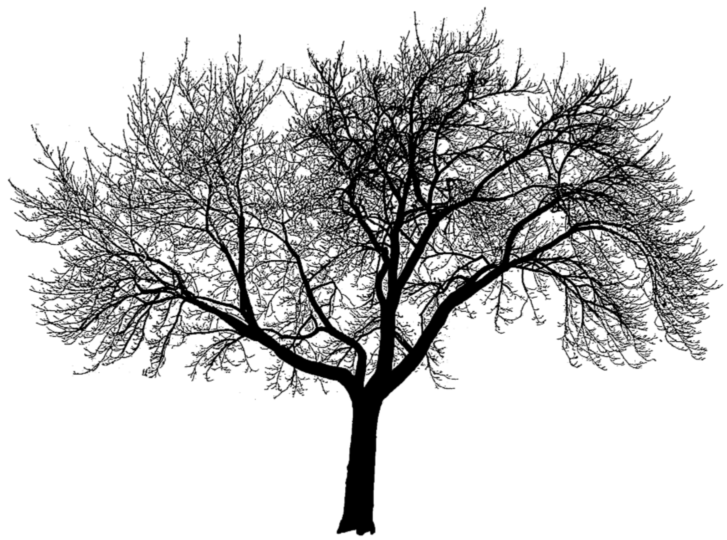

Figure 236: fractal tree

and get the result

```
reality:fdc_mac boccio$ ./fdc  -i tree.tga -b1 200 -b2 8 -n 1 -bf 1.2 -p
FDC - Fractal Dimension Calculator
Initialising GLUT
Creating windows
Creating menus (Right mouse button)
Reading file "tree.tga"
   Image dimensions: 2048 x 1536 x 24
Creating screen version
   Full screen size: 2560 x 1440
   Display size:     1024 x 768
   Scale factor:     2
Processing starting
   Box size:  200 [  1], Total count = 47
   Box size:  166 [  2], Total count = 63
   Box size:  138 [  3], Total count = 85
```

```
   Box size:  114 [  4], Total count = 120
   Box size:   95 [  5], Total count = 170
   Box size:   79 [  6], Total count = 232
   Box size:   65 [  7], Total count = 333
   Box size:   54 [  8], Total count = 467
   Box size:   45 [  9], Total count = 653
   Box size:   37 [ 10], Total count = 928
   Box size:   30 [ 11], Total count = 1384
   Box size:   24 [ 12], Total count = 2100
   Box size:   20 [ 13], Total count = 2947
   Box size:   16 [ 14], Total count = 4511
   Box size:   13 [ 15], Total count = 6638
   Box size:   10 [ 16], Total count = 10842
   Box size:    8 [ 17], Total count = 16332
Processing finished
Estimated fractal dimension:  1.833
```

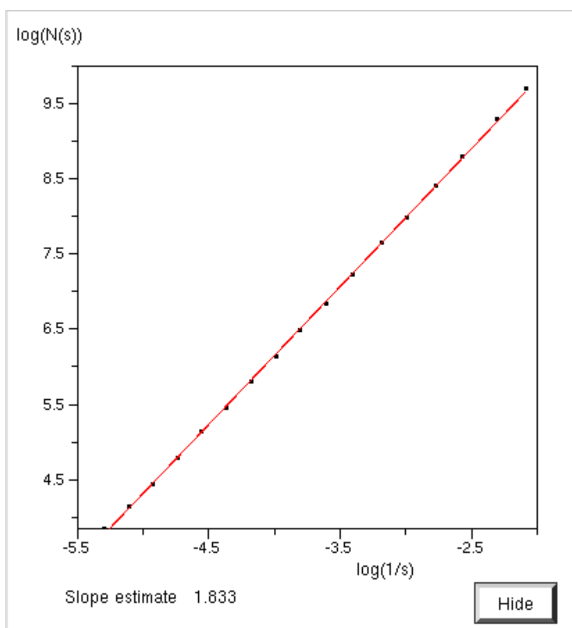

Figure 237: log(N)/log(r) plot

What about a picture of a real trees? I found a picture of a real tree and put it into the program

Figure 238: real tree

and the result is

```
reality:fdc_mac boccio$ ./fdc  -i tree-569.tga -b1 200 -b2 8 -n 1 -bf 1.2 -p
FDC - Fractal Dimension Calculator
Initialising GLUT
Creating windows
Creating menus (Right mouse button)
Reading file "tree-569.tga"
   Image dimensions: 450 x 326 x 24
Creating screen version
   Full screen size: 2560 x 1440
   Display size:     450 x 326
   Scale factor:     1
Processing starting
   Box size:  200 [  1], Total count = 6
   Box size:  166 [  2], Total count = 7
   Box size:  138 [  3], Total count = 11
   Box size:  114 [  4], Total count = 17
```

```
   Box size:   95 [  5], Total count = 19
   Box size:   79 [  6], Total count = 23
   Box size:   65 [  7], Total count = 34
   Box size:   54 [  8], Total count = 50
   Box size:   45 [  9], Total count = 57
   Box size:   37 [ 10], Total count = 79
   Box size:   30 [ 11], Total count = 105
   Box size:   24 [ 12], Total count = 172
   Box size:   20 [ 13], Total count = 227
   Box size:   16 [ 14], Total count = 341
   Box size:   13 [ 15], Total count = 486
   Box size:   10 [ 16], Total count = 761
   Box size:    8 [ 17], Total count = 1073
Processing finished
Estimated fractal dimension:  1.563
```

The log(N)/log(r) graph looks like:

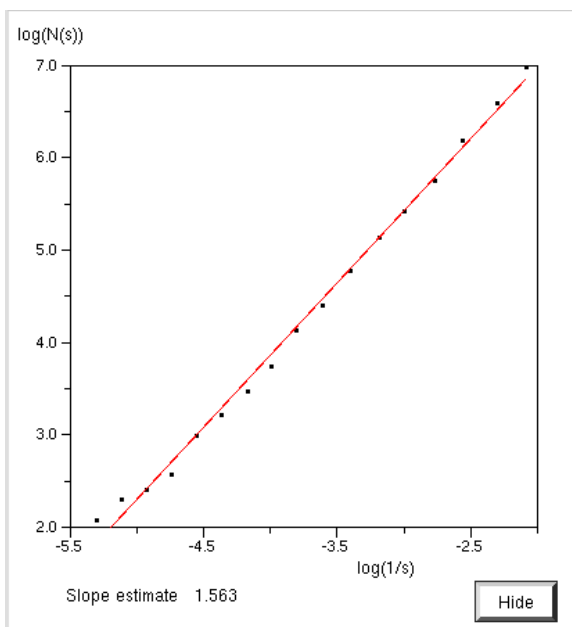

Figure 239: log(N)/log(r) plot

10. Rotations

Part 1

We now turn to rotational motion.

Our goal is to understand the general rotational motion of a rigid body under any combination of applied forces.

Chasle's theorem states that

> **Any displacement of a rigid body can be decomposed into two independent motions:**
>
> **A single translation of the CM + a single rotation about the CM.**

We already know how to handle the translational motion.

For the rotational motion, we start by considering a particle and then move on to systems of particles and finally to rigid bodies.

Angular Momentum of a Particle

Formal definition: for a particle of momentum $\vec{p}$ and position vector $\vec{r}$ (wrt some coordinate system), its angular momentum $\vec{L}$ is defined as

$$\vec{L} = \vec{r} \times \vec{p} \tag{10.1}$$

Since its properties are so strange and we lack any intuition about its behavior, we will discuss this new quantity in excruciating detail.

Direction: $\vec{r}$ and $\vec{p}$ determine a plane as shown (say the $x - y$ plane)

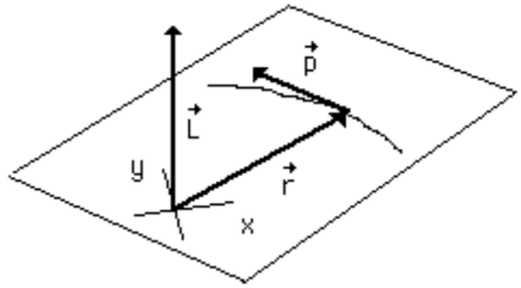

Figure 240:

If $\vec{r} = r\hat{r}$ and $\vec{p} = p_r\hat{r} + p_\theta\hat{\theta}$, then using (from earlier definitions) $\hat{r} \times \hat{r} = 0$, $\hat{\theta} \times \hat{\theta} = 0$ and $\hat{r} \times \hat{theta} = \hat{k} = -\hat{\theta} \times \hat{r}$, we have

$$\vec{L} = \vec{r} \times \vec{p} = (r\hat{r}) \times (p_r\hat{r} + p_\theta\hat{\theta}) = rp_\theta\hat{r} \times \hat{\theta} = rp_\theta\hat{k} \tag{10.2}$$

So the direction of the angular momentum is **perpendicular** to the plane containing the two vectors $\vec{r}$ and $\vec{p}$.

What happens if the particle was moving in the opposite direction?

There is a so-called R(ight)H(and)-rule implied

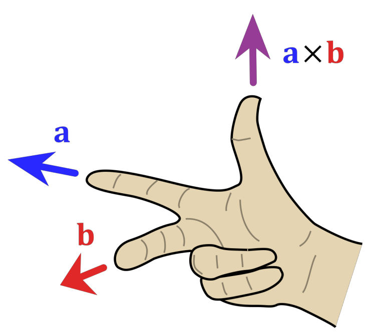

Figure 241:

which follows from the mathematical rules

$$\begin{aligned} \hat{i} \times \hat{j} = \hat{k}\hat{j} \times \hat{k} = \hat{i} \quad & \hat{k} \times \hat{i} = \hat{j} \\ \hat{r} \times \hat{\theta} = \hat{k}\hat{\theta} \times \hat{k} = \hat{k} \quad & \hat{k} \times \hat{r} = \hat{\theta} \end{aligned} \tag{10.3}$$

Now consider the geometry below:

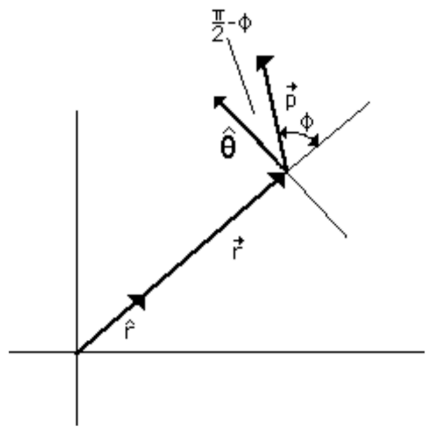

Figure 242:

so that

$$p_\theta = p\cos\left(\frac{\pi}{2} - \phi\right) = p\sin\phi$$

and the magnitude of the angular momentum vector is $L = rp\sin\phi$ as it should be.

Angular Momentum of a Particle Moving in a Straight Line

Considering the diagram below:

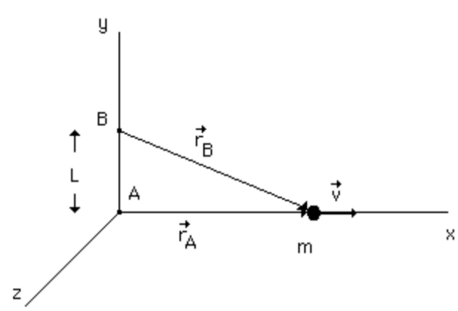

Figure 243:

we have

$$\vec{v} = v\hat{i} \;\; , \;\; \vec{r}_A = x\hat{i} \;\; , \;\; \vec{r}_B = x\hat{i} - L\hat{j}$$

Thus,

$$\vec{L}_A = \vec{r}_A \times m\vec{v} = mxv\hat{i} \times \hat{i} = 0 \tag{10.4}$$

$$\vec{L}_B = \vec{r}_B \times m\vec{v} = m(x\hat{i} - L\hat{j}) \times v\hat{i} = -mLv\hat{j} \times \hat{i} = mLv\hat{k} \tag{10.5}$$

So we find that the angular momentum depends on the choice of origin and that having a nonzero angular momentum does not require rotational motion!

Example - Conical Pendulum

Consider a concical pendulum as illustrated below

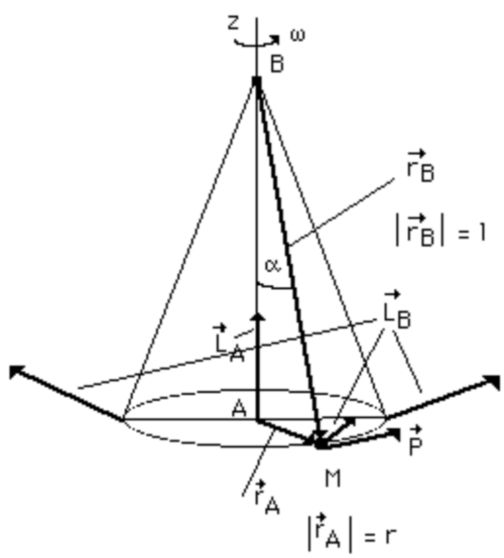

Figure 244:

For point A, we have

$$\vec{p} = Mv\hat{\theta} = Mr\omega\hat{\theta}$$

$$\vec{L}_A = \vec{r}_A \times \vec{p} = (r\hat{r}) \times (Mr\omega\hat{\theta}) = Mr^2\omega\hat{k}$$

The angular momentum wrt A is a **constant vector**!

For point B, we have

$$\vec{p} = Mv\hat{\theta} = Mr\omega\hat{\theta}$$

$$\vec{L}_B = \vec{r}_B \times \vec{p} = (r\hat{r} - L\cos\alpha\hat{k}) \times (Mr\omega\hat{\theta}) = Mr^2\omega\hat{k} + Mr\omega L\cos\alpha\hat{r}$$

It points as shown in the diagram (at different times).

It is clearly not a constant vector!

Its magnitude is constant and equal to

$$L_B = \sqrt{M^2r^4\omega^2 + M^2r^2\omega^2L^2\cos^2\alpha} \quad , \quad r = L\sin\alpha$$

$$L_B = \sqrt{M^2r^2\omega^2L^2\sin^2\alpha + M^2r^2\omega^2L^2\cos^2\alpha} = MrL\omega$$

It is always perpendicular to both $\vec{p}$ and $\vec{r}_B$ and this direction is changing as the particle moves around the circle.

If we think in terms of $x-$, $y-$, and $z-$components, the $z-$component is a constant but the $x-$ and $y-$components are not.

Why do we get this result?

What is different about points A and B?

Torque

Let us now consider the time rate of change of the angular momentum.

$$\begin{aligned}\frac{d\vec{L}}{dt} &= \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} \\ &= m\vec{v} \times \vec{v} + \vec{r} \times \vec{F} \\ &= \vec{r} \times \vec{F} = \text{ torque } = \vec{\tau} \end{aligned} \tag{10.6}$$

The same considerations about directions and magnitudes apply to the torque as they did to the angular momentum.

The direction of the torque is perpendicular to the plane containing the two vectors $\vec{r}$ and $\vec{F}$.

Examples:

Consider the figure below:

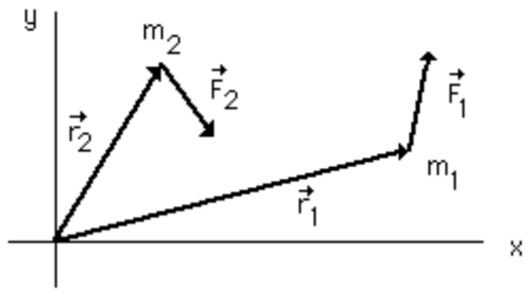

Figure 245:

The RH-rule says that the torque $\vec{\tau}_1 = \vec{r}_1 \times \vec{F}_1$ points out of the paper and the torque $\vec{\tau}_2 = \vec{r}_2 \times \vec{F}_2$ points into the paper(torques wrt thhe origin).

Torque and force are **very different** quantities.

Consider the three cases below.

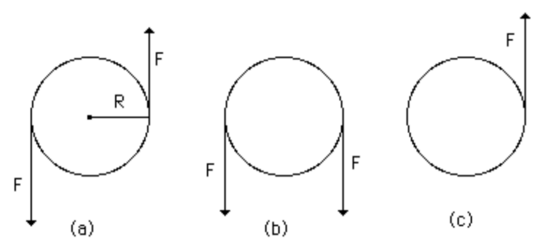

Figure 246:

Case (a): total force $=0$; total torque = $2RF$ (out of paper)

Case (b): total force = $2F$ (down); total torque $= 0$

Case (c): total force = F (up); total torque $=RF$ (out of paper)

We see that the torque is always perpendicular to its associated force.

We also see that we can have a net nonzero torque when the total force is zero and a zero net torque when the total force is not zero.

The equation of motion for torque

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau} \tag{10.7}$$

shows that if the torque = 0, the angular momentum is a constant vector, i.e., we have conservation of angular momentum ... **the 3rd of the great conservation laws of mechanics**.

The real power of this statement will not be clear until we apply these new ideas to extended bodies made up of many particles (instead of just a single particle).

Example: Consider a particle moving under the action of a central force $\vec{F}(\vec{r}) = f(r)\hat{r}$.

We get

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = rf(r)\hat{r} \times \hat{r} = 0 \tag{10.8}$$

Hence the angular momentum of the particle is a **constant vector** if the force is central.

Since the angular momentum starts off perpendicular to the plane containing the position and momentum vectors and its direction must be constant, it remains perpendicular to this same plane for all time and hence for all central forces the **motion remains in that plane!**

As we will see this is **a very powerful tool!!!!**

If we define the direction perpendicular to the plane (the direction of the angular momentum) as the z-direction, then L_z = constant,

$$L_z = m|\vec{r} \times \vec{v}| = m|(r\hat{r}) \times (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})| = mr^2\dot{\theta}|\hat{r} \times mr^2\dot{\theta}\hat{\theta}| \tag{10.9}$$

Now as the particle moves, in a time Δt the position vector sweeps out area as shown

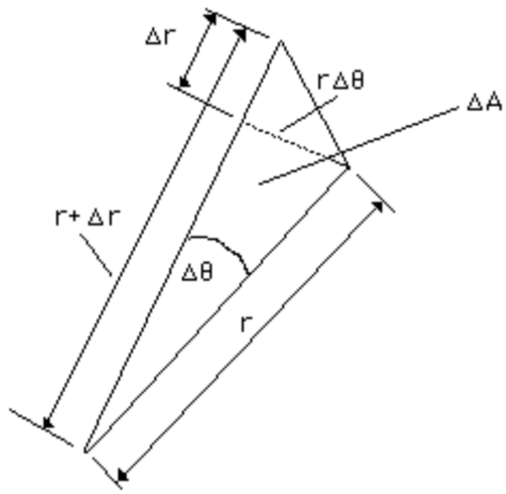

Figure 247:

We have

$$\Delta A = \frac{1}{2}r^2\Delta\theta + \frac{1}{2}r\Delta r\Delta\theta \approx \frac{1}{2}r^2\Delta\theta \tag{10.10}$$

Why? So we get ,

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{L_z}{2m} = \text{ constant} \tag{10.11}$$

This will later turn out to be one of **Kepler's laws** for orbital motion!

Now let us return to the particle moving in a straight line and conical pendulum examples and look at the torque(s) involved.

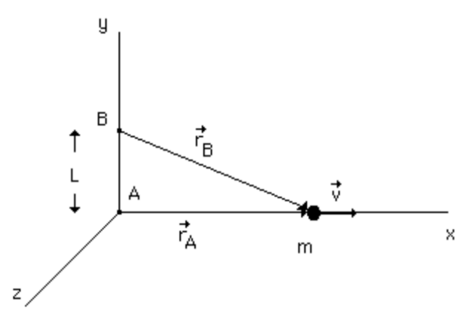

Figure 248:

We had

$$\vec{L}_A = 0 \quad , \quad \vec{L}_B = mLc\hat{k}$$

Suppose that the particle is slowing down because of a friction force $\vec{f} = -f\hat{i}$.

There is no torque about origin A due to this frictional force so $\vec{L}_A$ is constant and $= 0$.

The torque about B is given by

$$\vec{\tau}_B = \vec{r}_B \times \vec{f} = (x\hat{i} - l\hat{j}) \times (-f\hat{i}) = -fL\hat{k}$$

As the block slows down, the angular momentum direction does not change, but its magnitude decreases. We have

$$\vec{L}_B = mLv\hat{k} \to \Delta\vec{L}_B = mL\Delta v\hat{k}$$

$$\frac{\Delta\vec{L}_B}{\Delta t} = L\left(m\frac{\Delta v}{\Delta t}\right)\hat{k} \to \frac{d\vec{L}_B}{dt} = L\left(m\frac{dv}{dt}\right)\hat{k} = L(-f)\hat{k} = -mf\hat{k} = \vec{\tau}_B$$

Remember that we must always use the **same origin** for both the torque and the angular momentum.

For the conical pendulum we have a more complicated situation

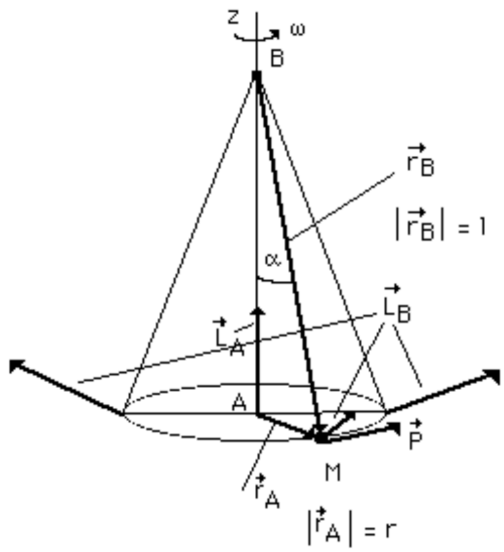

Figure 249:

The forces on the mass M are the tension T in the string and the weight.

In the vertical direction

$$T\cos\alpha = Mg = 0 \text{ (no acceleration)}$$

Therefore, the total (net) force on M is

$$\vec{F} = -T\sin\alpha\hat{r}$$

The torque about A is $= 0$ (force and position vectors in same direction) and hence the angular momentum about A is constant (as we found earlier).

With respect to B, we have

$$\begin{aligned}\vec{\tau}_B &= \vec{r}_B \times \vec{F} = (r\hat{r} - K\cos\alpha\hat{k}) \times (-T\sin\alpha\hat{r}) \\ &= LT\cos\alpha\sin\alpha\hat{\theta} = MgL\sin\alpha\hat{\theta}\end{aligned}$$

It is tangent to the circular path (in the plane of motion).

Now we have from earlier

$$\vec{L}_B = Mr^2\omega\hat{k} + Mr\omega L\cos\alpha\hat{r}$$

and it is easy to see that

$$\frac{d\vec{L}_B}{dt} = \frac{d}{dt}\left(Mr^2\omega\hat{k} + Mr\omega L\cos\alpha\hat{r}\right) = Mr\omega L\cos\alpha\frac{d}{dt}(\hat{r})$$
$$=$$

and using

$$T\sin\alpha = Mr\omega^2 = Mg\tan\alpha$$

we get

$$\frac{d\vec{L}_B}{dt} = Mr\omega^2 L\cos\alpha\hat{\theta} = MgL:\sin\alpha\hat{\theta} = \vec{\tau}_B$$

as required.

Torque Due to Gravity

Let us figure this out for an extended body from first principles.

Consider the situation below:

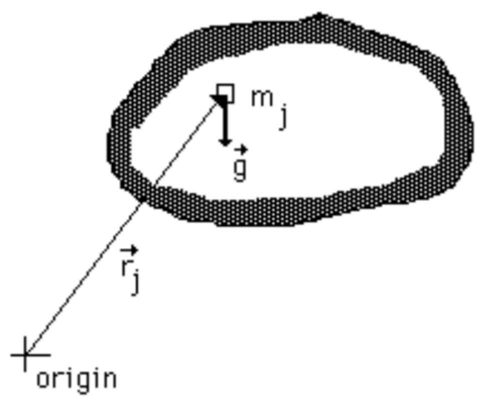

Figure 250:

The torque about the origin $\vec{\tau}_j$ due to thej^{th} element of mass is

$$\vec{\tau}_j = \vec{r}_j \times m_j\vec{g} \tag{10.12}$$

The total torque is the **vector sum** of all the individual torques.

$$\vec{\tau} = \sum_j \vec{\tau}_j = \sum_j \vec{r}_j \times m_j\vec{g} = \left(\sum_j m_j\vec{r}_j\right) \times \vec{g} \tag{10.13}$$

Now the definition of the CM position is

$$\vec{R} = \frac{\sum_j m_j \vec{r}_j}{\sum_j m_j} = \frac{\sum_j m_j \vec{r}_j}{M} \tag{10.14}$$

Therefore,

$$\vec{\tau} = \left(\sum_j m_j \vec{r}_j \right) \times \vec{g} = M\vec{R} \times \vec{g} = \vec{R} \times \vec{W} \tag{10.15}$$

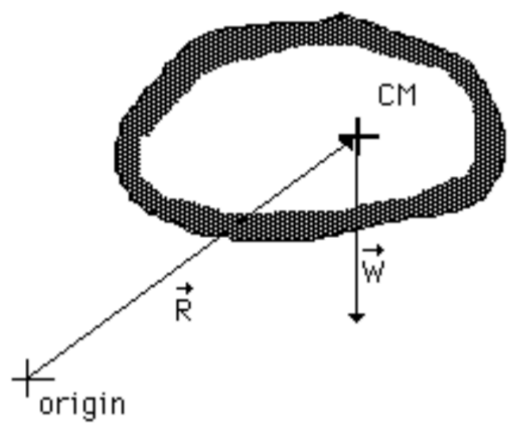

Figure 251:

This says that the torque due to gravity about **any origin** is

$$\vec{\tau} = \vec{R} \times \vec{W} \tag{10.16}$$

The shape or density of the body does not matter!!

Example:

A 3000 lb car is parked on a 30° slope, facing uphill.

The center of mass of the car is halfway between the front and rear wheels and is 2 ft above the ground. The wheels are 8 ft apart. Find the normal force exerted by the road on the front wheels and on the rear wheels.

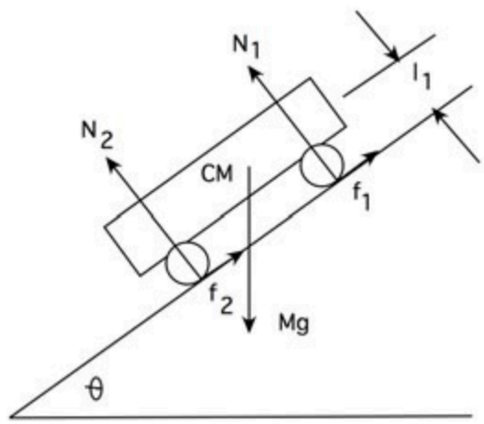

Figure 252:

The system is in equilibrium.

We have from balanced forces

$$N_1 + N_2 = Mg\cos\theta$$

$$f_1 + f_2 = Mg\sin\theta$$

and from the torque about the CM

$$0 = N_1\ell_2 - N_2\ell_2 + f_1\ell_1 + f_2\ell_1 = (N_1 - N_2)\ell_2 + (f_1 + f_2)\ell_1$$

or

$$0 = (N_1 - N_2)\ell_2 + (Mg\sin\theta)\ell_1$$

We then have

$$N_1 + N_2 = Mg\cos\theta$$

$$N_1 - N_2 = -(Mg\sin\theta)\frac{\ell_1}{\ell_2}$$

Solving we get

$$N_1 = \frac{1}{2}Mg\left(\cos\theta - \frac{\ell_1}{\ell_2}\sin\theta\right)$$

$$N_2 = \frac{1}{2}Mg\left(\cos\theta + \frac{\ell_1}{\ell_2}\sin\theta\right)$$

Putting in numbers we get

$$N_1 = 924\,lb \quad , \quad N_2 = 1674\,lb$$

Part 2

Fixed Axis Rotation

Our discussion of the rotational motion of rigid bodies will center mainly around a special case .... rotation about a fixed axis.

Fixed axis here means that the direction of the axis is always in the same direction .... that is, the axis can translate, but not change direction.

An example is a wheel rolling down the road in a straight line without wobble or twisting.

We choose this fixed direction to be the z-axis.

We consider only rigid bodies ... this means that the spatial relationships between all the mass elements making up the body is fixed.

When a rigid body rotates about an axis, every particle in the body remains at a fixed distance from the axis(each moves in a circle).

Suppose we choose the coordinate system shown below,

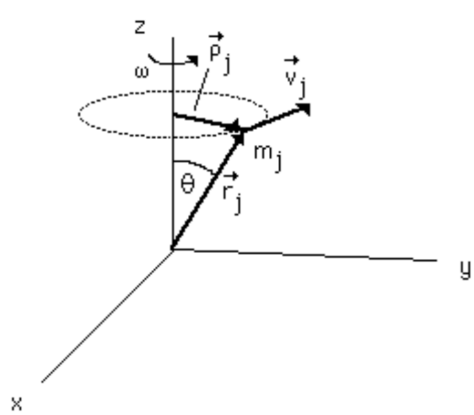

Figure 253:

where we have chosen the origin of the coordinate system to lie on the axis of rotation.

For rotation about the z-axis with angular velocity ω in the sense shown(CCW) we introduce the angular velocity vector $\vec{\omega} = \omega\hat{k}$, where the direction of this vector correlates (CCW = z-up) to the sense of rotation about the axis.

We then have for the j^{th} particle in the body (as it rotates about the z-axis)

$$\vec{v}_j = \vec{\omega} \times \vec{r}_j \tag{10.17}$$

which gives a velocity vector in the correct direction as shown.

Note that because the body is rigid, **all particles** making it up will have the **same angular velocity vector**.

We then have the result that

$$v_j = |\vec{v}_j| = |\vec{\omega} \times \vec{r}_j| = \omega r_j \sin\theta = \omega\rho_j \tag{10.18}$$

In this case, we have

$$\rho_j^2 = x_j^2 + y_j^2 \tag{10.19}$$

since the z-axis is the axis of rotation.

The distance from the origin is

$$r_j^2 = x_j^2 + y_j^2 + z_j^2 \tag{10.20}$$

The angular momentum of the j^{th} particle is given by

$$\begin{aligned}\vec{L}_j &= \vec{r}_j \times m_j\vec{v}_j = m_j(x_j\hat{\imath} + y_j\hat{\jmath} + z_j\hat{k}) \times (v_{j,x}\hat{\imath} + v_{j,x}\hat{\jmath}) \\ &= m_j(-z_jv_{j,y}\hat{\imath} + z_jv_{j,x}\hat{\jmath} + (x_jv_{j,y} - y_jv_{j,x})\hat{k})\end{aligned} \tag{10.21}$$

Because the axis is fixed in direction during our discussions, we need only consider the z-component of the angular momentum (the component along the axis of rotation).

That will change if the axis of rotation is not fixed (covered in a more advance course).

$$L_{j,z} = m_j(x_jv_{j,y} - y_jv_{j,x}) \tag{10.22}$$

Now the vectors $\vec{\rho}_j$ and $\vec{v}_j$ are perpendicular to each other and in the $x - y$ plane as shown

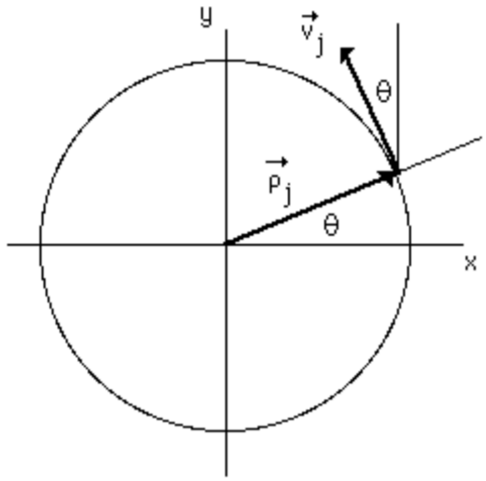

Figure 254:

and hence we can write

$$\begin{aligned} L_{j,z} &= m_j(x_j v_{j,y} - y_j v_{j,x}) = m_j \rho_j v_j (\cos\theta\cos\theta - \sin\theta(-\sin\theta)) \\ &= m_j \rho_j v_j = m_j \rho_j^2 \omega \end{aligned} \tag{10.23}$$

The z-component of the total angular momentum is the **algebraic sum** of the z-components for all the particles

$$L_z = \sum_j L_{j,z} = \left(\sum_j m_j \rho_j^2\right)\omega = I_z \omega = I\omega \tag{10.24}$$

where

$$I = \sum_j m_j \rho_j^2 \tag{10.25}$$

is a **purely geometrical quantity** called the **moment of inertia**.

I depends both on the distribution of mass and the location of the axis of rotation.

For a continuously distributed mass(rather than particles glued together), we replace the sum by an integral to get

$$I = \int \rho^2 dm = \int (x^2 + y^2) dm = \int (x^2 + y^2)\mu dV \tag{10.26}$$

where μ is the density and dV is a volume element.

This can be complicated to evaluate for non-symmetrical bodies.

For bodies with a high degree of symmetry, however, the evaluation of the integral is straightforward.

Examples:

Uniform Thin Hoop

Since the hoop (shown below) is thin, we have

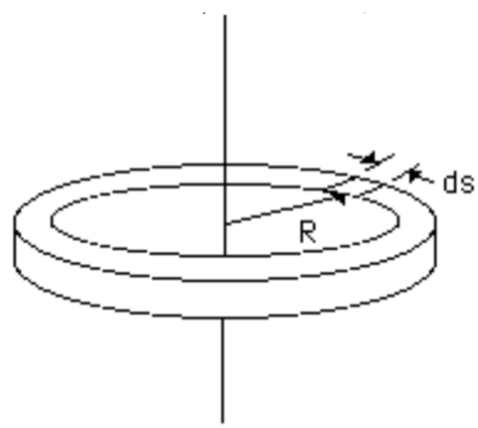

Figure 255:

We then have

$$dm = \lambda ds \quad , \quad \lambda = \frac{M}{2\pi R}$$

$$I = \int R^2 dm = \lambda R^2 \int_o^{2\pi R} ds = \frac{M}{2\pi R} R^2 (2\pi R) = MR^2$$

Uniform Disk

A uniform disk as shown below

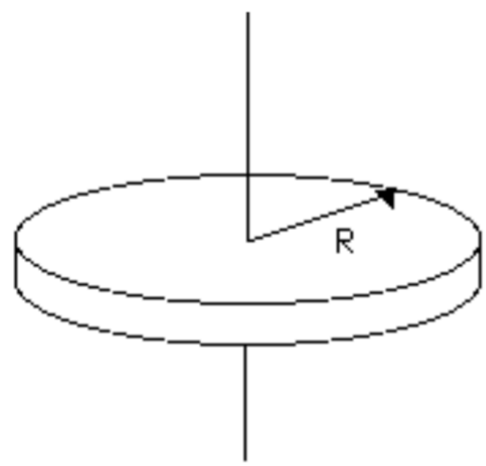

Figure 256:

is a collection of thin hoops as shown below

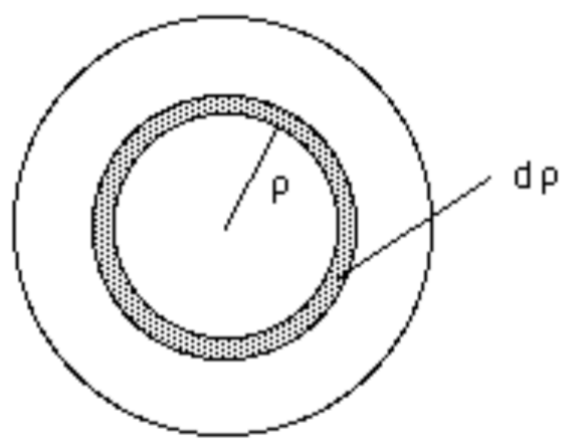

Figure 257:

This gives

$$dm = \frac{dA}{A}M = M\frac{2\pi\rho d\rho}{\pi R^2} = \frac{2M\rho d\rho}{R^2}$$

$$I = \int \rho^2 dm = \frac{2M}{R^2}\int_0^R \rho^3 d\rho = \frac{1}{2}MR^2$$

It is less then the thin hoop. Why?

Uniform Thin Rod:

Assume the axis is through center (perpendicular to rod as shown

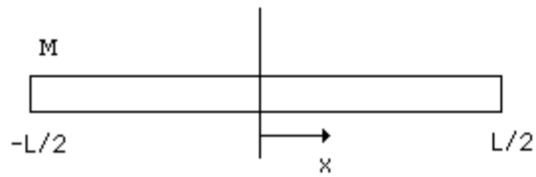

Figure 258:

$$I = \int_{-L/2}^{L/2} x^2 dm = \int_{-L/2}^{L/2} x^2 \frac{M}{L} dx = \frac{1}{12} M L^2$$

Uniform Thin Rod:

Assume the axis at one end(perpendicular to rod) as shown

Figure 259:

$$I = \int_0^L x^2 dm = \int_0^L x^2 \frac{M}{L} dx = \frac{1}{3} M L^2$$

Sphere:

A sphere is a collection of thin disks.

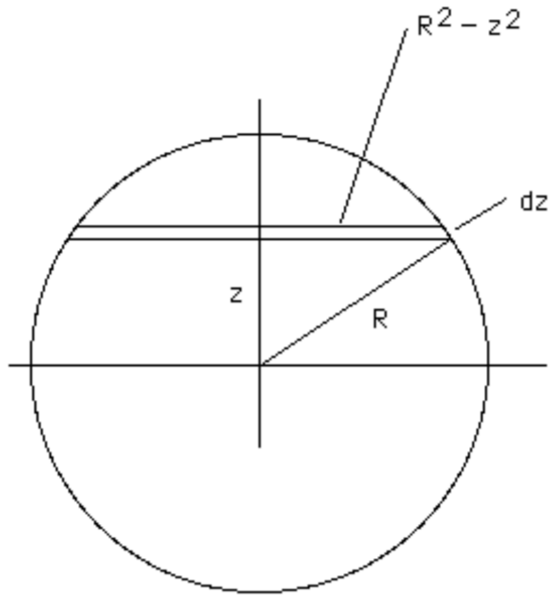

Figure 260:

Therefore we get

$$dm_{disk} = \frac{dV}{V}M = M\frac{\pi(R^2 - z^2)dz}{\frac{4}{3}\pi R^3}$$

$$dI_{disk} = (R^2 - z^2)dm_{disk} = M\frac{\pi(R^2 - z^2)^2 dz}{\frac{4}{3}\pi R^3}$$

$$I = 2\int_{z=0}^{z=R} dI_{disk} = \frac{3M}{4R^3}\int_0^R (R^2 - z^2)^2 dz = \frac{3M}{4R^3}\int_0^R (R^4 - 2R^2z^2 + z^4)dz$$
$$= \frac{3M}{4R^3}\left(R^5 - \frac{2}{3}R^5 + \frac{1}{5}R^5\right) = \frac{2}{5}MR^2$$

Parallel Axis Theorem disk

This theorem tells us the moment of inertia I about any axis provided we know the moment of inertia I_0 about a parallel axis through the CM.

Consider the moment of inertia of a body about an axis (call it the z-axis) as shown.

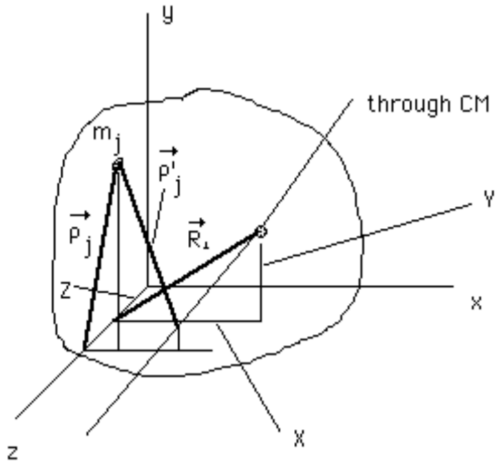

Figure 261:

The vector from the z-axis to the j^{th} particle is

$$\vec{\rho}_j = x_j\hat{\imath} + y_j\hat{\jmath}$$

and

$$I = \sum_j m_j \rho_j^2$$

If the CM is at

$$\vec{R} = X\hat{\imath} + Y\hat{\jmath} + Z\hat{k}$$

then the perpendicular vector from the z-axis is

$$\vec{R}_\perp = X\hat{\imath} + Y\hat{\jmath} \quad (\text{ magnitude } = L)$$

If the vector from the axis through the CM to the j^{th} particle is $\vec{\rho}_j'$, then the moment of inertia about the CM is

$$I_0 = \sum_j m_j \rho_j'^2$$

Now

$$\vec{\rho}_j = \vec{\rho}_j' + \vec{R}_\perp$$

so that

$$
\begin{aligned}
I &= \sum_j m_j \rho_j^2 = \sum_j m_j (vec\rho'_j + \vec{R}_\perp)^2 = \sum_j m_j \left(\rho_j'^2 + 2\vec{\rho}'_j \cdot \vec{R}_\perp + R_\perp^2 \right) \\
&= \sum_j \rho_j'^2 + 2\vec{R}_\perp \cdot \sum_j m_j \vec{\rho}'_j + R_\perp^2 \sum_j m_j \\
&= I_0 + 2M\vec{R}_\perp \cdot \vec{R}_{CM,CM} + MR_\perp^2 = I_0 + ML^2
\end{aligned}
\tag{10.27}
$$

which is the parallel axis theorem.

Remember L is the separation between the z-axis and a parallel axis through the CM.

Dynamics of Pure Rotation about an Axis

Earlier we showed that the motion of a system of particles is simple to describe if we distinguish between external forces and internal forces acting on the particles.

The internal forces cancel by Newton's 3rd law and the momentum changes only because of external forces.

This leads to the law of conservation of momentum – **the momentum of an isolated system is a constant vector.**

What about rotational motion?

Can we distinguish between internal and external torques?

There is no way to prove that the sum of the internal torques is zero.

Newton's laws are no help.

So we must rely on experiment ... the angular momentum of an **isolated** system has **NEVER** been observed to change spontaneously which implies that the sum of the internal torques must always $= 0$.

So we will assume that only external torques change the angular momentum of a rigid body.

What does it mean to make this assumption?

We are considering fixed axis rotation with no translation which is pure rotation.

Suppose we have a body rotating with angular velocity ω about the z-axis.

From earlier we then have

$$L_z = I\omega \tag{10.28}$$

The equation of motion

$$\vec{\tau} = \frac{d\vec{L}}{dt} \tag{10.29}$$

implies that

$$\tau_z = \frac{dL_z}{dt} = \frac{d(I\omega)}{dt} = I\frac{d\omega}{dt} = I\alpha \tag{10.30}$$

where

$$\alpha = \frac{d\omega}{dt} \tag{10.31}$$

is called the **angular acceleration**.

We will use this equation in the same manner as we used Newton's 2nd law

$$F = ma \quad , \quad a = \frac{dv}{dt} \tag{10.32}$$

which has remarkable similarity.

Don't be deceived into thinking, however, that the motion generated by the two different sets of equations will be similar!

We can also derive the kinetic energy associated with the rotational motion.

$$L = \sum_j \frac{1}{2} m_j v_j^2 = \sum_j \frac{1}{2} m_j \rho_j^2 \omega^2 = \frac{1}{2}\left(\sum_j m_j \rho_j^2\right)\omega^2 = \frac{1}{2} I\omega^2 \tag{10.33}$$

Atwood's Machine with Massive Pullies

Consider the pulley-mass arrangement below. **The pulley now has a mass**.

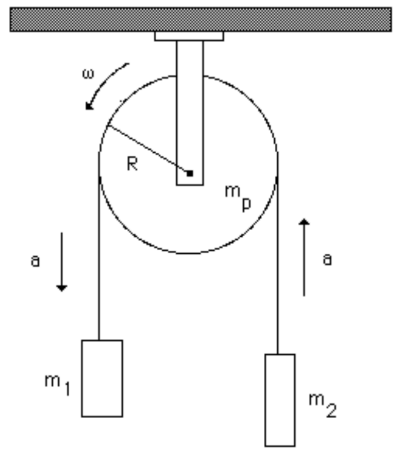

Figure 262:

The force diagrams are as shown below:

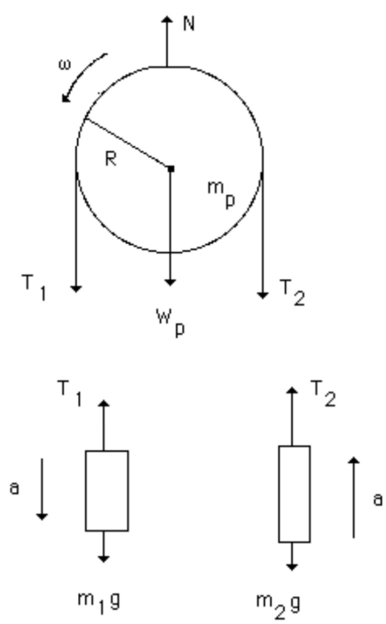

Figure 263:

The equations of motion are:

$$W_1 - T_1 = m_1 a \quad , \quad T_2 - W_2 = m_2 a$$

$$\tau = T_1 R - T_2 R = I\alpha$$

$$N - T_1 - T_2 = W_p$$

In this simple case, each tension vector is perpendicular to its radius vector and hence

$$\vec{\tau}_1 = \vec{R}_1 \times \vec{T}_1 = (-R\hat{x}) \times (-T_1\hat{y}) = RT_1\hat{z}$$

$$\vec{\tau}_2 = \vec{R}_2 \times \vec{T}_2 = (R\hat{x}) \times (-T_2\hat{y}) = -RT_2\hat{z}$$

$$\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2 = (RT_1 - RT_2)\hat{z}$$

or we can use a sign convention related to sense of rotation.

In our equation, we have assume torques in the positive z-direction or out of the paper are positive.

Also positive corresponds to a torque that would cause a CCW rotation.

Our definition of α above implies that α must be positive CCW.

There is a constraint relating a and α assuming the rope does not slip.

The velocity of the rope is the velocity of a point on the rim of the pulley or

$$v = R\omega$$

which gives

$$a = \frac{dv}{dt} = R\frac{d\omega}{dt} = R\alpha$$

We can now eliminate the tensions and the angular acceleration to get

$$W_1 - W_2 - (T_1 - T_2) = (m_1 + m_2)a$$

$$T_1 - T_2 = \frac{T\alpha}{R} = \frac{Ia}{R^2}$$

$$W_1 - W_2 - \frac{Ia}{R^2} = (m_1 + m_2)a$$

For the disk

$$I = \frac{1}{2} m_p R^2$$

which gives

$$a = \frac{(m_1 - m_2)g}{m_1 + m_2 + \frac{m_p}{2}}$$

Simple Pendulum

The simple pendulum is shown below.

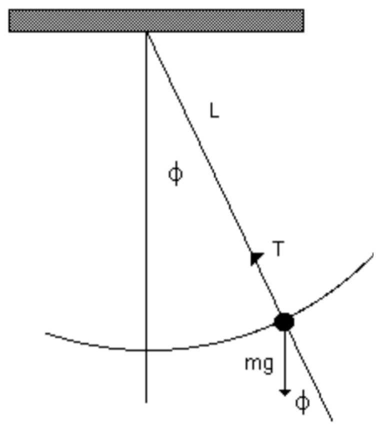

Figure 264:

Using Newton's laws, we have these equations of motion:

$$\begin{aligned} m\vec{a} &= ma_r\hat{r} + ma_\phi\hat{\phi} = m(\ddot{r} - r\dot{\phi}^2)\hat{r} + m(2\dot{r}\dot{\phi} + r\ddot{\phi})\hat{\phi} \\ &= (mg\cos\phi - T)\hat{r} - mg\sin\phi\hat{\phi} \end{aligned}$$

or

$$mg\cos\phi - T = m(\ddot{r} - r\dot{\phi}^2)$$
$$-mg\sin\phi = m(2\dot{r}\dot{\phi} + r\ddot{\phi})$$

Now, $\dot{r} = \ddot{r} = 0, r = L$, so that we have

$$T = g\cos\phi - L\dot{\phi}^2$$

$$\ddot{\phi} + \frac{g}{L}\sin\phi = 0$$

The first equation simply determines the tension T,

The 2nd equation is the equation of motion of the pendulum.

We could also derive this result much faster (because this is really a pure rotation) using torques....

$$I = mL^2 \ , \ \alpha = \ddot{\phi} \ , \ \tau = -mgL\sin\phi$$

$$\tau = I\alpha$$

$$-mgL\sin\phi = mL^2\ddot{\phi}$$

$$\ddot{\phi} + \frac{g}{L}\sin\phi = 0$$

The equation cannot be solved exactly as written .. it is a nonlinear equation.

However, if the pendulum only has small oscillations ... it never swings very far from the vertical or $\phi \ll 1$.

In this approximation we have $\sin\phi \approx \phi$ and the equation of motion becomes

$$\ddot{\phi} + \frac{g}{L}\phi = 0$$

This is the equation for simple harmonic motion (SHM) and thus the solution is

$$\phi(t) = A\sin\omega t + B\cos\omega t$$

where

$$\omega = \sqrt{\frac{g}{L}} = \frac{2\pi}{T} = 2\pi\nu$$

$T = \frac{1}{\nu}$ = oscillation period and ν = frequency.

So the period of the pendulum is

$$T = 2\pi\sqrt{\frac{L}{g}}$$

Some simple solutions are the same as with the spring:

Initial conditions:

$$\phi(0) = \phi_0 \quad , \quad \dot{\phi}(0) = 0$$

gives

$$\phi(t) = \phi_0 \cos \omega t$$

Initial conditions:

$$\phi(0) = 0 \quad , \quad \dot{\phi}(0) = \omega_0$$

gives

$$\phi(t) = \frac{\omega_0}{\omega} \sin \omega t$$

Physical Pendulum

Let us now consider the physical pendulum shown below.

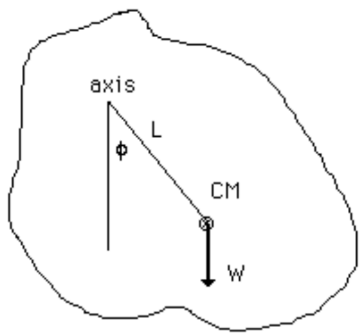

Figure 265:

It can have any shape.

It is rotating about some axis a.

Its mass is m and the distance from the CM to the axis of rotation a is L.

The moment of inertia about the axis of rotation is I_a and the moment of inertia about an axis through the CM is I_0 (these are parallel axes).

The motion is pure rotation about an axis a.

The only torque is that due to gravity and thus we have

$$-Lmg \sin \phi = I_a \ddot{\phi}$$

Making the small angle approximation we have

$$I_a\ddot{\phi} + mgL\phi = 0$$

with solutions

$$\phi(t) = A\sin\omega t + B\cos\omega t \quad , \quad \omega = \sqrt{\frac{mgL}{I_a}}$$

If we write

$$I_0 = mk^2 \quad , \quad k = \text{ radius of gyration}$$

and using

$$I_a = I_0 + mL^2 = m(k^2 + L^2)$$

by the parallel axis theorem, we then have

$$\omega = \sqrt{\frac{gL}{k^2 + L^2}}$$

The Door Stop

The banging of a door against its stop can tear loose the hinges due to torques about the stop.

It is possible to minimize this effect as follows.

Consider the door stop arrangement below.

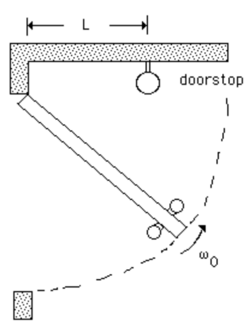

Figure 266:

The forces on the door during impact are as shown below.

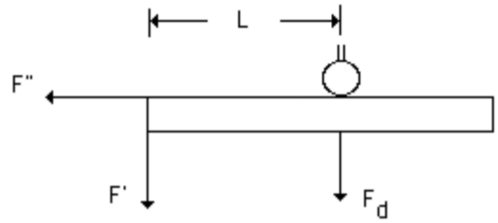

Figure 267:

F_d is due to the stop, F' and F'' are due to the hinge.

F'' is the small radial force that provides the radial centripetal acceleration of the rotating door, and F' is large and a reaction to the impact at the stop....this is the force that damages the hinge (tears it loose).

To minimize stress on the hinge we need to make F' as small as possible.

We solve this problem by considering the angular momentum of the door about the hinges and the linear momentum of the CM.

We have (during the impact)

$$dL = \tau dt \rightarrow L_{final} - L_{initial} = \int_{t_i}^{t_f} \tau dt$$

Now, $L_{initial} = I\omega_0$, I = moment of inertia about the hinges.

The door comes to rest at the end so that $L_{final} = 0$.

The torque on the door during the collision is $\tau = -LF_d$ so we get

$$I\omega_0 = L\int_{t_i}^{t_f} F_d dt$$

over the duration of the collision.

The CM motion obeys (during the impact)

$$P_{final} - P_{initial} = \int_{t_i}^{t_f} F_{total} dt$$

in the y-direction.

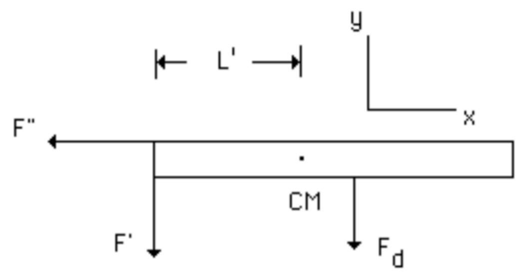

Figure 268:

Before the collision $P_{initial} = MV_0 = ML'\omega_0$

We also have $P_{final} = 0$.

So that(using the angular momentum result above) we have

$$-ML'\omega_0 = \int_{t_i}^{t_f} (F_d + F')dt$$

$$-ML'\omega_0 - \frac{I\omega_0}{L} = \int_{t_i}^{t_f} F'dt$$

We minimize this(actually make it $= 0$) by choosing

$$L = \frac{I}{ML'}$$

If the door is uniform and of width w, then

$$I = \frac{Mw^2}{3} \quad , \quad L' = \frac{w}{2} \rightarrow L = \frac{2}{3}w$$

This distance L = **center of percussion**.

When hitting a tennis ball (or a baseball), it is important to hit it at the **"sweet spot"** or at the center of percussion to avoid a large reaction on the hand that would cause a painful sting.

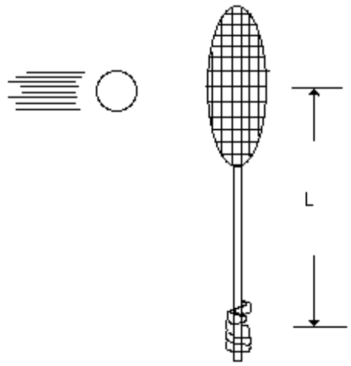

Figure 269:

Part 3

Motion Involving Both Translation and Rotation

The key to solving these systems is that the motion always breaks up into **"translation of the CM"** + **"rotation about the CM"**.

We continue our restricted discussion so that the axis of rotation remains constant in direction(we again choose the z-axis).

We now calculate the angular momentum about the z-axis and show that it can be written as two terms, **the angular momentum due to rotation of the body about its CM + the angular momentum due to the translational motion of the CM with respect to the origin of an inertial coordinate system**.

Again we treat the body as a bunch of particles glued together.

We assume N particles with mass m_j, position vectors $\vec{r}_j$ (wrt an inertial frame), $j = 1, 2, 3, ..., N$.

The angular momentum of the body is

$$\vec{L} = \sum_{j=1}^{N} m_j \vec{r}_j \times \frac{d\vec{r}_j}{dt} \tag{10.34}$$

The CM position is given by

$$\vec{R} = \frac{\sum_j m_j \vec{r}_j}{M} \tag{10.35}$$

where $M = \sum_j m_j$ = total mass.

The position vectors in the CM frame are

$$\vec{r}_j = \vec{R} + \vec{r}'_j \tag{10.36}$$

Introducing the CM coordinates we have

$$\begin{aligned}\vec{L} &= \sum_{j=1}^{N} m_j \vec{r}_j \times \frac{d\vec{r}_j}{dt} = \sum_{j=1}^{N} m_j (\vec{R} + \vec{r}'_j) \times \frac{d(\vec{R} + \vec{r}'_j)}{dt} \\ &= \sum_{j=1}^{N} m_j \vec{R} \times \frac{d\vec{R}}{dt} + \sum_{j=1}^{N} m_j \vec{R} \times \frac{d\vec{r}'_j}{dt} + \sum_{j=1}^{N} m_j \vec{R}'_j \times \frac{d\vec{R}}{dt} + \sum_{j=1}^{N} m_j \vec{r}'_j \times \frac{d\vec{r}'_j}{dt}\end{aligned} \tag{10.37}$$

What a mess!!!

Consider the 2nd and 3rd terms....

$$\sum_{j=1}^{N} m_j \vec{R} \times \frac{d\vec{r}'_j}{dt} = \vec{R} \times \frac{d}{dt} \sum_{j=1}^{N} m_j vecr'_j = \vec{R} \times \frac{d}{dt} \vec{R}_{CM,CM} = 0 \tag{10.38}$$

$$\sum_{j=1}^{N} m_j \vec{R}'_j \times \frac{d\vec{R}}{dt} = \vec{R}_{CM,CM} \times \frac{d\vec{R}}{dt} = 0 \tag{10.39}$$

Thus, the angular momentum becomes

$$\vec{L} = \sum_{j=1}^{N} m_j \vec{R} \times \frac{d\vec{R}}{dt} + \sum_{j=1}^{N} m_j \vec{r}'_j \times \frac{d\vec{r}'_j}{dt} = M\vec{R} \times \vec{V} + \sum_{j=1}^{N} m_j \vec{r}'_j \times \frac{d\vec{r}'_j}{dt} \tag{10.40}$$

where $\vec{V} = \frac{d\vec{R}}{dt}$ = velocity of the CM with respect to the inertial system.

The 1st term represents the angular momentum due to translational motion of the CM.

The last term is the angular momentum due to motion about the CM.

The only way for particles of a rigid body to move wrt the CM is for the body to **rotate as a whole**.

Let us look at the z-component only. We have

$$L_z = (M\vec{R} \times \vec{V})_z + \left(\sum_{j=1}^{N} m_j \vec{r}'_j \times \frac{d\vec{r}'_j}{dt}\right)_z \qquad (10.41)$$

For rotation about the x-axis the 2nd term can be simplified.

The body has angular velocity $\vec{\omega} = \omega\hat{k}$about the CM and since the origin of the $\vec{r}'_j$ is the CM, the 2nd term can be handled exactly as we did earlier for pure rotation

$$\left(\sum_{j=1}^{N} m_j \vec{r}'_j \times \frac{d\vec{r}'_j}{dt}\right)_z = \sum_{j=1}^{N} L_{z,j} = \sum_{j=1}^{N} m_j \rho'^2_j \omega = I_0 \omega \qquad (10.42)$$

where $\vec{\rho}'_j$ is the perpendicular vector to m_j from an axis in the z-direction through the CM and $I_0 = \sum_{j=1}^{N} m_j \rho'^2_j$ is the moment of inertia of the body about this axis as shown below.

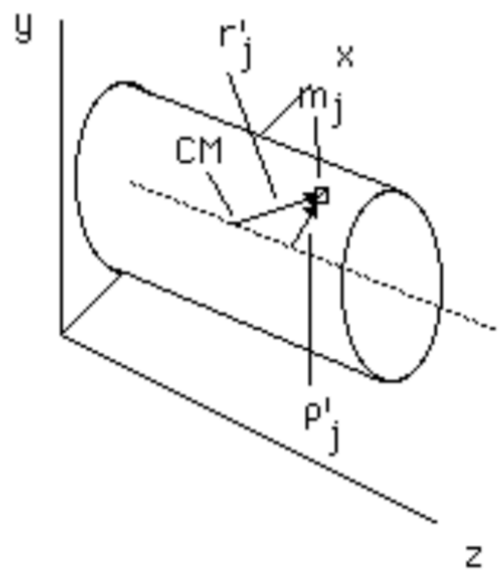

Figure 270:

Putting it all together we have

$$L_z = I_0\omega + (M\vec{R} \times \vec{V})_z \tag{10.43}$$

which is the result we stated earlier.

The angular momentum breaks up into two parts - the angular momentum of the CM and the angular momentum wrt to the CM.

They are often called orbital and spin angular motion in analogy with the figure below.

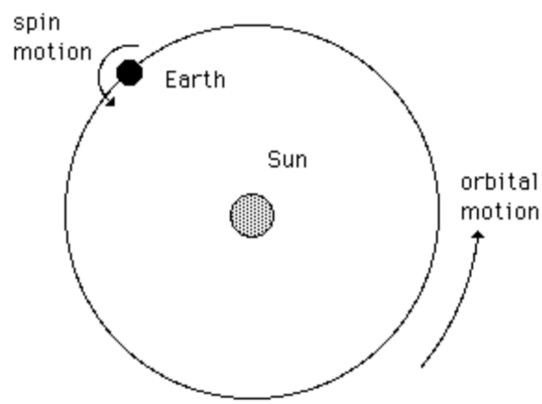

Figure 271:

The spin angular momentum is independent of choice of origin - in some sense it is intrinsic to the body.

Orbital angular momentum is origin dependent and can be made to disappear by an origin choice.

This result is valid even if the CM is accelerating since we calculated everything wrt to an inertial frame.

Angular Momentum of a Rolling Wheel

A uniform wheel of mass m and radius b is rolling uniformly without slipping. See figure below.

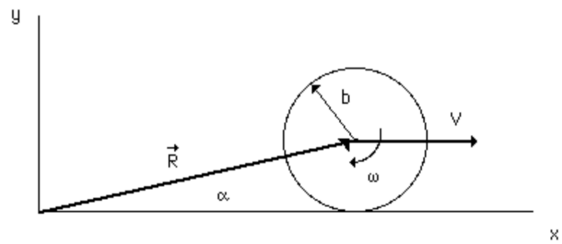

Figure 272:

The moment of inertia is $I_o = \frac{1}{2}mb^2$ and the angular momentum **about** the CM is

$$\vec{L}_0 = \sum_j \vec{\rho}'_j \times m_j \vec{v}'_j = \sum_j \rho'_j \hat{r} \times m_j(-\rho'_j \omega)\hat{\theta}$$
$$= -\left(\sum_j m_j \rho'^2_J\right)\omega\hat{k} = -I_0\omega\hat{k}$$

Why is this negative?

It is parallel to the z-axis and into the paper(minus sign).

Now we calculate the angular momentum of the CM wrt to the origin

$$\vec{L}_{CM} = M\vec{R} \times \vec{V} = -MRV\sin\alpha\hat{k} = -MVb\hat{k}$$

Therefore, the total z-component of angular momentum (actually all of it in this case) is

$$L_z = -\frac{1}{2}mb^2\omega - mb^2\omega = -\frac{3}{2}mb^2\omega$$

The torque also naturally divides into two components.

The torque on the body is

$$\vec{\tau} = \sum_j \vec{r}_j \times \vec{f}_j = \sum_j (\vec{r}'_j + \vec{R}) \times \vec{f}_j = \sum_j \vec{r}'_j \times \vec{f}_j + \vec{R} \times \sum_j \vec{f}_j = \sum_j \vec{r}'_j \times \vec{f}_j + \vec{R} \times \vec{F}$$

where $\vec{F} = \sum_j \vec{f}_j$ is the total external force.

The 2nd term is the torque due the total external force acting at the CM.

The 1st term is the torque about the CM due to various external forces.

For fixed axis rotation,, $\vec{\omega} = \omega\hat{k}$, and we have

$$\tau_z = \left(\sum_j \vec{r}'_j \times \vec{f}_j\right)_z + (\vec{R} \times \vec{F})_z = \tau_0 + (\vec{R} \times \vec{F})_z$$

where τ_0 is the z-component of the torque about the CM.

Now we obtained earlier that

$$L_z = I_0\omega + (M\vec{R} + \vec{V})_z$$

which gives

$$\begin{aligned}\tau_z &= \frac{dL_z}{dt} = I_0\frac{d\omega}{dt} + \frac{d}{dt}(M\vec{R} + \vec{V})_z = I_0\frac{d\omega}{dt} + (M\vec{R} + \vec{a})_z \\ &= I_0\alpha + (M\vec{R} + \vec{a})_z = I_0\alpha + (\vec{R} \times \vec{F})_z\end{aligned}$$

Hence, using our earlier expression for τ_z we find that

$$\tau_0 = I_0\alpha$$

This states that the rotational motion about the CM depends ONLY on the torque about the CM and is independent of the translational motion.

This is correct even if the axis is accelerating!!

This is a rather amazing and powerful result.

Finally we look at the kinetic energy.

We get

$$\begin{aligned}K &= \sum_j \frac{1}{2}m_j v_j^2 = \sum_j \frac{1}{2}m_j\left(\vec{v}_{j,cm} + \vec{V}\right)^2 = \sum_j \frac{1}{2}m_j\vec{v}_{j,cm}^2 + \sum_j \frac{1}{2}m_j\vec{V}^2 + \sum_j \frac{1}{2}m_j 2\vec{v}_{j,cm} \cdot \vec{V} \\ &= \sum_j \frac{1}{2}\rho_{j,cm}^2\omega^2 + \frac{1}{2}MV^2 + \left(\sum_j m_j\vec{v}_{j,cm}\right) \cdot \vec{V} = \frac{1}{2}I_0\omega^2 + \frac{1}{2}MV^2 + \frac{1}{M}\hat{P}_{cm} \cdot \vec{V} \\ &= \frac{1}{2}I_0\omega^2 + \frac{1}{2}MV^2 \end{aligned} \tag{10.44}$$

The 1st term = kinetic energy of spin **about** the CM and the 2nd term = orbital kinetic energy of the CM motion and we have used $\vec{P}_{cm} = 0$.

Summary of Relations for Fixed Axis Rotation

I. Pure rotation about an axis – no translation

$$L = I\omega \ , \ \tau = I\alpha \ , \ K = \frac{1}{2}I\omega^2 \tag{10.45}$$

II. Rotation and Translation

$$L_z = I_0\omega + (M\vec{R} + \vec{V})_z \ , \ \tau_z = I_0\alpha + (\vec{R} \times \vec{F})_z \tag{10.46}$$

$$\tau_0 = I_0\omega \ , \ , \ K = \frac{1}{2}I_0\omega^2 + \frac{1}{2}MV^2 \tag{10.47}$$

Disk on Ice

A disk of mass m and radius b is pulled with a constant force F by a thin tape wound around its circumference.

The disk slides on the ices without friction.

What is the motion?

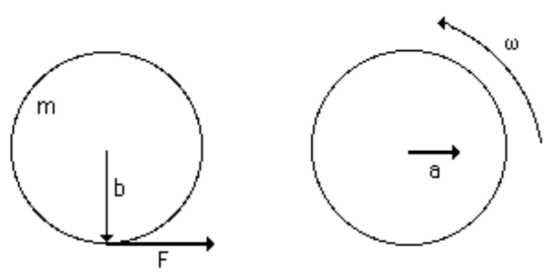

Figure 273:

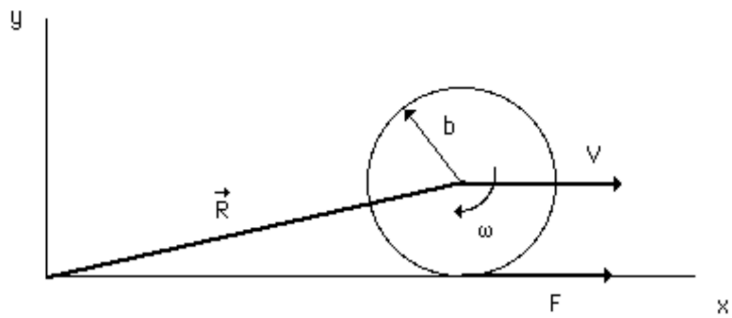

Figure 274:

METHOD 1:

About the CM we have

$$\text{rotation: } \tau_0 = bF = I_0\alpha \to \alpha = \frac{bF}{I_0} \;,\; \text{translation: } a = \frac{F}{m}$$

METHOD 2:

We choose a coordinate system with origin along the line of action of the force).

The torque about the origin is

$$\tau_z = I_0\alpha + (\vec{R} \times \vec{F})_z = bF = bF = 0$$

as we expect.

This means that angular momentum about the origin is conserved.

The angular momentum about the origin is

$$L_z = I_0\omega + (M\vec{R} \times \vec{V})_z = I_0\omega - bMV$$

Now

$$\frac{dL_z}{dt} = 0 = I_0\alpha = bMa = I_0\alpha - bF \to \alpha = \frac{bF}{I_0} \quad \text{(as before)}$$

Drum Rolling down a Plane

A uniform drum of mass M and radius b rolls without slipping down an inclined plane as shown.

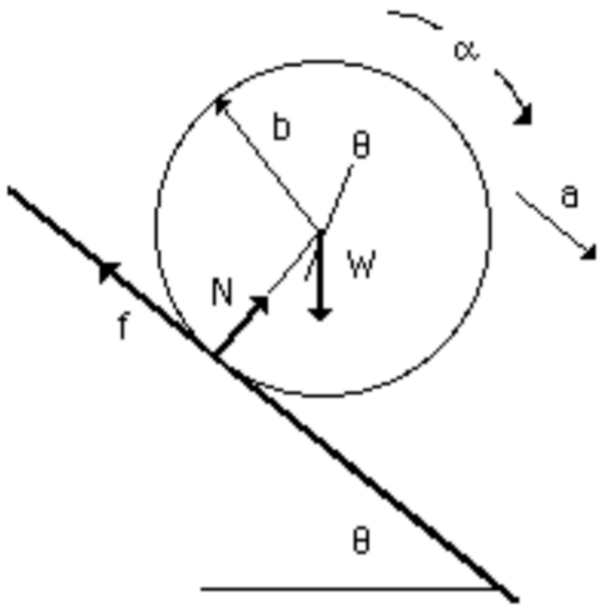

Figure 275:

Find acceleration down the plane. The moment of inertia of the drum about its axis is $I_0 = \frac{1}{2}Mb^2$.

METHOD 1:

The forces acting on the drum are shown above.

The translation of the CM is given by

$$Mg\sin\theta - f = Ma$$

The rotation about the CM is given by

$$bf = I_0\alpha$$

For rolling without slipping we have $a = b\alpha$.

Eliminating f we get

$$Mg\sin\theta - I_0\frac{\alpha}{b} = Ma = Mg\sin\theta - \frac{1}{2}Ma$$

$$a = \frac{2}{3}g\sin\theta$$

METHOD 2:

Choose a coordinate system with origin at A (on the plane) as shown below.

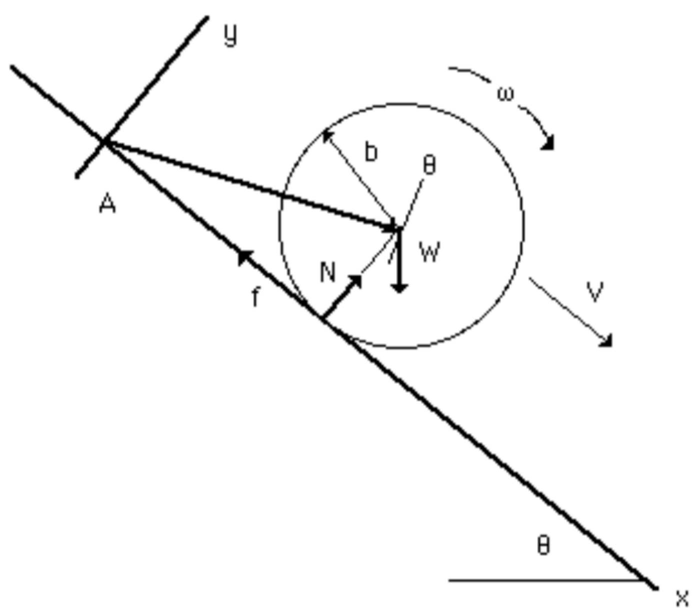

Figure 276:

The torque about A is

$\tau_z = I_0\alpha + (\vec{R}\times\vec{F})_z = -R_z f + R_z(f - Mg\sin\theta) + R_z(N - Mg\cos\theta) = -bMg\sin\theta$

since $R_z = b$ and $N = W\cos\theta$.

More details are shown below.

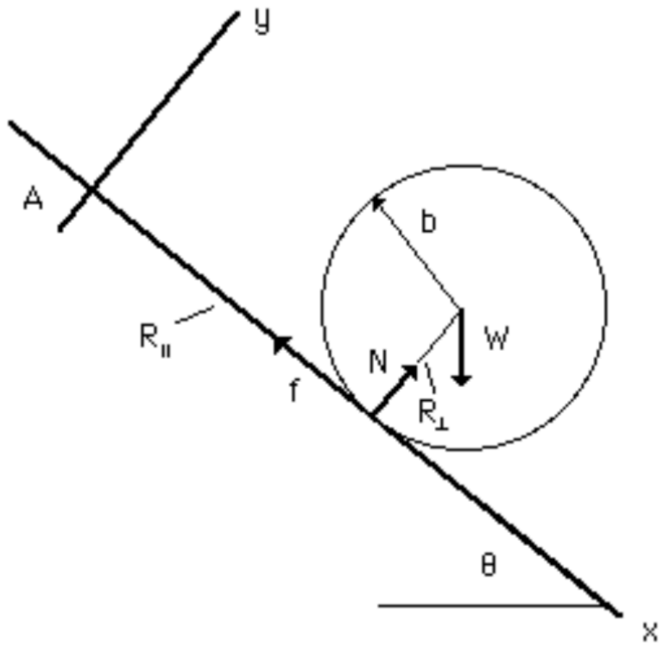

Figure 277:

The angular momentum about A is

$$L_z = -I_0\omega + (M\vec{R} \times \vec{V})_z = -\frac{1}{2}Mb^2\omega - Mb^2\omega = -\frac{3}{2}Mb^2\omega$$

in the same way as earlier. Then we have

$$\frac{dL_z}{dt} = \tau_z = -bMg\sin\theta = -\frac{3}{2}Mb^2\frac{d\omega}{dt}$$

$$g\sin\theta = \frac{3}{2}\alpha \to \alpha = \frac{2}{3}\frac{g\sin\theta}{b}$$

For rolling without slipping

$$a = b\alpha = \frac{2}{3}g\sin\theta$$

as before.

An even more direct evaluation of the torque would have followed by choosing a different origin, namely, the point of contact as shown below.

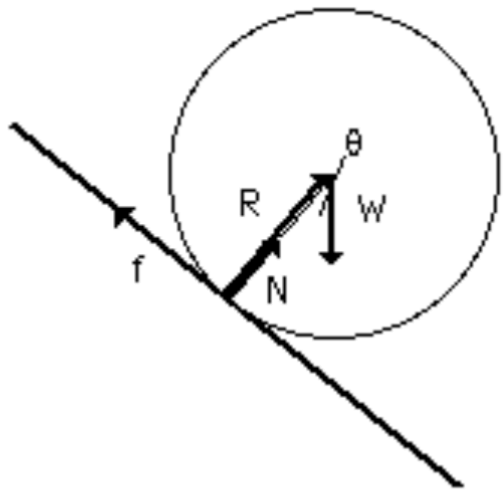

Figure 278:

The only contribution comes from W (N and f pass through the new origin).

We then directly obtain the torque give above.

Work-Energy Theorem

We now generalize the work-energy theorem.

As with our earlier discussions, the translational motion of the CM gives

$$\begin{aligned} W_{ba} &= \int_{a,path}^{b} \vec{F} \cdot d\vec{r} = \\ &= \\ &= \frac{1}{2} M V_b^2 - \frac{1}{2} M V_a^2 \end{aligned} \tag{10.48}$$

This result is independent of path!

Now we consider the work associated with the rotational kinetic energy.

$$\tau_0 = I_) \alpha = I_0 \frac{d\omega}{dt} \tag{10.49}$$

$$\tau_0 d\theta = I_0 \frac{d\omega}{dt} d\theta = I_0 \frac{d\omega}{dt} \omega dt = d\left(\frac{1}{2} I_0 \omega^2\right) \qquad (10.50)$$

Integrating we get

$$\int_a^b \tau_0 d\theta = \int_a^b d\left(\frac{1}{2} I_0 \omega^2\right) = \frac{1}{2} I_0 \omega_b^2 - \frac{1}{2} I_0 \omega_a^2 \qquad (10.51)$$

This integral obviously represents the work done by an applied torque.

We generalize the work-energy theorem to include this rotational work term and the rotational kinetic energy terms.

The new kinetic energy is then

$$K = \frac{1}{2} M V^2 + \frac{1}{2} I_0 \omega^2 \qquad (10.52)$$

Drum Rolling down a Plane - Energy Method

Again consider the drum rolling down the plane as shown.

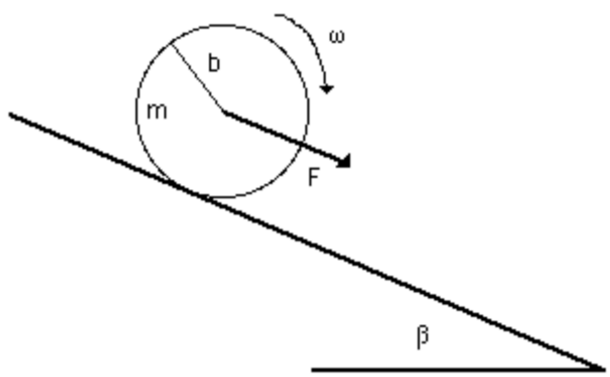

Figure 279:

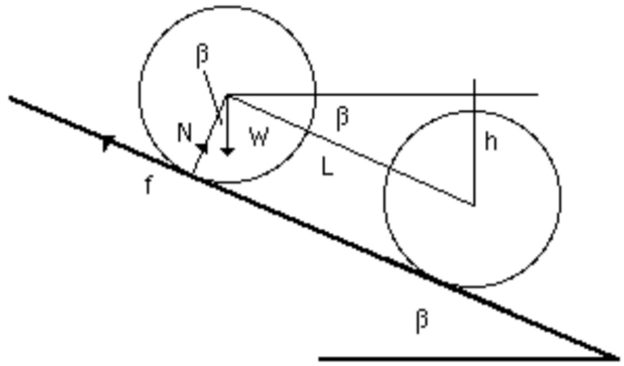

Figure 280:

Drum is released from rest, rolls without slipping, and descends a distance h. What is its speed v?

The translational energy equation gives

$$\int_a^b \vec{F}\cdot d\vec{r} = (W\sin\beta - f)L = \frac{1}{2}MV_b^2 - \frac{1}{2}MV_a^2 = \frac{1}{2}MV^2 \quad L = \frac{h}{\sin\beta}$$

The rotational energy equation gives

$$\int_a^b \tau d\theta = fb(\theta_b - \theta_a) = fb\theta = \frac{1}{2}I_0\omega_b^2 - \frac{1}{2}I_0\omega_a^2 = \frac{1}{2}I_0\omega^2$$

where θ = rotation angle.

Since we have no slipping, we have

$$b\theta = L \rightarrow b\frac{d\theta}{dt} = b\omega = \frac{dL}{dt} = V$$

Substituting all back into the equations we get

$$fL = \frac{1}{2}I_0\omega^2 = \frac{1}{2}I_0\frac{V^2}{b^2}$$

$$Wh = \frac{1}{2}\left(M + \frac{I_0}{b^2}\right)V^2 = \frac{1}{2}\left(M + \frac{M}{2}\right)V^2 = \frac{3}{4}MV^2$$

$$V = \sqrt{\frac{4gh}{3}}$$

Note that the friction force does not cause a loss of energy here (it is not dissipative).

In fact, the torque due to f actually increases the rotational energy.

If slipping occurs this is no longer the case.

Two More Examples

(1) A Yo-Yo of mass M has an axle of radius b and a spool of radius R as shown below.

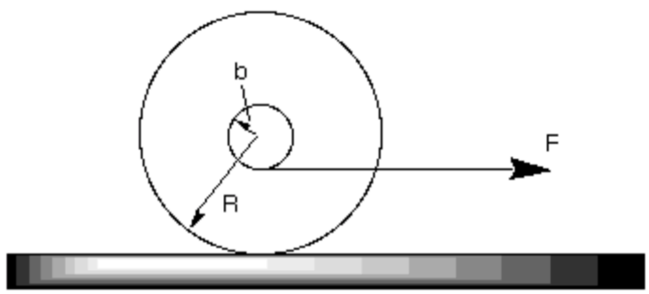

Figure 281:

The Yo-Yo is placed upright on a table and the string is pulled with a horizontal force F as shown.

The coefficient of friction between the Yo-Yo and the table is μ. What is the maximum value of F such that the Yo-Yo will roll without slipping?

$$F - f = MA \quad , \quad fR - Fb = I\alpha$$

$$A = R\alpha \rightarrow \text{ rolling/noslipping}$$

$$fR - Fb = \frac{1}{2}MR^2\frac{A}{R} = \frac{1}{2}MRA$$

$$f = \frac{1}{3}F\left(1 + \frac{2b}{R}\right) \leq \mu N = \mu Mg \Rightarrow F_{max} + \frac{3\mu Mg}{1 + \frac{2b}{R}}$$

Suppose the string makes an angle θ with the horizontal instead of being horizontal as shown.

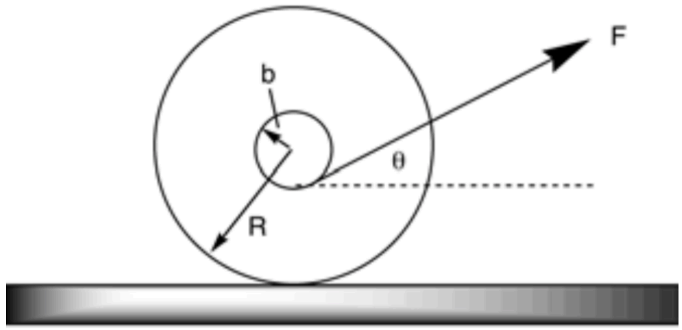

Figure 282:

For what value of θ does the Yo-Yo have no tendency to rotate - it only slides?

$$F \text{ give}$$

$$F\cos\theta - f = MA \text{ (not needed)}$$

$$F\sin\theta + N = Mg$$

There is no tendency to rotate when the torque = 0.

Hence,

$$Fb = fR$$

$$f = \mu N = \mu(Mg - F\sin\theta) \text{ yo-yo not rotating}$$

$$Fb = \mu(Mg - F\sin\theta)R$$

$$\sin\theta = \frac{\mu Mg - Fb}{\mu FR}$$

(2) A section of steel pipe of large diameter and relatively thin wall is mounted as shown on a flat-bed truck.

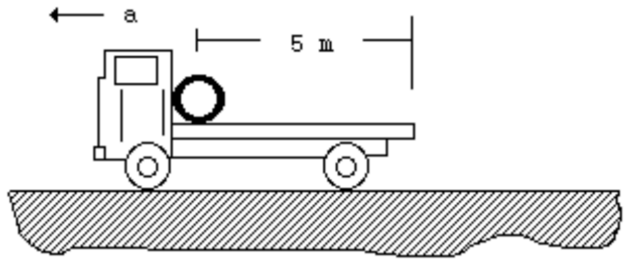

Figure 283:

The driver of the truck, not realizing that the pipe has not been lashed in place, starts up the truck with a constant acceleration of $a = 0.5g$.

As a result, the pipe rolls backward (relative to the truck bed) without slipping and falls to the ground.

The length of the truck is 5 meters.

(a) With what horizontal velocity does it strike the ground?

If no movement

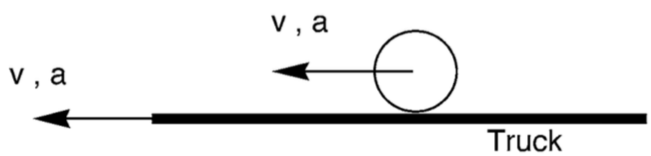

Figure 284:

which says that they stay together.

If movement

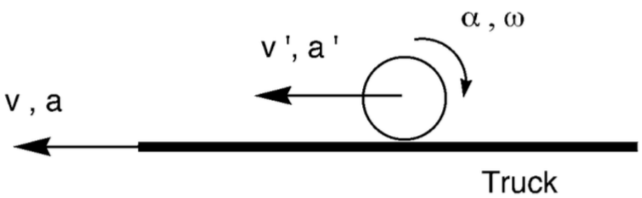

Figure 285:

where

$$a' = r\alpha \quad , \quad \omega = -\frac{v'}{r}$$

which corresponds to pure rolling.

If $a' < a$, then it will eventually fall off the back and relative to the ground it looks like

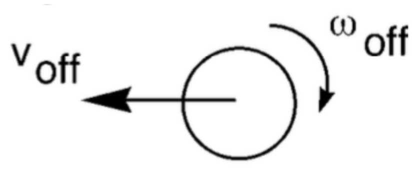

Figure 286:

Note the directions.

The situation is such that it looks like a force = ma is acting on the cylinder.

Therefore we have

$$\text{work } = \int madx = \frac{1}{2}mv_{off}^2 + \frac{1}{2}I\omega_{off}^2$$

$$m(0.5g)(5m) = \frac{1}{2}mf_{off}^2 + \frac{1}{2}r^2\left(-\frac{v_{off}}{r}\right)^2 - mv_{off}^2$$

$$v_{off} = -\sqrt{\frac{5g}{2}} = -5 \text{ m/s} \to \omega_{off} = -\frac{v_{off}}{r} = +\sqrt{\frac{5g}{2r^2}}$$

Note that v_{off} is in the direction of the truck motion and the sense of ω_{off} is as shown in the diagram!

(b) How far does it skid before beginning to roll without slipping if the coefficient of friction between the pipe and the ground is 0.3?

The angular momentum with respect to the contact point (on the ground) is conserved (no torques about this point since all forces pass through this point).

When the cylinder is rolling we have

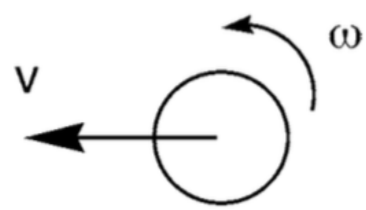

Figure 287:

and therefore (be careful about directions)

$$\vec{L}_i = -mrv_{off}\hat{k} + I\omega_{off}\hat{k} = 0$$

$$\vec{L}_f = -mrv\hat{k} - I\omega\hat{k} = 0 \quad , \quad v = r\omega \text{ (pure rolling)}$$

where $I = I_{contact} = I_{cm} + mr^2 + 2mr^2$ using the parallel-axis theorem.

Thus,

$$-3mrv = 0 \rightarrow v = \omega = 0$$

when pure rolling occurs.

We then have

$$v_f^2 = 0 = v_{off} - 2\frac{f}{m}s$$

$$\frac{5g}{2} = 2\frac{\mu mg}{m}s \rightarrow s = \text{ distance it skids } = \frac{5}{4\mu} = 4.16\,m$$

An alternative method.......

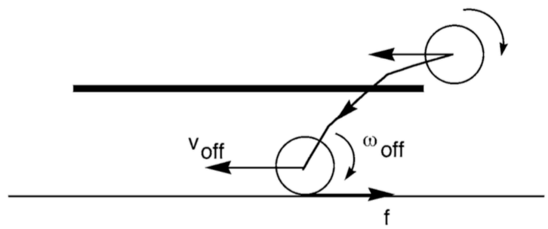

Figure 288:

In this situation we have

$$\omega(t) = \omega_{off} - \int_0^t \frac{rf}{I} dt = +\sqrt{\frac{5g}{2r^2}} - \frac{1}{mr}\int_0^t f dt$$

$$v(t) = v_{off} - \frac{1}{m}\int_0^t f dt = -\sqrt{\frac{5g}{2}} - \frac{1}{m}\int_0^t f dt$$

We have pure rolling when

$$v(t) = r\omega(t)$$

or

$$-\sqrt{\frac{5g}{2}} + \frac{1}{m}\int_0^t f dt = +\sqrt{\frac{5g}{2}} - \frac{1}{m}\int_0^t f dt$$

$$\frac{1}{m}\int_0^t f dt = \sqrt{\frac{5g}{2}}$$

Therefore, we get rolling when

$$\omega = +\sqrt{\frac{5g}{2r^2}} - \sqrt{\frac{5g}{2r^2}} = 0$$

$$v = -\sqrt{\frac{5g}{2}} + \sqrt{\frac{5g}{2}} = 0$$

as before.

More examples:

(1) A wheel is attached to a fixed shaft, and the system is free to rotate without friction.

To measure the moment of inertia of the wheel-shaft system, a tape of negligible mass wrapped around the shaft is pulled with a known force F.

When a length L of the tape has unwound, the system is rotating with angular speed ω_0.

Find the moment of inertia of the system, I_0.

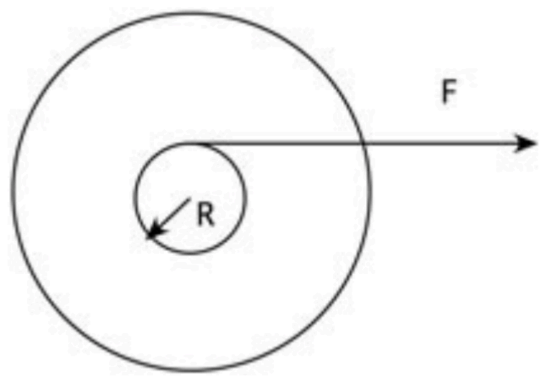

Figure 289:

We have

$$\tau = I_0 \frac{d\omega}{dt} \rightarrow \frac{d\omega}{dt} = \frac{\tau}{I_0} = \frac{FR}{I_0} \rightarrow \omega = \left(\frac{FR}{I_0}\right) t$$

and

$$\theta = \int \omega dt = \frac{1}{2}\left(\frac{FR}{I_0}\right) t^2$$

At $t = t_0$, a length L has been unwound.

Therefore

$$L = R\theta = \frac{1}{2}\left(\frac{FR}{I_0}\right) t_0^2 \rightarrow t_0 = \sqrt{\frac{2LI_0}{FR^2}}$$

Now, $\omega = \omega_0$ at $t = t_0$ so that

$$\omega_0 = \frac{FR}{I_0} t_0 = \sqrt{\frac{2LF}{I_0}}$$

Therefore,

$$I_0 = \frac{2LF}{\omega_0^2} = 400 \text{ kg-m}^2$$

(2) A drum of mass M and radius R is suspended from another drum also of mass M and radius R, which is free to rotate about its axis (attached to the ceiling).

The suspension is in the form of a massless metal tape wound around the outside of each drum and free to unwind, as shown.

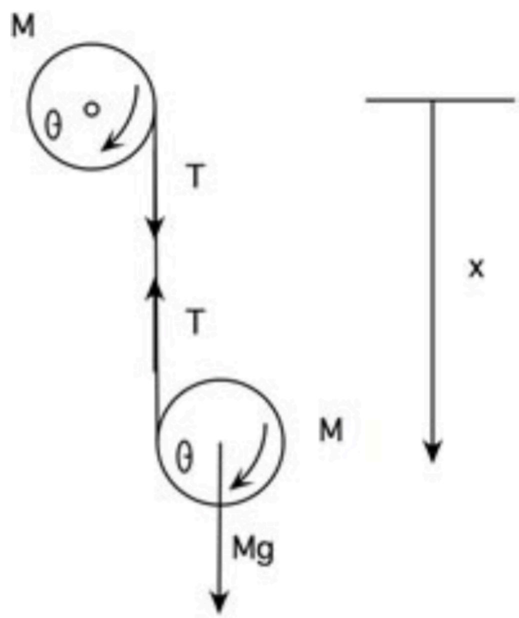

Figure 290:

Gravity is directed downward.

Both drums are initially at rest.

Find the initial acceleration of the falling drum, assuming that it moves straight downward.

Both drums turn through the same angle and therefore have the same angular acceleration (the torques about the centers are identical).

We then have

$$x = \ell_0 + 2R\theta \rightarrow \ddot{x} = A = 2R\alpha$$

Lower drum:

$$Mg - T = MA$$

$$TR = \frac{1}{2}MR^2\alpha$$

Upper drum:

$$TR = \frac{1}{2}MR^2\alpha$$

Solving, we get

$$Mg = \frac{5}{4}MA \to A = \frac{4}{5}g$$

(3) A plank of length 2ℓ and mass M lies on a frictionless plane.

A ball of mass m and speed v_0 strikes its end as shown.

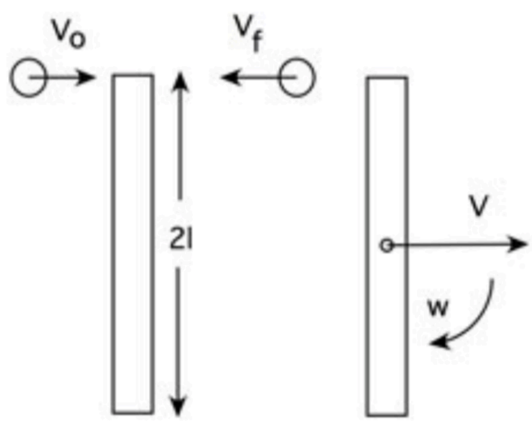

Figure 291:

Find the final velocity of the ball, v_f, assuming that the mechanical energy is conserved and that v_f is along the original line of motion.

Momentum and energy conservation:

$$mv_0 = Mv = mv_f$$

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}Mv^2 + \frac{1}{2}I_0\omega^2$$

About the CM - angular momentum conservation

$$mv_0\ell = -mv_f\ell + I_0\omega$$

where

$$I_0 = \frac{1}{12}M(2\ell)^2 = \frac{1}{3}M\ell^2$$

We find

$$\left(1 + \frac{4m}{M}\right)v_f^2 + \left(\frac{8m}{M}v_0\right)v_f - \left(1 - \frac{4m}{M}\right)v_0^2 = 0$$

which gives

$$v_f = \left(\frac{1 - \frac{4m}{M}}{1 + \frac{4m}{M}}\right)v_0$$

Find v_f assuming that the stick is pivoted at the lower end.

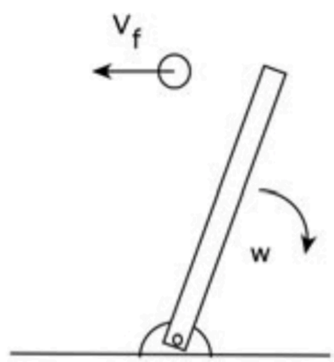

Figure 292:

Momentum need not be conserved because forces act at the pivot.

Energy is conserved.

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}I_{pivot}\omega^2$$

Take angular momentum about the pivot.

$$mv_0(2\ell) = -mv_f(2\ell) + I_{pivot}\omega$$

Therefore,

$$\omega = \frac{2m\ell(v_0 + v_f)}{I_{pivot}}$$

$$I_{pivot} = \frac{1}{3}M(2\ell)^2 = \frac{4}{3}M\ell^2$$

We then have

$$\left(1 + \frac{3m}{M}\right)v_f^2 + \left(\frac{6m}{M}v_0\right)v_f - \left(1 - \frac{3m}{M}\right)v_0^2 = 0$$

which gives

$$v_f = \left(\frac{1 - \frac{3m}{M}}{1 + \frac{3m}{M}}\right)v_0$$

Part 4

Rigid Body Motion

It is clear that we will not need to change any of our analysis of fixed axis rotational motion if we assume that the angular velocity is a vector along the axis of rotation with magnitude = the angular velocity.

With this assumption we can clearly write

$$\vec{v} = \vec{\omega} \times \vec{r} \tag{10.53}$$

as shown in the figure.

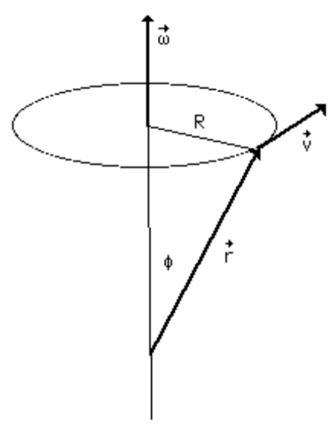

Figure 293:

We have

$$\vec{r} = R\hat{r} + r\cos\phi\hat{k} \quad , \quad \vec{\omega} = \omega\hat{k} \tag{10.54}$$

$$\vec{v} = \vec{\omega} \times \vec{r} = R\omega\hat{\theta} \tag{10.55}$$

as it should.

Example : Rotating Skew Rod

Angular Momentum

Consider the system shown below: a simple rigid body consisting of two particles of mass m separated by a massless rod of length 2ℓ.

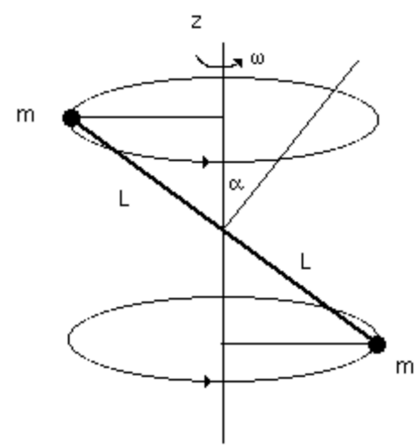

Figure 294:

The midpoint of the rod is attached to a vertical axis which rotates with angular speed ω.

The rod is skewed at angle α.

What is the angular momentum of the system?

$$\vec{L} = \sum_j \vec{r}_j \times \vec{p}_j = \vec{r}_a \times \vec{p}_a + \vec{r}_b \times \vec{p}_b$$

This has the direction(s) shown below.

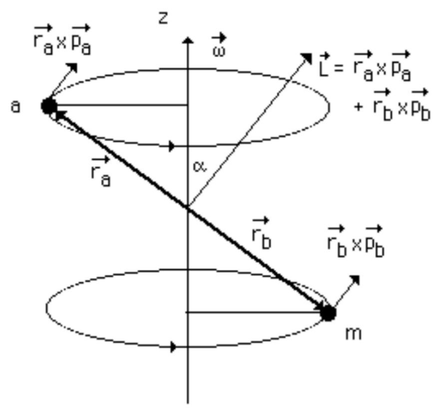

Figure 295:

The magnitude is

$$
\begin{aligned}
|\vec{L}| &= |\vec{r}_a \times \vec{p}_a + \vec{r}_b \times \vec{p}_b| = |\vec{r}_a \times m(\vec{\omega} \times \vec{r}_a) + \vec{r}_b \times m(\vec{\omega} \times \vec{r}_b)| = 2m|\vec{r}_a \times m(\vec{\omega} \times \vec{r}_a)| \\
&= 2m\left|\left(\ell \cos\alpha \hat{r}_{ah} + \ell \sin\alpha \hat{k}\right) \times (\omega\hat{k} \times \left(\ell \cos\alpha \hat{r}_{ah} + \ell \sin\alpha \hat{k}\right)\right|) \\
&= 2m\omega\ell^2\left|\left(\cos\alpha \hat{r}_{ah} + \sin\alpha \hat{k}\right) \times (\hat{k} \times \left(\cos\alpha \hat{r}_{ah} + \sin\alpha \hat{k}\right)\right|) \\
&= 2m\omega\ell^2\left|\left(\cos\alpha \hat{r}_{ah} + \sin\alpha \hat{k}\right) \times \cos\alpha\hat{\theta}\right| = 2m\omega\ell^2 \cos\alpha\left|\left(\cos\alpha \hat{r}_{ah} + \sin\alpha \hat{k}\right) \times \hat{\theta}\right| \\
&= 2m\omega\ell^2 \cos\alpha\left|\left(\cos\alpha \hat{r}_{ah} - \sin\alpha \hat{k}\right)\right| = 2m\omega\ell^2 \cos\alpha
\end{aligned}
$$

$\vec{L}$ is perpendicular to the skew rod and in the plane of the rod and the z-axis.

The tip of the $\vec{L}$ vector turns with the rod and describes a horizontal circle.

Note that $\vec{L}$ and $\vec{\omega}$ are not parallel. T

his is generally true for nonsymmetric bodies.

Torque

The angular momentum above is constant in magnitude but changes in direction.

$\vec{L}$ is fixed wrt the rod and rotates in space with the rod.

The torque can be calculated using

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

We can do it in terms of components (see diagram below).

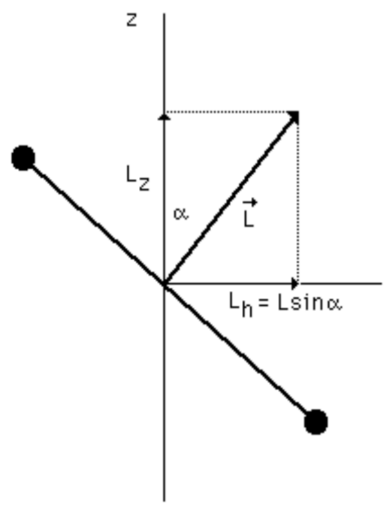

Figure 296:

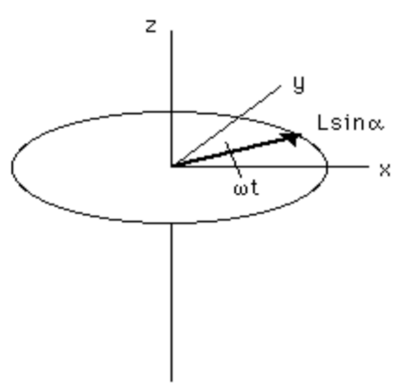

Figure 297:

The component L_z is parallel to the z-axis and is $= L\cos\alpha$.

It is constant.

The horizontal component, $L_h = L\sin\alpha$ swings with the rod.

We choose the $x - y$ axes such that L_h coincides with the x-axis at $t = 0$.

Then at time t we have:

$$L_x = L_h \cos\omega t = L\sin\alpha\cos\omega t \quad , \quad L_y = L_h \sin\omega t = L\sin\alpha\sin\omega t$$

and

$$\vec{L} = L\sin\alpha(\hat{\imath}\cos\omega t + \hat{\jmath}\sin\omega t) + L\cos\alpha\hat{k}$$

The torque is

$$\vec{\tau} = \frac{d\vec{L}}{dt} = L\sin\alpha(-\hat{\imath}\sin\omega t + \hat{\jmath}\cos\omega t)$$

So we get

$$\tau_x = -L\omega\sin\alpha\sin\omega t = -2m\ell^2\omega\sin\alpha\cos\alpha\sin\omega t$$

$$\tau_y = L\omega\sin\alpha\cos\omega t = -2m\ell^2\omega\sin\alpha\cos\alpha\cos\omega t$$

$$\tau = \sqrt{\tau_x^2 + \tau_y^2} = \omega L\sin\alpha$$

Note that $\tau = 0$ for $\alpha = 0$ or $\alpha = \pi/2$. Why?

The Gyroscope

We now attempt to understand the motion of a gyroscope using the basic concepts of angular momentum, torque and the time derivative of a vector.

We concentrate on only one motion of the gyroscope, namely, **uniform precession**.

A gyroscope consists of a spinning flywheel and a suspension mechanism that allows the axis of rotation of the flywheel (its axle) to take on various orientations without constraint.

Schematically it looks like

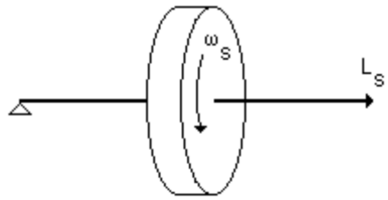

Figure 298:

The triangle is the pivot.

The direction of spin rotation is as shown.

If the gyroscope is released horizontally with one end support by the pivot, it wobbles off horizontally and then settles down to "uniform precession" ... where the axle slowly rotates about the vertical with angular velocity Ω as shown.

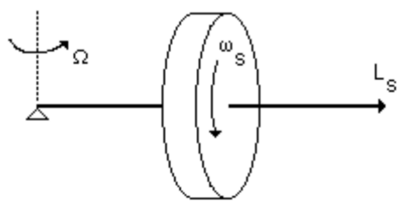

Figure 299:

Question #1

Why doesn't the gyroscope just fall down vertically?

A possible answer....look at the force diagram below.

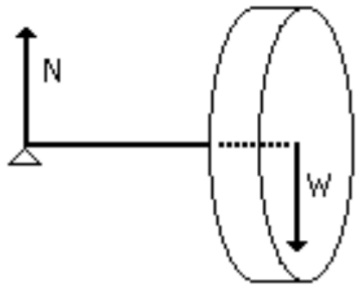

Figure 300:

The total vertical force is $N - W$, where N = vertical force from the pivot and W = weight. If $N = W$, the CM cannot fall.

While this is correct, it is not a satisfactory explanation. We have asked the wrong question.

Instead we should ask why it doesn't rotate about the pivot?

As a matter of fact, if it is not spinning or spinning slowly, it does this exactly...it just rotates about the pivot.

It only precesses if the flywheel is spinning very rapidly.

In this case, the large angular momentum of the wheel somehow dominates the motion.

Nearly all of the gyroscope's angular momentum lies in $\vec{L}_s$, the spin angular momentum. $\vec{L}_s$ is directed along the axle and has a magnitude $L_s = I_0\omega$, where I_0 is the moment of inertia of the flywheel about the axle.

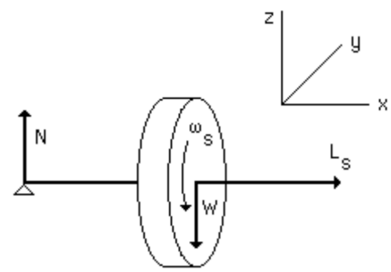

Figure 301:

When the gyroscope precesses about the z-axis, it has a small orbital angular momentum in the z-direction.

For uniform precession, this is constant in magnitude and direction and plays no dynamical role in the motion of the gyroscope.

$\vec{L}_s$ always points along the axle.

As the gyroscope precesses, $\vec{L}_s$ rotates with it as in the figure below.

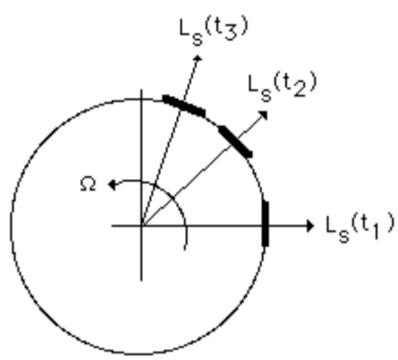

Figure 302:

The torque about the pivot point is given by

$$\vec{\tau} = \vec{d} \times \vec{W} = -W d\hat{r} \times \hat{k} = W d\hat{\theta} = \frac{d\vec{L}}{dt}$$

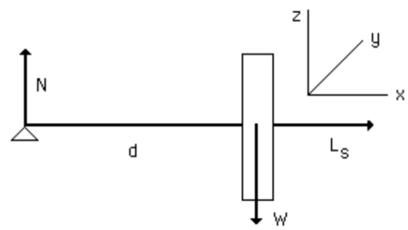

Figure 303:

or $\vec{\tau}$ is perpendicular to $\vec{L}_s$ as shown below.

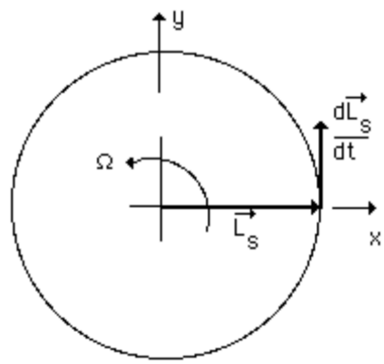

Figure 304:

Therefore

$$\vec{\tau} dt = W d\hat{\theta} dt = d\vec{L}_s$$

is perpendicular to $\vec{L}_s$.

This says that

$$d\vec{L}_s \cdot \vec{L}_s = 0 = d(\vec{L}_s \cdot \vec{L}_s) = d(L_s^2)$$

or L_s = magnitude of $\vec{L}_z$ = constant.

This means that the $\vec{L}_s$ vector moves in a circle about the z-axis as shown above. The direction of the rotation is counter-clockwise.

What is the precessional angular velocity?

Assume that the vector turns through an angle $d\phi$ during the time interval dt as shown below:

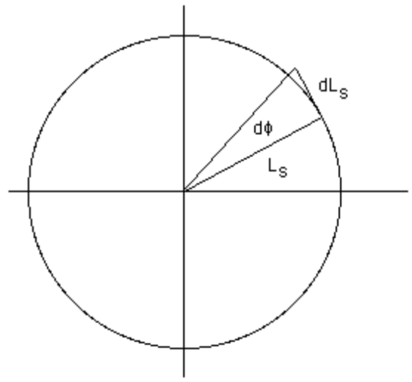

Figure 305:

then we have

$$d\phi = \frac{dL_s}{L_s} = \frac{\tau dt}{L_s} = \frac{Mdgdt}{I_0\omega_s}$$

or Ω = precessional angular velocity

$$\Omega = \frac{d\phi}{dt} = \frac{Mgd}{I_0\omega_s}$$

This equation indicates that Ω increases as the flywheel slows down (ω_s decreases).

This does not continue forever.

Eventually, uniform precession gives way to violent and erratic motion.

This occurs when the precessional angular momentum becomes comparable to the spin angular momentum and we can no longer neglect it.

Note the general rule:

$$\tau = \Omega L_s$$

This derivation assumed that the axle was horizontal.

It turns out that the precessional angular velocity is independent of the angle of the axle.

We can see this as follows.

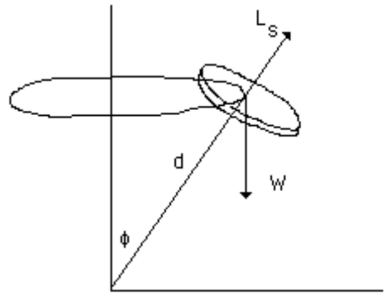

Figure 306:

In this case

$$d\phi = \frac{dL_{s,horiz}}{L_{s,horiz}} = \frac{\tau dt}{L_s \sin\phi} = \frac{Mgd\sin\phi dt}{I_0\omega_s \sin\phi}$$

$$\Omega = \frac{d\phi}{dt} = \frac{Mgd}{I_0\omega_s}$$

The dependence on the angle of the axle drops out!

Examples:

(1) Consider the diagram below:

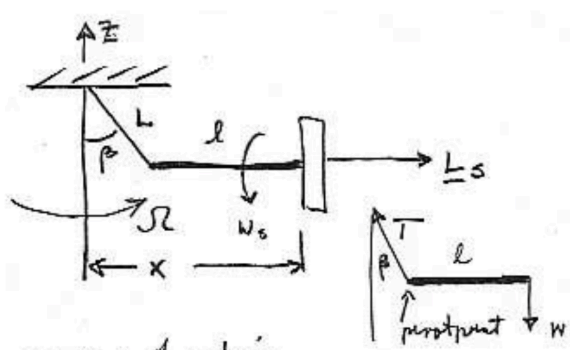

Figure 307:

What is the angle β?

The equations of motions are

$$T\cos\beta - W = 0 \text{ there is no acceleration vertically}$$

$$T\sin\beta = M\Omega^2 x \text{ circular motion}$$

The angular equation (wrt the pivot point) is

$$\tau_z = \frac{dL_z}{dt} = \omega L_s = W\ell = (T\cos\beta\ell) \qquad L_s = I_0\omega_s$$

For $\sin\beta \approx \beta$ we have

$$T = W \quad , \quad \beta = \frac{M\Omega^2 x}{W} = \frac{\Omega^2 x}{g}$$

$$\Omega = \frac{T\ell}{L_s} = \frac{W\ell}{L_s} = \frac{W\ell}{I_0\omega_s}$$

Therefore,

$$\beta = \frac{W^2\ell^2 x}{gI_o^2\omega_s^2}$$

but

$$x = \ell + L\beta$$

This implies that

$$\beta = \frac{W^2\ell^2(\ell + L\beta)}{g(I_0\omega_s)^2} = \frac{W^2\ell^3\left(1 + \frac{L}{\ell}\beta\right)}{g(I_0\omega_s)^2} = \beta_0\left(1 + \frac{L}{\ell}\beta\right) \quad \beta_0 = \frac{M^2 g\ell^3}{(I_o\omega_s)^2}$$

so that

$$\beta\left(1 - \frac{L}{\ell}\beta_0\right) = \beta_0 \quad \text{or} \quad \beta = \frac{\beta_0}{1 - \frac{L}{\ell}\beta_0}$$

(2) Consider the diagram below:

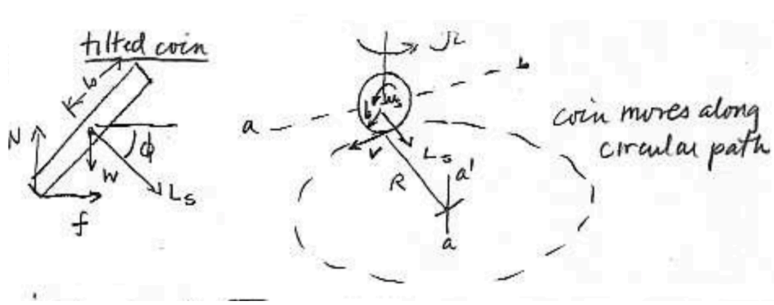

Figure 308:

The coin is accelerating.

We use the CM as origin(required) for the torque.

The equations are:

$$N = \text{ normal force } = Mg$$

$$f = \text{ frictional force } = \frac{MV^2}{R}$$

$$L_s = \text{ spin angular momentum } = I\omega = \frac{1}{2}Mb^2\frac{V}{b} = \frac{1}{2}Mbv$$

where the factor $\frac{V}{b}$ follows because of pure rolling.

All directions are in the diagram above.

$\Omega =$ angular speed of coin rotation about the vertical axis aa' $= \dfrac{V}{R}$ (pure rotation)

$$\Omega L_s \cos\phi = \text{ rate of change of angular momentum in horizontal plane}$$

The torque equation along the a-b axis is

$$Nb\sin\phi - fb\cos\phi = \Omega L_s \cos\phi$$

so that

$$\tan\phi = \frac{3V^2}{2gR}$$

11. Central Forces

Definition: A central force is defined by the equation

$$\vec{F} = f(r)\hat{r} \tag{11.1}$$

CM + Relative One-Body Motion

Consider an isolated, interacting (via a central force) 2-particle system as shown:

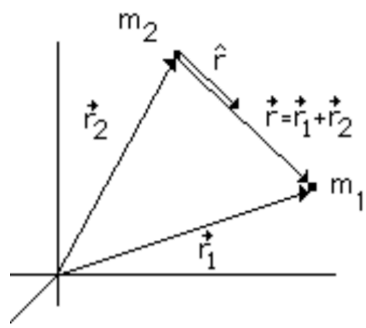

Figure 309:

where

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \quad , \quad r = |\vec{r}_1 - \vec{r}_2| \tag{11.2}$$

The equations of motion are:

$$m_1\ddot{\vec{r}}_1 = f(r)\hat{r} \quad , \quad m_2\ddot{\vec{r}}_2 = -f(r)\hat{r} \tag{11.3}$$

$$f(r) < 0 \rightarrow \text{ attractive-force} \quad , \quad f(r) > 0 \rightarrow \text{ repusive-force} \tag{11.4}$$

These equations are **coupled differential equations** and are difficult to solve.

We can, however, uncouple the motions.

The position of the CM is given by:

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \tag{11.5}$$

From the equations of motion we have:

$$\ddot{\vec{R}} = \frac{m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2}{m_1 + m_2} = \frac{f(r)\hat{r} - f(r)\hat{r}}{m_1 + m_2} = 0 \tag{11.6}$$

$$\vec{R} = \vec{V}t + \vec{R}_0 \tag{11.7}$$

$$\ddot{\vec{r}}_1 - \ddot{\vec{r}}_2 = \left(\frac{1}{m_1} - \frac{1}{m_2}\right) f(r)\hat{r} \tag{11.8}$$

or

$$\mu\ddot{\vec{r}} = f(r)\hat{r} \tag{11.9}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \text{ reduced-mass} \tag{11.10}$$

The motion of the two-particle system breaks up into a constant velocity translational motion of the CM plus a relative motion equation (an effective one-body problem) with a **reduced mass**.

If we can solve these two equations for $\vec{R}$ and $\vec{r}$ then we have a solution for the motion of the original masses.

$$\vec{r}_1 = \vec{R} + \left(\frac{m_2}{m_1 + m_2}\right)\vec{r} + \vec{R} + \vec{r}_{1,CM} \tag{11.11}$$

$$\vec{r}_2 = \vec{R} + \left(\frac{m_1}{m_1 + m_2}\right)\vec{r} + \vec{R} + \vec{r}_{2,CM} \tag{11.12}$$

General Properties

Before looking at some specific forcing functions $f(r)$, we derive some general properties valid for all central forces.

Since

$$\vec{\tau} = \vec{r} \times \vec{F} = r f(r) \hat{r} \times \hat{r} = 0 = \frac{d\vec{L}}{dt} \tag{11.13}$$

angular momentum is conserved (= constant vector) for central force motion.

This means that direction of $\vec{L}$ is fixed and since $\vec{r}$ and $\vec{v}$ are perpendicular to $\vec{L}$, the motion must always be in the same plane as it starts in (defined by $\vec{r}$ and $\vec{v}$).

We choose this plane to be the $x - y$ plane and hence the angular momentum is in the z-direction.

In this case our equations of motion become

$$\mu\ddot{\vec{r}} = \mu(\ddot{r} - r\dot{\theta}^2)\hat{r} + \mu(2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} = f(r)\hat{r} \tag{11.14}$$

or

$$\mu(\ddot{r} - r\dot{\theta}^2) = f(r) \tag{11.15}$$

$$\mu(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0 \tag{11.16}$$

The constant magnitude of the angular momentum is given by ℓ where

$$\ell = |\vec{r}\times\vec{p}| = |r\hat{r}\times\mu\vec{v}| = |r\hat{r}\times\mu(v_r\hat{r}+v_\theta\hat{\theta})| = |r\hat{r}\times\mu(\dot{r}\hat{r}+r\dot{theta}\hat{\theta})| \tag{11.17}$$

$$\ell = \mu r v_\theta = \mu r^2\dot{\theta} \tag{11.18}$$

This result could also have been obtained by using the θ component of Newton's 2nd law:

$$\mu(2\dot{r}\dot{\theta}+r\ddot{\theta}) = 0 = \frac{1}{r}\frac{d}{dt}(\mu r^2\dot{\theta}) \tag{11.19}$$

$$\frac{d}{dt}(\mu r^2\dot{\theta}) = 0 \to \mu r^2\dot{\theta} = \ell = \text{ constant} \tag{11.20}$$

This is a conservative force so the total energy is also conserved.

The total energy is

$$E = \frac{1}{2}\mu v^2 + U(r) = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 + U(r) \tag{11.21}$$

$$U(r) - U(r_0) = -\int_{r_0}^{r} f(r)dr \tag{11.22}$$

If we use the value of ℓ to eliminate $\dot{\theta}$ we get

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r) = \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r) \tag{11.23}$$

$$U_{eff}(r) = \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r) = \text{ effective potential energy} \tag{11.24}$$

The formal solution of these equations is

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{\mu}(E - U_{eff}(r))} = \sqrt{\frac{2}{\mu}\left(E - \frac{1}{2}\frac{\ell^2}{\mu r^2} - U(r)\right)} \tag{11.25}$$

or

$$\int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{\mu}(E - U_{eff}(r))}} = t - t_0 \tag{11.26}$$

and

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{\ell}{\mu r^2} \tag{11.27}$$

or

$$\theta - \theta_0 = \int_{t_0}^{t} \frac{\ell}{\mu r^2} dt \tag{11.28}$$

These formal solutions for $r(t)$ and $\theta(t)$ are often very difficult to carry out.

In general we are not interested in the coordinates as functions of time, but instead are interested in the equation of the path of motion, which is given by the function $r(\theta)$.

This function is the solution of the equations

$$\left(\frac{dr}{dt}\right)^2 = \frac{2}{\mu}\left(E - \frac{1}{2}\frac{\ell^2}{\mu r^2} - U(r)\right) \tag{11.29}$$

$$dt = \mu r^2 d\theta \tag{11.30}$$

$$\left(\frac{dr}{d\theta}\right)^2 = 2\mu r^4 \left(E - \frac{1}{2}\frac{\ell^2}{\mu r^2} - U(r)\right) \tag{11.31}$$

If we define a new variable

$$u = \frac{1}{r} \tag{11.32}$$

we get the equations

$$\frac{dr}{d\theta} = \frac{d\frac{1}{u}}{d\theta} = \frac{d\frac{1}{u}}{du}\frac{du}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta} \tag{11.33}$$

$$\frac{1}{u^4}\left(\frac{du}{d\theta}\right)^2 = \frac{2\mu}{u^4}\left(E - \frac{1}{2}\frac{\ell^2}{\mu r^2} - U(r = \frac{1}{u})\right) \tag{11.34}$$

$$\left(\frac{du}{d\theta}\right)^2 = 2\mu\left(E - \frac{1}{2}\frac{\ell^2}{\mu r^2} - U(r = \frac{1}{u})\right) \tag{11.35}$$

At the level we are studying these equations we will assume from now on that

$$m_2 = M \ \ , \ \ m_1 = m \ \ , \ \ M \gg m \to \mu \approx m \tag{11.36}$$

$$\vec{r}_{2,CM} \approx 0 \ \ , \ \ \vec{r}_{1,CM} \approx \vec{r} \tag{11.37}$$

and our equations are just equations for a small mass m orbiting about a very large mass M which exerts a force $f(r)$ on it.

Can we qualitatively figure out the motion of the system?

Consider the equations

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{mr^2} + U(r) = K_{radial} + U_{eff}(r) \tag{11.38}$$

$$\dot{\theta} = \frac{\ell}{mr^2} \tag{11.39}$$

As we did with the one-dimensional case, suppose we plot the effective potential, choose an E value and define the effective radial force by

$$f_{eff}(r) = -\frac{dU_{eff}(r)}{dr} \tag{11.40}$$

then we can discuss the radial motion qualitatively.

Once we understand the radial motion we can also determine the angular motion using equation (11.39) above.

Procedure:

1. Given $f(r)$, determine $U_{eff}(r)$.

   Let us consider the case of motion in a gravitational field where

   $$f(r) = -\frac{RMm}{r^2} \rightarrow U(r) = -\frac{GMm}{r} \tag{11.41}$$

   We then have

   $$U_{eff}(r) = \frac{\ell^2}{2mr^2} - \frac{GMm}{r} \tag{11.42}$$

2. Plot the effective potential as a function of r in range $[0, \infty]$.

   The various terms in $U_{eff}(r)$ will look like the figure below:

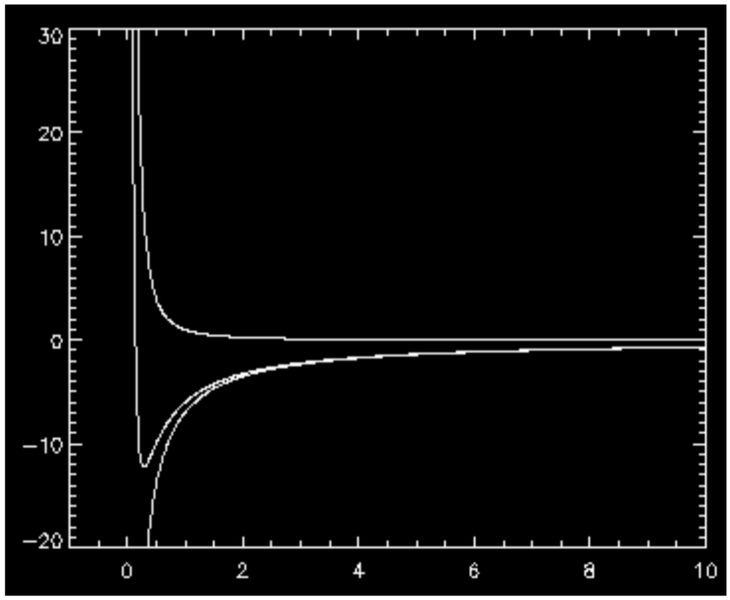

Figure 310:

where the different curves represent:

- top curve = $\frac{\ell^2}{2mr^2}$ term
- bottom curve = $-\frac{GMm}{r}$ term
- middle curve = $U_{eff}(r)$ = sum of other two curves

What would this this look like for a repulsive inverse square potential?

As before the difference between the energy value and the effective potential energy curve is the radial kinetic energy (> 0).

For a given energy, there are unphysical regions where the radial kinetic energy would become < 0.

If the slope of the effective potential energy curve is > 0 the effective radial force is towards $r = 0$ (attractive).

If the slope of the effective potential energy curve is < 0 the effective radial force is towards $r = \infty$ (repulsive).

If the slope of the effective potential energy curve is = 0 the effective radial force is = 0.

Cases 1 and 2: $E > 0$; always repulsive — r will decrease to minimum called the turning point(where energy = effective potential energy)and then increase; discuss simultaneous angular motion to show unbounded motion → hyperbolic orbit.

What is the difference between the attractive and repulsive case?

Cases 4 and 5: $E < 0$; alternates between repulsive and attractive; r oscillates between two turning points where energy = effective potential energy → elliptical orbit.

Case 3: $E = 0$ limit of 1 and 2 and 4 and 5 → a parabola; unbounded but not unbounded.

Case 6: E = minimum; r = single value → circular motion when angular motion considered.

Planetary Motion

Let us now derive the exact motion for planets in a gravitational field.

We have

$$\left(\frac{du}{d\theta}\right)^2 = 2\mu\left(E - \frac{1}{2}\frac{\ell^2}{\mu r^2} - U(r = \frac{1}{u})\right) \tag{11.43}$$

where now

$$U(r) = -\frac{GMm}{r} = -GMmu \tag{11.44}$$

so that

$$\left(\frac{du}{d\theta}\right)^2 = 2\mu\left(E - \frac{1}{2}\frac{\ell^2}{\mu}u^2 + GMmu\right) \tag{11.45}$$

Our qualitative discussion and experimental observation indicates that we should look for a solution that is a conic section.

The general equation for conic sections

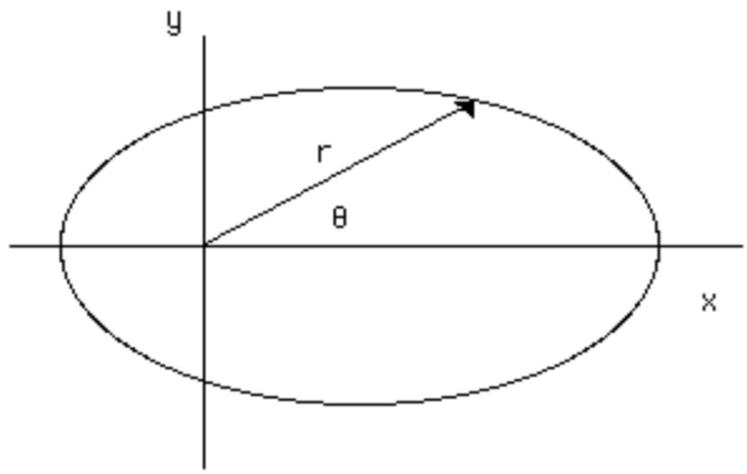

Figure 311:

in plane-polar coordinates is

$$r = \frac{r_0}{1 - \epsilon \cos\theta} \tag{11.46}$$

where

$r_0 =$ radius of circular orbit(minimum)
$\epsilon =$ eccentricity
$\epsilon > 1 \rightarrow$ hyperbola
$\epsilon = 1 \rightarrow$ parabola
$1 > \epsilon > 0 \rightarrow$ ellipse
$\epsilon = 0 \rightarrow$ circle

Another way to see this is to replace

$$r = \sqrt{x^2 + y^2} \quad , \quad \cos\theta = \frac{x}{r} \tag{11.47}$$

to get

$$(1 - \epsilon^2)x^2 - 2r_0\epsilon x + y^2 = r_0^2 \tag{11.48}$$

$$\left(\sqrt{1-\epsilon^2}x - \frac{r_0\epsilon}{\sqrt{1-\epsilon^2}}\right)^2 + y^2 = r_0^2\left(\frac{1}{1-\epsilon^2}\right) \tag{11.49}$$

We now check that a conic section represents a solution of the equation of motion:

$$r = \frac{r_0}{1-\epsilon\cos\theta} \quad , \quad u = \frac{1-\epsilon\cos\theta}{r_0} \tag{11.50}$$

$$\left(\frac{du}{d\theta}\right)^2 = \epsilon^2\sin^2\theta = 2\mu\left(E - \frac{1}{2}\frac{\ell^2}{\mu}\left(\frac{1-\epsilon\cos\theta}{r_0}\right)^2 + GMM\frac{1-\epsilon\cos\theta}{r_0}\right) \tag{11.51}$$

Special cases:

$$\theta = 0 \to 0 = E - \frac{1}{2}\frac{\ell^2}{\mu}\left(\frac{1-\epsilon}{r_0}\right)^2 + GMM\frac{1-\epsilon}{r_0} \tag{11.52}$$

$$\theta = \pi/2 \to \epsilon^2 = 2\mu\left(E - \frac{1}{2}\frac{\ell^2}{\mu}\left(\frac{1}{r_0}\right)^2 + GMM\frac{1}{r_0}\right) \tag{11.53}$$

or

$$r_0 = \frac{\ell^2}{GMm^2} \quad , \quad \epsilon = \sqrt{1 + \frac{2E\ell^2}{G^2M^2m^3}} \tag{11.54}$$

Again we see that

$$\begin{aligned}
&\epsilon > 1 \to E > 0 \to \text{ hyperbola} \\
&\epsilon = 1 \to E = 0 \to \text{ parabola} \\
&1 > \epsilon > 0 \to 0 > E > -\frac{G^2M^2m^3}{2\ell^2} \to \text{ ellipse} \\
&\epsilon = 0 \to E = \frac{G^2M^2m^3}{2\ell^2} \to \text{ circle}
\end{aligned}$$

Does the circle case make sense?

Consider regular circular motion from earlier:

$$m\frac{v^2}{r} = \frac{GMm}{r^2} \to v = \sqrt{\frac{GM}{r}}$$

$$\ell = mvr \to \ell = m\sqrt{GMr} \to r = \frac{\ell^2}{GMm^2}$$

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{1}{2}\frac{GMm}{r} - \frac{GMm}{r} = -\frac{GMm}{2r}$$

$$E = -\frac{G^2M^2m^3}{2\ell^2}$$

Note that there is an inverse relationship between velocity and radius when in circular orbit.

Elliptical Orbits

Properties of an Ellipse

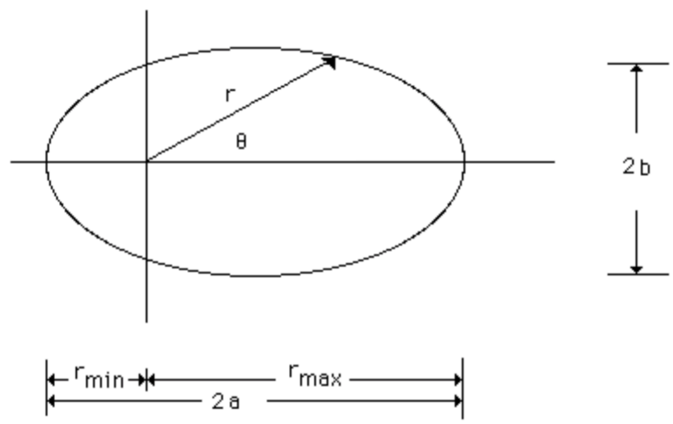

Figure 312:

we have

$$r = \frac{r_0}{1 - \epsilon \cos\theta} \tag{11.55}$$

$$r_{max} = \frac{r_0}{1 - \epsilon} \tag{11.56}$$

$$r_{min} = \frac{r_0}{1 + \epsilon} \tag{11.57}$$

$$2a = r_{max} + r_{min} = \frac{2r_0}{1 - \epsilon^2} = -\frac{GMm}{E} \tag{11.58}$$

$$E = -\frac{GMm}{2a} \tag{11.59}$$

$$\frac{r_{max}}{r_{min}} = \frac{1 + \epsilon}{1 - \epsilon} \tag{11.60}$$

These imply that the length of the semi-major axis a is independent of the angular momentum ℓ and depends only on the energy E.

The angular momentum ℓ does affect the semi-minor axis b, however.

$$b = \frac{r_0}{\sqrt{1-\epsilon^2}} = \frac{\frac{\ell^2}{GMM^2}}{\sqrt{\frac{2E\ell^2}{G^2M^2m^3}}} = \frac{\ell}{\sqrt{2mE}} \tag{11.61}$$

Note how the ellipse changes shape as functions of E and ℓ.

Satellite Orbits

A satellite of mass 2000 kg is in an elliptical orbit about the earth.

At perigee (point of closest approach) it has an altitude of 1100 km and at apogee(point of farthest distance) its altitude is 4100 km.

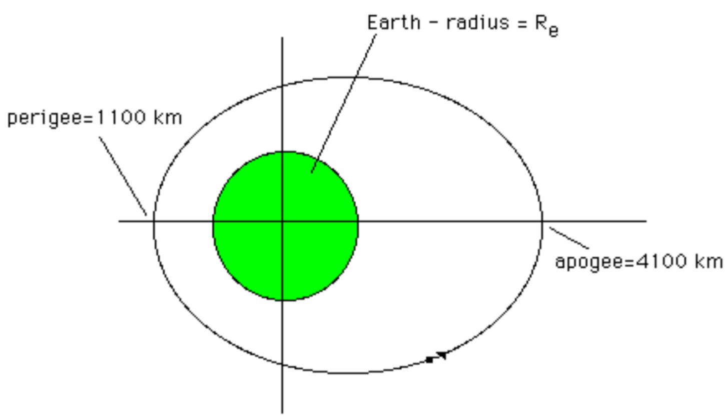

Figure 313:

What are the satellite's energy and angular momentum and how fast is it traveling at apogee and perigee?

We have

$$R_e = 6400\,km$$

$$2a = r_{max} + r_{min} = (4100 + 6400 + 1100)\,km + (1100 + 6400)\,km = 1.8 \times 10^7\,m \tag{11.62}$$

$$E = -\frac{GM_em}{2a} = -\frac{mgR_e^2}{2a} = -4.5 \times 10^{10}\,J \tag{11.63}$$

$$E_i = \text{ initial energy before launch from earth surface } = -\frac{GM_e m}{R_e} = -12.5{\times}10^{10}\, J \tag{11.64}$$

$$\Delta E = E - E_i = \text{ energy need to put into orbit } = 8 \times 10^{10}\, J \tag{11.65}$$

$$r_{max} = \frac{r_0}{1-\epsilon} \quad , \quad r_{min} = \frac{r_0}{1+\epsilon} \to \frac{r_{max}}{r_{min}} = \frac{1+\epsilon}{1-\epsilon} \tag{11.66}$$

$$\epsilon = \frac{r_{max} - r_{min}}{r_{max} + r_{min}} = \frac{1}{6} \tag{11.67}$$

$$\epsilon = \sqrt{1 + \frac{2E\ell^2}{G^2 M_e^2 m^3}} \to \ell = \sqrt{G^2 M_e^2 m^3 \frac{\epsilon^2 - 1}{2E}} = 1.2 \times 10^{14}\, \frac{kg \cdot m}{sec^2} \tag{11.68}$$

$$E = \frac{1}{2} m v^2 - \frac{GM_e m}{r} \to v = \sqrt{\frac{2}{m}\left(E + \frac{GM_e m}{r}\right)} \tag{11.69}$$

$$v_p = v_{perigee} = \sqrt{\frac{2}{m}\left(E + \frac{GM_e m}{r_{min}}\right)} = 7900\, \frac{m}{sec} \tag{11.70}$$

$$v_a = v_{apogee} = \sqrt{\frac{2}{m}\left(E + \frac{GM_e m}{r_{max}}\right)} = 5600\, \frac{m}{sec} \tag{11.71}$$

Alternatively, by conservation of angular momentum

$$m v_p r_p = m v_a r_a \to v_a = v_p \frac{r_p}{r-a} \tag{11.72}$$

since the velocity and radius vectors are perpendicular at those points.

What maneuvers would actually be required to launch from the surface of the Earth and insert a satellite into orbit?

Satellite Maneuver

Suppose that we are in a circular orbit around the earth of radius r_i and we want to transfer to different circular orbit with radius r_f.

What is the sequence of necessary steps?

This is call a Hohman transfer (illustrated below).

It is the least energy required transfer method.

This is the way we went to the moon in 1969.

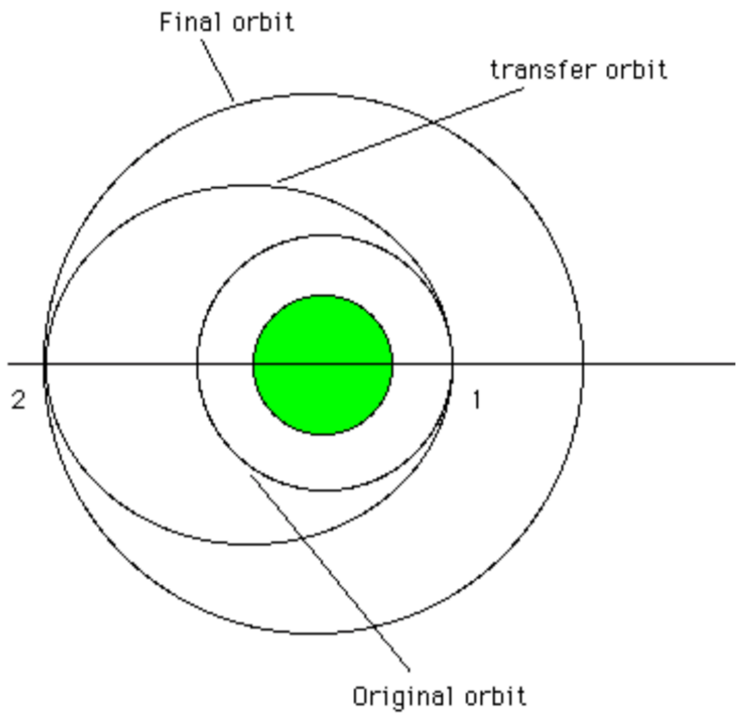

Figure 314:

Initial orbit properties

A circle

$$r = r_1 \quad , \quad v = \sqrt{\frac{GM_e}{r_1}}$$

$$\ell = mrv = mr_1\sqrt{\frac{GM_e}{r_1}}$$

$$E = \frac{1}{2}mv^2 - \frac{GM_e m}{r} = -\frac{GM_e m}{2r_1}$$

Transfer orbit properties

An ellipse

$$r_p = r_1 \;\;,\;\; r_a = r_2 \;\;,\;\; 2a = r_a + r_p = r_1 + r_2$$

$$E = -\frac{GM_em}{2a} = -\frac{GM_em}{r_a + r_p}$$

$$E_a = \frac{1}{2}mv_a^2 - \frac{GM_em}{r_a} = -\frac{GM_em}{r_a + r_p}$$

$$v_a = \sqrt{\frac{2}{m}\left(\frac{GM_em}{r_a} - \frac{GM_em}{r_a + r_p}\right)} = \sqrt{\frac{2GM_e}{r_p}\left(\frac{r_a}{r_a + r_p}\right)}$$

$$\ell_a = mr_av_a = mr_a\sqrt{\frac{2GM_e}{r_a}\left(\frac{r_p}{r_a + r_p}\right)}$$

$$E_p = \frac{1}{2}mv_p^2 - \frac{GM_em}{r_p} = -\frac{GM_em}{r_a + r_p}$$

$$v_p = \sqrt{\frac{2}{m}\left(\frac{GM_em}{r_p} - \frac{GM_em}{r_a + r_p}\right)} = \sqrt{\frac{2GM_e}{r_p}\left(\frac{r_a}{r_a + r_p}\right)}$$

$$\ell_p = mr_pv_p = mr_a\sqrt{\frac{2GM_e}{r_p}\left(\frac{r_a}{r_a + r_p}\right)}$$

Final orbit properties

A circle

$$r = r_2 = r_a \quad , \quad v = \sqrt{\frac{GM_e}{r_2}}$$

$$\ell = mrv = mr_2\sqrt{\frac{GM_e}{r_2}}$$

$$E = \frac{1}{2}mv^2 - \frac{GM_em}{r} = -\frac{GM_em}{2r_2}$$

Initial orbit to transfer orbit

We must fire our rockets to change

$$E_{initial} \rightarrow E_{transfer}$$

or

$$v_{initial} \to v_{transfer\,at\,perigee} \quad , \quad \ell_{initial} \to \ell_{transfer}$$

So we have

$$\Delta E = -\frac{GM_e m}{r_a + r_p} - \left(-\frac{GM_e m}{2r_p}\right) = \frac{GM_e m}{2r_p}\frac{r_a - r_p}{r_a + r_p} > 0$$

So we have to speed up.

Transfer orbit to final orbit

We must fire our rockets to change

$$E_{transfer} \to E_{final}$$

or

$$v_{transfer\,at\,apogee} \to v_{final}$$

or

$$\ell_{transfer} \to \ell_{final}$$

So we have

$$\Delta E = -\frac{GM_e m}{2r_a} - \left(-\frac{GM_e m}{r_a + r_p}\right) = \frac{GM_e m}{2r_a}\frac{r_p - r_a}{r_a + r_p} < 0$$

So we have to slow down.

In order to rendezvous with an object in the final orbit, we must launch at the correct time from the initial orbit.

The transfer orbit is one-half of an ellipse.

Therefore the transfer orbit time is one-half the elliptical orbit period (see Kepler's laws later)

$$T = \frac{1}{2}\tau = \frac{1}{2}\frac{2\pi}{\sqrt{GM_e}}a^{3/2} = \frac{\pi}{\sqrt{GM_e}}(r_i + r_f)^{3/2}$$

This means that the object in the final circular orbit must be behind the object in the initial circular orbit by an angle given by

$$\Delta = w_f T = \frac{v_f}{r_f}\frac{\pi}{\sqrt{GM_e}}(r_i + r_f)^{3/2} = \frac{\sqrt{\frac{GM_e}{r_f}}}{r_f}\frac{\pi}{\sqrt{GM_e}}(r_i + r_f)^{3/2} = \pi\left(1 + \frac{r_i}{r_f}\right)^{3/2}$$

Discuss moon shot — figure below.

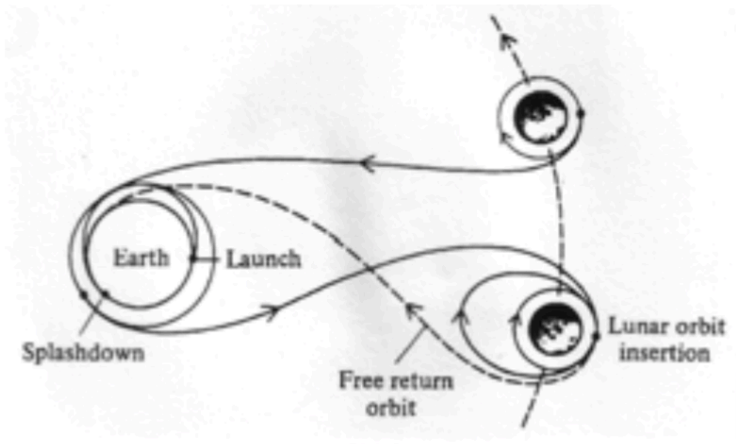

Figure 315:

Kepler's laws

#1: All planets move in elliptical orbits with the sun at one focus. (PROVED already)

#2: The radius vector from the sun to a planet sweeps out equal areas in equal times.

This follows from angular momentum conservation.

$$\frac{1}{2}r^2\dot{\theta} = \frac{\ell}{2m} = \text{ constant } = \frac{\frac{1}{2}r(rd\theta)}{dt} = \frac{dA}{dt} \tag{11.73}$$

#3: The period of revolution T of a planet about the sun is related to the major axis $2a$ of the ellipse by

$$T^2 = ka^3 \tag{11.74}$$

where k is the same constant for all the planets. #2 says that

$$dA = \frac{1}{2}r^2 d\theta = \frac{\ell}{2m}dt$$

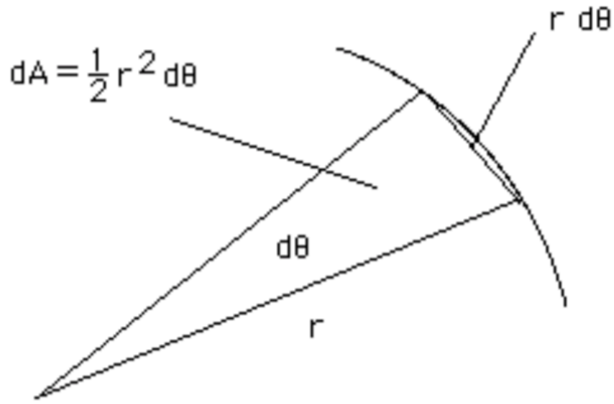

Figure 316:

$$A = \text{ area swept out in 1 period } T = \frac{\ell}{2m} T = \text{ area of the ellipse } = \pi ab$$

$$T = \frac{2\pi mab}{\ell} = \frac{2\pi ma\left(\frac{\ell}{\sqrt{-2mE}}\right)}{\ell}$$

$$T^2 = \frac{4\pi^2 m^2 a^2}{(-2mE)} = 4\pi^2 ma^2 \frac{a}{GMm} = 4\frac{\pi^2}{GM}a^3 \tag{11.75}$$

Perturbed Circular Orbits

Suppose we have a satellite of mass m that is orbiting the earth in a circular orbit of radius r_0.

The orbit is given a slight perturbation by firing the rocket engine briefly in such a way that the energy changes but not the angular momentum.

This can be done by firing the rocket (exerting a force) only in the radial direction.

This kind of perturbation does not change the effective potential curve (which depends on the angular momentum).

So we simply move from the minimum to a higher energy level which corresponds to elliptical motion.

If the final energy is not much different than the initial (or circular orbit energy) energy than the value of r never differs much from r_0.

We can treat this problem as an additional small motion in the radial direction.

We look at U_{eff} and expand it in a power series about r_0.

For small deviations we will have a parabolic potential and provided that the effective spring constant is positive we will have SHM about r_0.

$$U_{eff}(r) = -\frac{GMm}{r} + \frac{\ell^2}{2mr^2}$$

The minimum occurs when $r = r_0$r or

$$\left.\frac{dU_{eff}}{dr}\right|_{r=r_0} = \frac{GMm}{r_0^2} - \frac{\ell^2}{mr_0^3}$$

$$\ell = m\sqrt{MGr_0} = mv_0r_0 \to v_0 = \sqrt{\frac{GM}{r_0}}$$

as earlier for circular motion.

The effective spring constant is given by

$$\left.\frac{dU_{eff}^2}{dr^2}\right|_{r=r_0} = -\frac{2GMm}{r_0^3} + \frac{3\ell^2}{mr_0^4} = \frac{GMm}{r_0^3}$$

which gives an angular frequency of oscillations

$$\omega = \sqrt{\frac{k_{eff}}{m}} = \frac{\ell}{mr_0^2}$$

The radial motion is given by

$$r = r_0 + A\sin\omega t + B\cos\omega t$$

Since $r(0) = r_0$ we must have $B = 0$, so we have

$$r = r_0 + A\sin\omega t$$

We can now derive the orbit equation:

$$\dot{\theta} = \frac{\ell}{mr_0^2} \to \theta = \frac{\ell}{mr_0^2} t = \omega t$$

to the same level of approximation as the radial motion.

Surprise - the frequency of radial perturbation oscillations is the same as the circular orbit frequency.

Thus we have

$$r = r_0 + A\sin\theta = r_0\left(1 + \frac{A}{r_0}\sin\theta\right) = \frac{r_0}{1 - \frac{A}{r_0}\sin\theta} \quad , \quad \frac{A}{r_0} \ll 1$$

which is an ellipse as we predicted.

Orbit Equation: Force Law from Orbit Information

$$m(\ddot{r} - r\dot{\theta}^2) = f(r)$$

$$m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0 \to \ell = mr^2\dot{\theta} = \text{ constant}$$

Change variable

$$u = \frac{1}{r}$$

Algebra gives

$$\frac{du}{d\theta} = \frac{du}{dr}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} = -\frac{1}{r^2}\frac{\dot{r}}{\dot{\theta}}$$

Using the equation for ℓ we get

$$\frac{du}{d\theta} = -\frac{m}{\ell}\dot{r}$$

Then

$$\frac{d^2u}{d\theta^2} = \frac{d}{d\theta}\left(-\frac{m}{\ell}\dot{r}\right) = \frac{dt}{d\theta}\frac{d}{dt}\left(-\frac{m}{\ell}\dot{r}\right) = -\frac{m}{\ell\dot{\theta}}\ddot{r} = -\frac{m^2}{\ell^2}r^2\ddot{r}$$

Thus,

$$\ddot{r} = -\frac{\ell^2}{m^2}u^2\frac{d^2u}{d\theta^2}$$

$$r\dot{\theta}^2 = \frac{\ell^2}{m^2}u^3$$

Then the radial equation of motion becomes

$$m(\ddot{r} - r\dot{\theta}^2) = f(r)$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{\ell^2}\frac{1}{u^2}f(\frac{1}{u})$$

$$\frac{d^2\left(\frac{1}{r}\right)}{d\theta^2} + \frac{1}{r} = -\frac{m}{\ell^2}r^2 f(r) \tag{11.76}$$

This form of the equation of motion is particularly useful if we wish to find the force law that causes a particular motion or orbit equation.

Examples:

(1) Suppose we have an elliptical orbit...(we know the answer to this one!)

$$r = \frac{r_0}{1 - \epsilon\cos\theta} \rightarrow u = \frac{1 - \epsilon\cos\theta}{r_0}$$

$$\frac{d^2\left(\frac{1}{r}\right)}{d\theta^2} + \frac{1}{r} = -\frac{m}{\ell^2}r^2 f(r) = \frac{\epsilon\cos\theta}{r_0} + \frac{1}{r} = \frac{1}{r_0}$$

$$f(r) = -\frac{\ell^2}{mr_0}\frac{1}{r^2}$$

So we get the inverse square field back again!!!

(2) Suppose that we have an exponential orbit.....

$$r = ke^{\alpha\theta} \rightarrow u = \frac{1}{k}e^{-\alpha\theta}$$

$$\frac{d^2\left(\frac{1}{r}\right)}{d\theta^2} + \frac{1}{r} = -\frac{m}{\ell^2}r^2 f(r) = \frac{\alpha^2}{r} + \frac{1}{r}$$

$$f(r) = -\frac{\ell^2}{m}(1 + \alpha^2)\frac{1}{r^3}$$

Thus, an inverse cube force produces an exponential spiral!!!!!

(3) Suppose that we have a θ^2 orbit ....

$$r = k\theta^2 \rightarrow u = \frac{1}{k}\theta^{-2}$$

$$\frac{d^2\left(\frac{1}{r}\right)}{d\theta^2} + \frac{1}{r} = -\frac{m}{\ell^2}r^2 f(r) = \frac{6}{k}\theta^{-4} + \frac{1}{r} = \frac{6k}{r^2} + \frac{1}{r}$$

$$f(r) = -\frac{\ell^2}{m}\left(\frac{1}{r^3} + \frac{6k}{r^4}\right)$$

It takes that weird looking potential to produce this type of spiral orbit!!

Mission to Uranus

Grand Tours of the Outer Planets

Several direct spacecraft missions have been made to the two nearest planetary neighbors of Earth - Venus and Mars.
12pt] A great difficulty in outer-planet exploration is the long time duration of direct flights.

Fortunately, the flight times for outer-planet missions can be considerably shortened by means of gravitational assists as the spacecraft swings by the planets en route.

In the late 1970s the outer planets lined up in a favorable configuration that permitted a single spacecraft to make a Grand Tour of the planets Jupiter, Saturn, Uranus, and Neptune.

The possibility of this four planet mission occurs only at 175 year intervals.

By utilizing the gravitational energy boost obtained from a Jupiter swing-by, the Grand Tour of these four planets was made in 12 years.

In comparison, the flight time for a direct mission to Neptune with equivalent launch energy would take 30 years.

The essential aspects of the gravity-assistance trajectory for the Grand Tour can be developed from the planetary-orbit equations we have derived.

By a similar mechanism a planetary swing-by can act as a brake.

The Mercury Mariner voyage of 1974 used the planet Venus to reduce energy allowing a subsequent close approach to the planet Mercury.

The earth's orbital velocity represents a substantial fraction of the minimum launch velocity needed to send a spacecraft to the outer parts of our solar system.

Thus in sending a spacecraft to the outer planets, the launch should be made in the direction of the earth's orbital velocity about the sun, as illustrated in the 2nd diagram below.

This velocity of the earth, in a nearly circular orbit of radius r_e and period τ_e about the sun, is

$$v_e = \omega_e r_e = \frac{2\pi}{\tau_e} r_e = \frac{6.16(1.5 \times 10^8\,km)}{(365 \times 24 \times 3600\,s)} = 30\,\frac{km}{s}$$

For a spacecraft of mass m at an initial distance r_e from the sun to completely escape the gravitational pull of the sun, the minimum initial velocity necessary (orbit=parabola $\rightarrow E = 0$) is determined by

$$E = 0 = \frac{1}{2} m v_{esc,sun}^2 - \frac{G m M_{sun}}{r_e}$$

From these two equations we find

$$v_{esc,sun} = \sqrt{\frac{2 G M_{sun}}{r_e}} = \sqrt{2} v_e = 42\,\frac{km}{s}$$

This relation between the escape velocity at a distance r from a gravitational source and the velocity in a circular orbit at radius r is always true; that is.,

$$v_{esc}(r) = \sqrt{2} v_e(r)$$

By making the launch from the moving earth, the initial velocity required for escape from the gravitational pull of the sun can be reduced to

$$(\sqrt{2} - 1) v_e = 12\,\frac{km}{s}$$

The spacecraft must have additional initial velocity to escape from the gravitational attraction of the earth.

Direct Uranus Mission

To send a direct mission to the planet Uranus with a minimum amount of propulsion energy, the spacecraft should be launched in the direction of the earth's orbital motion into an elliptical orbit about the sun with perihelion at the earth's orbit and aphelion(the point in the orbit of a planet, asteroid, or comet at which it is furthest from the sun) at the orbit of Uranus, as shown in the figure below.

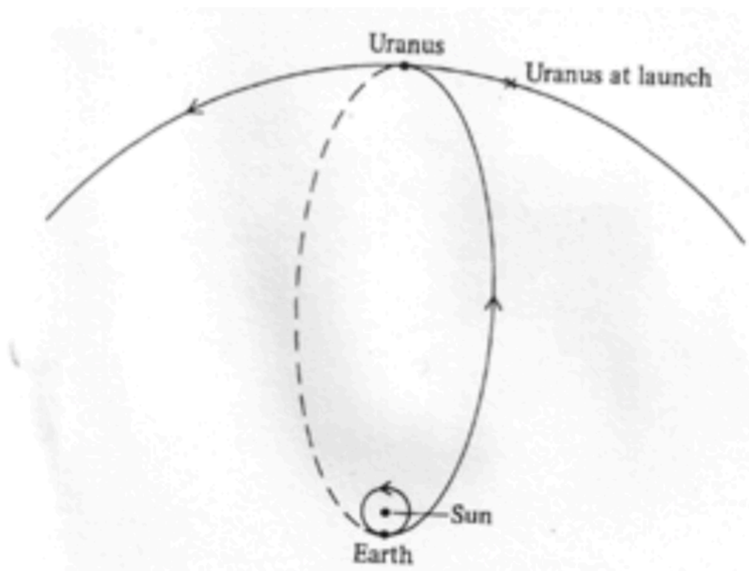

Figure 317:

The launch must be made at the proper time in order that Uranus and the spacecraft arrive together at the aphelion of the spacecraft's orbit.

The minimum and maximum values of the distance r from the sun on this spacecraft orbit are

$$r_{min} = 1\,AU \text{ (at Earth)}$$

$$r_{max} = 19.2 AU \text{ (at Uranus)}$$

where AU stands for the astronomical unit of length, namely, the sun to earth distance of $r = 1.5 \times 10^8\,km$.

The parameters r_0 and ϵ of our orbit equation can be determined from the minimum and maximum values of r,

$$r_{max} = \frac{r_0}{1-\epsilon} \quad , \quad r_{min} = \frac{r_0}{1+\epsilon} \rightarrow \frac{r_{max}}{r_{min}} = \frac{1+\epsilon}{1-\epsilon}$$

$$r_0 = \frac{2r_{min}r_{max}}{r_{min}+r_{max}} = \frac{38.4}{20.2} = 1.9$$

$$\epsilon = 1 - \frac{r_0}{r_{max}} = 1 - \frac{1.9}{19.2} = 0.9$$

Thus the spacecraft orbit from Earth to Uranus is

$$r = \frac{r_0}{1-\epsilon\cos\theta} = \frac{1.9}{1-0.9\cos\theta}$$

The semi-major axis of the orbit is

$$a = \frac{1}{2}(r_{min}+r_{max}) = 10.1\,AU$$

The velocity of the spacecraft at any point on the orbit can be found by

$$E = -\frac{GM_{sun}m}{2a} = \frac{1}{2}mv^2 = \frac{GM_{sun}m}{r}$$

$$v^2 = 2GM_{sun}\left(\frac{1}{r}-\frac{1}{2a}\right) = v_{esc}^2 r_e\left(\frac{1}{r}-\frac{1}{2a}\right)$$

$$v = v_{esc}\sqrt{r_e\left(\frac{1}{r}-\frac{1}{2a}\right)}$$

The perigee velocity necessary for insertion of the spacecraft at $r = r_e$ into the elliptical orbit to Uranus is (using $r = r_e$)

$$v_p = v_{esc,sun}\sqrt{1\left(\frac{1}{r}-\frac{1}{2a}\right)} = v_{esc,sun}\sqrt{\left(\frac{1}{1}-\frac{1}{20.2}\right)} = v_{esc,sun}\sqrt{\frac{19.2}{20.2}} = 41\,\frac{km}{s}$$

The time after earth launch at which the spacecraft reaches Uranus is just the half period $\tau/2$ of the elliptical orbit.

We can use Kepler's third law to calculate the time duration of this mission from the radius r_e and period τ_e of the earth's orbit about the sun.

$$\frac{\tau}{2} = \frac{\tau_e}{2}\left(\frac{a}{r_e}\right)^{3/2} = 16\,years$$

Gravity Boost Mission to Uranus

For the same launch energy as needed for the elliptical orbit, the duration of flight to Uranus can be cut from 16 years to about 5 years on a gravity-assistance orbit which swings by Jupiter, as we will now demonstrate.

The spacecraft is initially launched from Earth into an elliptical orbit about the sun.

For our Grand Tour comparison the first portion of the gravity boost orbit is taken to be the same as the direct mission to Uranus.

The launch time is chosen such that the spacecraft will make a close encounter with Jupiter, as illustrated in the figure below.

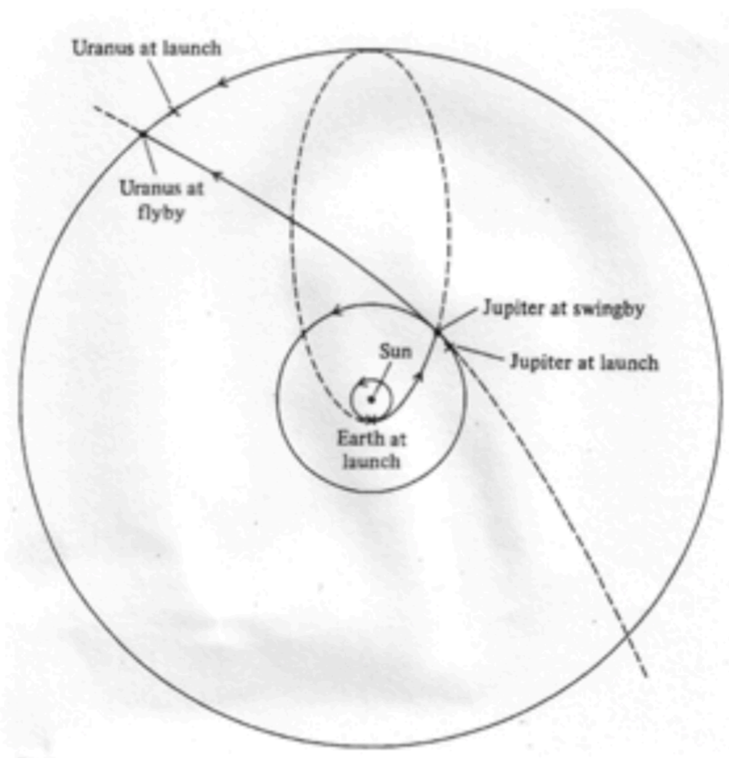

Figure 318:

As a result of the encounter the heliocentric velocity of the spacecraft is changed. In our discussion, we can neglect the slight change in the direction of Jupiter's velocity during the encounter, since the time duration of the encounter is short compared with Jupiter's period of revolution around the sun.

Until the spacecraft reaches the immediate vicinity of Jupiter, the spacecraft's orbit is governed by the strong gravitational field of the sun.

In the vicinity of Jupiter the sun's gravitational force on the spacecraft changes slowly compared to Jupiter's gravitational force so the spacecraft's orbit relative to Jupiter is essentially determined by Jupiter's gravitational field.

If we let $m\vec{v}_i$ and $m\vec{r}_f$ denote the spacecraft momenta in the heliocentric if (sun-centered) inertial frame just before and just after the Jovian encounter, we can write

$$\vec{r}_i = \vec{u}_i + \vec{V}_J$$
$$\vec{r}_f = \vec{u}_f + \vec{V}_J$$

where $\vec{V}_J$ is the velocity of Jupiter about the sun and $\vec{u}_i$ and $\vec{u}_f$ are the spacecraft velocities relative to Jupiter (i.e., in the reference frame in which Jupiter is at rest).

The change in velocity during the encounter is then given by

$$\Delta\vec{v} = \vec{r}_f - \vec{r}_i$$

and is the same in both frames

$$\Delta\vec{v} = \Delta\vec{u}$$

For an elastic collision in the CM (Jupiter fixed) frame

$$u_f = u_i$$

The change in squared heliocentric velocity measures the **gravity boost** in kinetic energy

$$\Delta K = \frac{1}{2}m(v_f^2 - v_i^2)$$

By squaring the two above equations

$$v_f^2 = u_f^2 + 2\vec{u}_f \cdot \vec{V}_J + V_J^2$$

$$v_i^2 = u_i^2 + 2\vec{u}_i \cdot \vec{V}_J + V_J^2$$

and then subtracting we obtain

$$v_f^2 - v_i^2 = 2\Delta\vec{u} \cdot \vec{V}_J = 2\Delta\vec{v} \cdot \vec{V}_J$$

The magnitude of the heliocentric velocity increases (or decreases) depending on whether the projection of $\Delta\vec{u}$ (or $\Delta\vec{v}$) on $\vec{V}_J$ is positive (or negative).

It is easiest to appreciate the implications of this in the planet rest frame.

If the spacecraft crosses the planet's orbit behind the planet, then $\vec{u} \cdot \vec{V}_J$ will be positive and a gravitational boost will result.

If, on the other hand, the spacecraft crosses the planet's path in front, $\vec{u} \cdot \vec{V}_J$ will be negative and the effect will be to brake the spacecraft.

The two situations are illustrated in the figure below by the actual cases of the Voyager mission with a gravitational boost from Jupiter, and the Mercury Mariner mission, which used Venus as a brake.

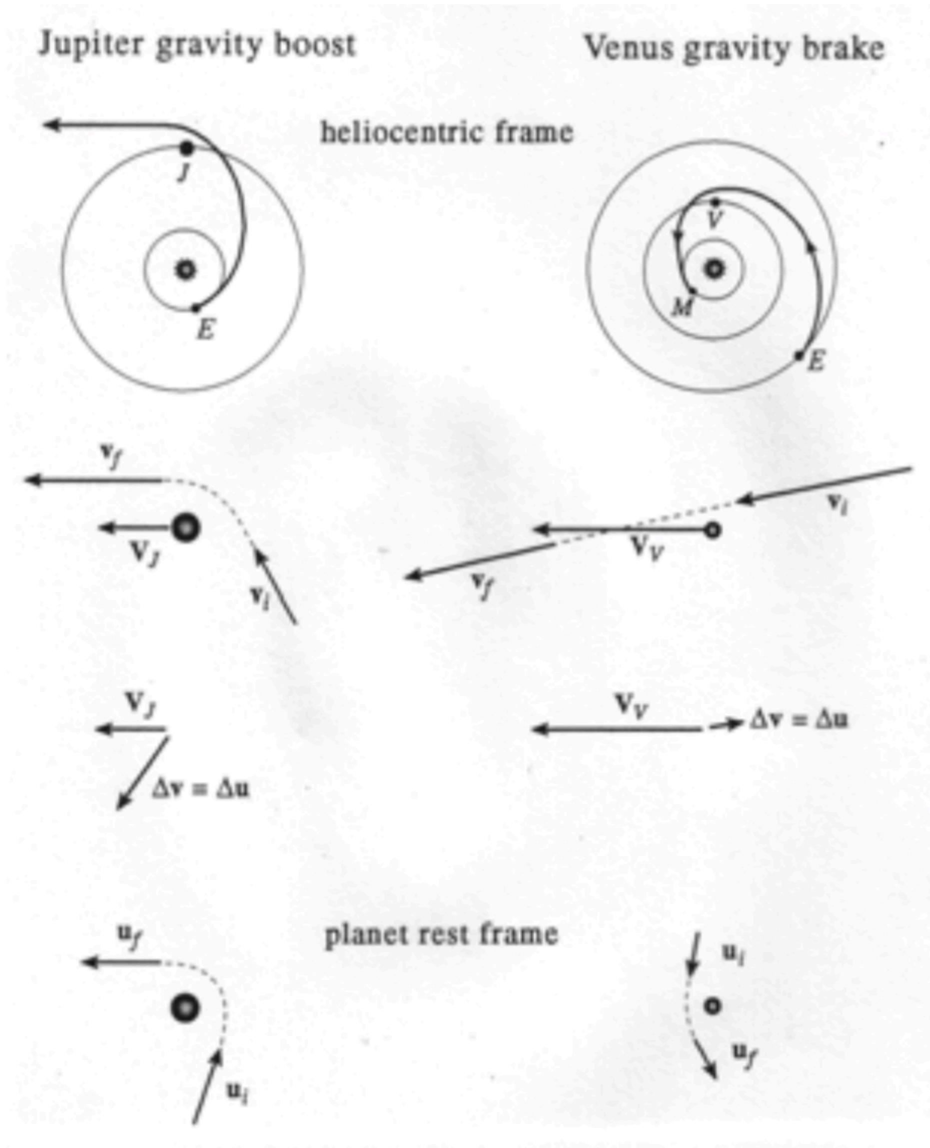

Figure 319:

For a given $\vec{v}_i$ the maximum gravity boost is achieved with $\vec{v}_f$ parallel to $\vec{V}_J$, as can be seen from the velocity diagram in the figure above.

Since $\vec{v}_f = \vec{u}_f + \vec{V}_J$, $\vec{u}_f$ is also parallel to $\vec{V}_J$ and $v_f = u_f + V_J$.

From $\vec{u}_f = \vec{v}_f + \vec{V}_J$ the value of u is

$$u = \sqrt{V_J^2 + v_i^2 - 2\vec{v}_i \cdot \vec{V}_J}$$

The magnitude of Jupiter's nearly circular orbital velocity is

$$V_J = \frac{2\pi}{\tau_J} r_J$$

where $\tau_J = 11.9$ years is Jupiter's period. The velocity v_i of the spacecraft as it nears Jupiter is

$$v_i = v_{esc,sun}\sqrt{\frac{1}{r_J} - \frac{1}{2a}}$$

The value of $\vec{v}_i \cdot \vec{V}_J = v_{i\theta}$ can be found from the conserved angular momentum of the spacecraft orbit, equating ℓ at the Jupiter encounter with its perigee value at earth launch

$$\frac{\ell}{m} = v_{i\theta} r_J = v_p r_e$$

with v_p given by

$$v_p = v_{esc,sun}\sqrt{1\left(\frac{1}{r_{min}} - \frac{1}{2a}\right)} = v_{esc,sun}\sqrt{\left(\frac{1}{1} - \frac{1}{20.2}\right)} = v_{esc,sun}\sqrt{\frac{19.2}{20.2}} = 41\,\frac{km}{s}$$

With $r_e = 1\,AU$, $r_J = 5.2\,AU$, the numerical values of the above quantities are

$$V_J = 13\,\frac{km}{s}$$

$$v_i = 16\,\frac{km}{s}$$

$$v_{i\theta} = \vec{v}_i \cdot \vec{V}_J = 8\,\frac{km}{s}$$

$$u = 14.7\,\frac{km}{s}$$

$$27.713\,\frac{km}{s}$$

The spacecraft leaves the region of Jupiter's influence with its exit velocity $\vec{v}_f$ parallel to Jupiter's velocity.

The outgoing orbit of the space-craft around the sun is another conic section with a turning point at the location of the encounter with Jupiter (that is, $\dot{r} = 0$ at $r = r_J$ since $\vec{v}_f$ is parallel to the circular orbit of Jupiter).

The type of new heliocentric conic section of the spacecraft orbit depends on the amount of velocity boost in the encounter.

Since the escape velocity $v_{esc,sun}$ from the solar system at Jupiter's orbit is

$$v_{esc,sun} = \sqrt{2}V_J$$

the new orbit of the spacecraft is related to the velocity v_f as follows:

$$v_f < \sqrt{2}V_J \rightarrow \text{ ellipse}$$

$$v_f = \sqrt{2}V_J \rightarrow \text{ parabola}$$

$$v_f > \sqrt{2}V_J \rightarrow \text{ hyperbola}$$

For the encounter considered above,

$$\frac{v_f}{\sqrt{2}V_J} = 1.5$$

and the new orbit is hyperbolic, as illustrated in the earlier figure.

The preceding analysis shows how gravity assisted dynamics works.

It remains to be shown that the distance of closest approach to Jupiter exceeds its radius.

In fact $r_{min} - 1.85R_J$ for this slingshot orbit.

Once we have worked out the orbit parameters, the time of flight in any portion of the orbit can be computed.

In our example these times are

$$t \text{ (Earth to Jupiter)} \approx 1.3\, years$$

$$t \text{ (Jupiter to Uranus)} \approx 3.7\, years$$

yielding a total trip time to Uranus of about 5 years compared with the 16 years required for a direct mission.

Of course, these numbers are approximate, since the gravitational influence of Jupiter and the sun on the spacecraft were treated independently.

Numerical methods can be used to make precise calculations of the orbit without such an approximation.

In the late 1970s two Grand Tour missions by NASA were launched, Voyagers 1 and 2 in 1979, which have provided vast amounts of new information on Jupiter, Saturn, Uranus, Neptune and their associated moons and rings.

Voyager 1 left the earth September 5, 1977; visited Jupiter March 5, 1979; Saturn November 12, 1980; then left the solar system.

Voyager 2 left the earth August 20, 1977; visited Jupiter July 9, 1979; Saturn August 2,5 1981; Uranus January 24, 1986; then Neptune on August 24, 1989, before leaving the solar system.

Other spacecraft to use gravitational boosts from Jupiter were the Pioneer 10 and 11, launched in 1973 and 1974, which flew by Jupiter and Saturn.

The trajectories for these Grand Tour missions are illustrated in the figure below.

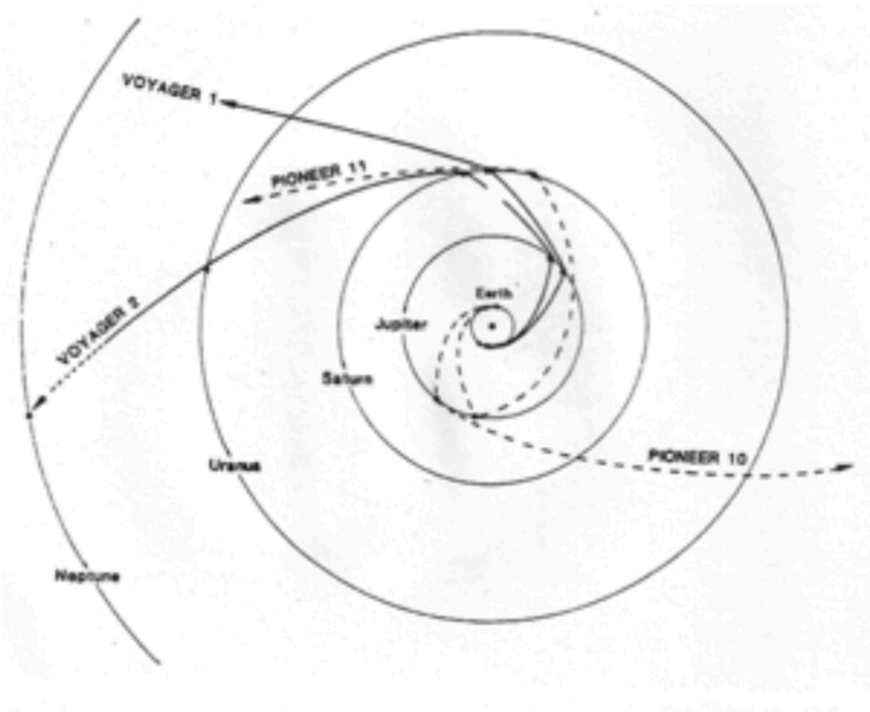

Figure 320:

After the last planetary encounter, the spacecraft on these missions continued to travel away from the sun, escaping the solar gravitational field and entering interstellar space.

Problem 1: A long-range rocket fired from the surface of the Earth (radius R) with velocity $\vec{v} = (v_r, v_\theta)$ See figure below.

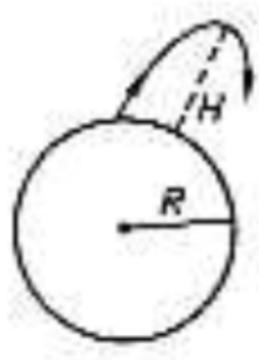

Figure 321:

Neglecting air friction and the rotation of the earth (but assuming the exact gravitational field), obtain an equation to determine the maximum height H achieved by the trajectory.

Solve it to lowest order in $\frac{H}{R}$ and verify that it gives a familiar result for the case that $\vec{v}$ is vertical.

Solution:

Both the angular momentum and the energy of the rocket are conserved under the action of gravity, a central force.

Considering the initial and final state when the rocket achieves maximum height, we have

$$mv_\theta = m(R+H)v'_\theta$$

$$\frac{1}{2}m(v_\theta^2 + v_r^2) - \frac{GMm}{R} = \frac{1}{2}mv'^2_\theta - \frac{GMm}{R+H}$$

where the primes refer to the final state at which the radial component of its velocity vanishes.

m and M are the masses of the rocket and the Earth, respectively.

Combining the above two equations we obtain

$$\frac{1}{2}m(v_\theta^2 + v_r^2) - \frac{GMm}{R} = \frac{1}{2}m\left(\frac{R}{R+H}\right)^2 v_\theta^2 - \frac{GMm}{R+H}$$

which gives the maximum height H.

Considering only the lowest order terms in $\frac{H}{R}$, we have

$$\frac{1}{2}m(v_\theta^2 + v_r^2) - \frac{GMm}{R} \approx \frac{1}{2}m\left(1 - \frac{2H}{R}\right)v_\theta^2 - \frac{GMm}{R}\left(1 - \frac{H}{|}R\right)$$

and hence

$$H \approx \frac{v_r^2 R}{2\left(\frac{GM}{R} - v_\theta^2\right)}$$

For vertical launching, $v_\theta = 0$, $v_r = v$, and if $\frac{H}{R}$ is small, we can consider g as a constant with $g = \frac{GM}{R^2}$.

We then obtain the familiar formula

$$H \approx \frac{v^2}{2\left(\frac{GM}{R^2}\right)} = \frac{v^2}{g}$$

Problem 2: A comet in an orbit about the Sun has a velocity of 10 km/sec at aphelion(the point in the orbit of a planet, asteroid, or comet at which it is furthest from the sun) and 80 km/sec at perihelion(the point in the orbit of a planet, asteroid, or comet at which it is closest from the sun). See figure below.

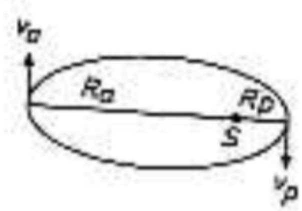

Figure 322:

If the earth's velocity in a circular orbit is 30 km/sec and the radius of its orbit is 1.5×10^8 km, find the aphelion distance R_a for the comet.

Solution:

Let v be the velocity of the Earth, R the radius of the Earth's orbit, m and m_s the masses of the earth and the Sun, respectively.

Then

$$\frac{mv^2}{R} = \frac{Gmm_s}{R^2}$$

or

$$Gm_s = Rv^2$$

By the conservation of energy and of angular momentum of the comet, we have

$$-\frac{Gm_e m_s}{R_a} + \frac{m_c v_a^2}{2} = -\frac{Gm_e m_s}{R_p} + \frac{m_c v_p^2}{2}$$

$$m_c R_a v_a = m_c R_p v_p$$

where m_c is the mass of the comet, and v_a and v_p are the velocities of the comet at aphelion and at perihelion, respectively.

The above equation then gives

$$R_a = \frac{2Gm_s}{v_a(v_a + v_p)} = \frac{2Rv^2}{v_a(v_a + v_p)} = 3 \times 10^8\, km$$

Problem3: The orbit of a particle moving under the influence of a central force is $r\theta$ = constant.

Determine the potential energy function as a function of r

Solution:

Consider a central force $\vec{F} = F(r)\hat{r}$ acting on a particle of mass m.

Newton's second law gives

$$F = m(\ddot{r} - r\dot{\theta}^2) \tag{11.77}$$

$$0 = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \tag{11.78}$$

in polar coordinates.

Equation(11.78) gives

$$r^2\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$$

or

$$r^2\dot{\theta} = \text{ constant } = h$$

or

$$\dot{\theta} = hu^2$$

by putting

$$r = \frac{1}{u}$$

Then as

$$\dot{r} = \frac{dr}{dt} = \dot{\theta}\frac{dr}{d\theta} = hu^2\frac{dr}{d\theta} = -h\frac{du}{d\theta}$$

$$\ddot{r} = \dot{\theta}\frac{d\dot{r}}{d\theta} = hu^2\frac{d}{d\theta}\left(-h\frac{du}{d\theta}\right) = -h^2u^2\frac{d^2u}{d\theta^2}$$

$$r\dot{\theta}^2 = \frac{1}{u}h^2u^4 = h^2u^3$$

Equation(11.77) becomes

$$F = -mh^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$

which is often known a Binet's formula.

In this problem, let $r = \frac{1}{u}$ and write the equation of the trajectory as

$$u = C\theta$$

where C is a constant.

Binet's formula then gives

$$F = -mh^2u^3 = -\frac{mh^2}{r^3}$$

The potential energy is by definition

$$V = -\int_{\infty}^{r} F(r)dr = \int_{\infty}^{r} \frac{mh^2}{r^3} = \left[-\frac{mh^2}{2r^2}\right]_{\infty}^{r} = -\frac{mh^2}{2r^2}$$

taking $r = \infty$ as the zero potential level.

12. Motion in a Noninertial Reference Frame

If a particle of mass m is accelerating at rate $\vec{a}_{in}$ with respect to an inertial reference frame and at rate $\vec{a}_{rot}$ with respect to a rotating reference frame, then we always have the equation of motion in the inertial frame

$$\vec{F}_{in} = \vec{F} = m\vec{a} = m\vec{a}_{in} \tag{12.1}$$

We would like to write the equation of motion in the rotating frame as

$$\vec{F}_{rot} = m\vec{a}_{rot} \tag{12.2}$$

If we let $\vec{a} = \vec{a}_{rot} + \vec{A}$, where $\vec{A}$ = the relative acceleration between the two frames, then we have

$$\vec{F}_{rot} = m\vec{a}_{rot} = m(\vec{a} - \vec{A}) = \vec{F} + \vec{F}_{fict} \tag{12.3}$$

where $\vec{F}_{fict} = -m\vec{A} =$ the "fictitious" force, which only appears in the rotating or noninertial reference frame.

Rotating Coordinate Systems

Let us consider two sets of coordinate axes.

Let one set be the “fixed” or inertial axes and let the other be another set that is rotating with respect to the inertial system.

We designate these axes as the “fixed” and “rotating” axes, respectively.

We use x_i' as coordinates in the fixed system and x_i as coordinates in the rotating system.

If we choose some point P as in the figure below, we have

$$\vec{r}' = \vec{R} + \vec{r} \tag{12.4}$$

where $\vec{r}'$ is the radius vector of P in the fixed system and $\vec{r}$ is the radius vector of P in the rotating system.

The vector $\vec{R}$ locates the origin of the rotating system in the fixed system. See figure below.

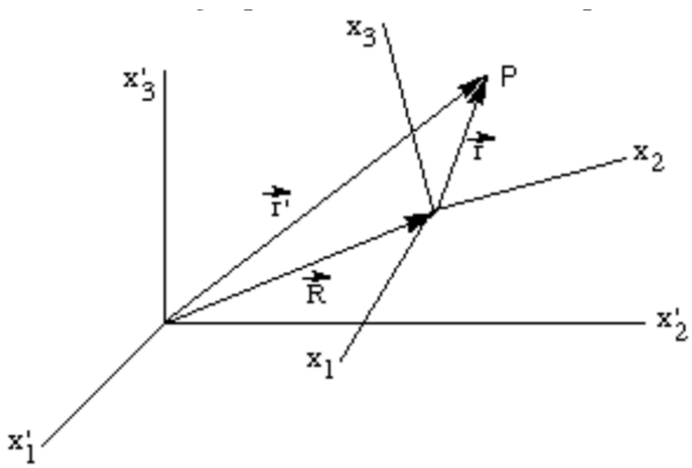

Figure 323:

If we assume that the point P is at rest in the rotating system, the rotating system has angular velocity $\vec{\omega}$ with respect to the fixed system and that

$$\vec{r} = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3 \tag{12.5}$$

then

$$\begin{aligned}\left(\frac{d\vec{r}}{dt}\right)_{fixed} &= \frac{d}{dt}(x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3) \\ &= \dot{x}_1\hat{e}_1 + \dot{x}_2\hat{e}_2 + \dot{x}_3\hat{e}_3 + x_1\dot{\hat{e}}_1 + x_2\dot{\hat{e}}_2 + x_3\dot{\hat{e}}_3 \\ &= \left(\frac{d\vec{r}}{dt}\right)_{rot} + x_1\dot{\hat{e}}_1 + x_2\dot{\hat{e}}_2 + x_3\dot{\hat{e}}_3 \end{aligned} \tag{12.6}$$

Now

$$\dot{\hat{e}}_i = \vec{\omega} \times \hat{e}_i --- \text{ think } \vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r} \tag{12.7}$$

so that

$$\left(\frac{d\vec{r}}{dt}\right)_{fixed} = \left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\omega} \times \vec{r} \tag{12.8}$$

In fact, this relation is true for any vector $\vec{Q}$ (not just the position vector $\vec{r}$),

$$\left(\frac{d\vec{Q}}{dt}\right)_{fixed} = \left(\frac{d\vec{Q}}{dt}\right)_{rot} + \vec{\omega} \times \vec{Q} \tag{12.9}$$

Note that the angular acceleration $\dot{\vec{\omega}}$ is the same in both the fixed and rotating systems:

$$\left(\frac{d\vec{\omega}}{dt}\right)_{fixed} = \left(\frac{d\vec{\omega}}{dt}\right)_{rot} + \vec{\omega} \times \vec{\omega} = \left(\frac{d\vec{\omega}}{dt}\right)_{rot} = \dot{\vec{\omega}} \tag{12.10}$$

We now can determine the velocity of P as measured in the fixed system.

We have

$$\left(\frac{d\vec{r}'}{dt}\right)_{fixed} = \left(\frac{d\vec{R}}{dt}\right)_{fixed} + \left(\frac{d\vec{r}}{dt}\right)_{fixed} = \left(\frac{d\vec{R}}{dt}\right)_{fixed} + \left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\omega} \times \vec{r} \tag{12.11}$$

Let us define

$$\vec{v}_f \equiv \dot{\vec{r}}_f \equiv \left(\frac{d\vec{r}'}{dt}\right)_{fixed} = \text{ velocity relative to the fixed axes} \tag{12.12}$$

$$\vec{V} \equiv r\dot{\vec{R}}_f \equiv \left(\frac{d\vec{R}}{dt}\right)_{fixed} = \text{ linear velocity of moving origin} \tag{12.13}$$

$$\vec{v}_r \equiv \dot{\vec{r}}_r \equiv \left(\frac{d\vec{r}}{dt}\right)_{rot} = \text{ velocity relative to the rotating axes} \tag{12.14}$$

and

$$\vec{\omega} = \text{ angular velocity of the rotating axes}$$

$$\vec{\omega} \times \vec{r} = \text{ velocity due to the rotation of the moving axes}$$

so that

$$\vec{v}_f = \vec{V} + \vec{v}_r + \vec{\omega} \times \vec{r} \tag{12.15}$$

Centrifugal and Coriolis Forces

Newton's 2nd law is valid in the inertial or fixed frame so that we can write

$$\vec{F} = m\vec{a}_f = m\left(\frac{d\vec{v}_f}{dt}\right)_{fixed} \tag{12.16}$$

where the differentiation is carried out with respect to the fixed system.

We then have (take derivative of Equation(12.15) in fixed frame)

$$\left(\frac{d\vec{v}_f}{dt}\right)_{fixed} = \left(\frac{d\vec{V}}{dt}\right)_{fixed} + \left(\frac{d\vec{r}_r}{dt}\right)_{fixed} + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{fixed} \tag{12.17}$$

We define the first term as

$$\left(\frac{d\vec{V}}{dt}\right)_{fixed} = \ddot{\vec{R}}_f = \text{ acceleration of origin in fixed system} \tag{12.18}$$

The second term becomes

$$\left(\frac{d\vec{r}_r}{dt}\right)_{fixed} = \left(\frac{d\vec{r}_r}{dt}\right)_{rot} + \vec{\omega} \times \vec{v}_r = \vec{a}_r + \vec{\omega} \times \vec{v}_r \tag{12.19}$$

where $\vec{a}_r$ = acceleration in the rotating coordinate system. The last term becomes

$$\vec{\omega} \times \left(\frac{d\vec{r}_r}{dt}\right)_{fixed} = \vec{\omega} \times \left(\frac{d\vec{r}_r}{dt}\right)_{rot} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) = \vec{\omega} \times \vec{v}_r + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \tag{12.20}$$

We then obtain

$$\vec{F} = m\vec{a}_f = m\ddot{\vec{R}}_f + m\vec{a}_r + m\dot{\vec{\omega}} \times \vec{r} + 2m\vec{\omega} \times \vec{v}_r + m\vec{\omega} \times (\vec{\omega} \times \vec{r}) \tag{12.21}$$

To an observer in the rotating coordinate system, however, the effective force on the particle is given by

$$\vec{F}_{eff} = m\vec{a}_r = \vec{F} - m\ddot{\vec{R}}_f - m\dot{\vec{\omega}} \times \vec{r} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m\vec{\omega} \times \vec{v}_r \tag{12.22}$$

The first term $\vec{F}$ is the external force acting on the particle as measured in the fixed inertial system.

The second term $-m\ddot{\vec{R}}_f$ and the third term $-m\dot{\vec{\omega}} \times \vec{r}$ result because of the translational and angular acceleration, respectively, of the moving coordinate system relative to the fixed system.

The fourth term $-m\vec{\omega} \times (\vec{\omega} \times \vec{r})$ is the **centrifugal** force term and reduces to $mr\omega^2$ for the case in which $\vec{\omega}$ is normal to the radius vector.

The minus sign indicates that the centrifugal force is directed outward from the center of rotation as shown in the figure below.

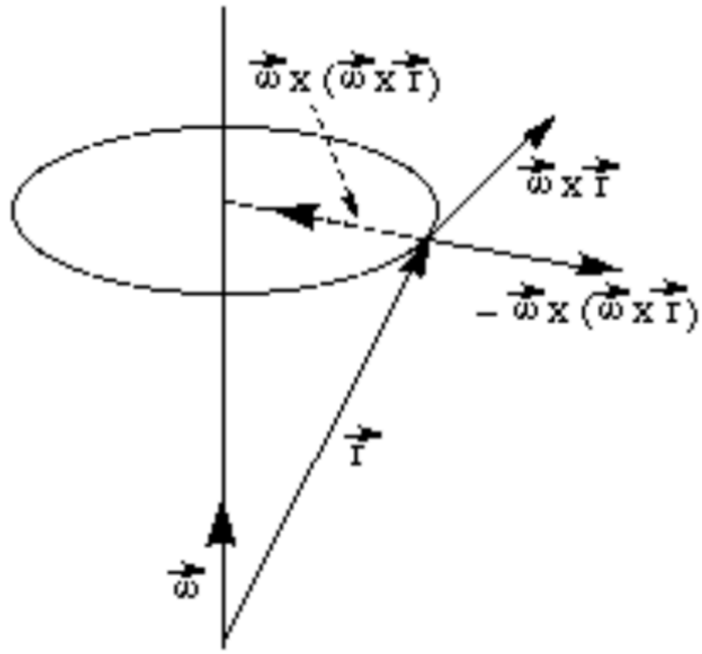

Figure 324:

In the fixed frame, as we saw in our work earlier in the course, this term corresponds to the inward radial (**centripetal**) acceleration $a = r\omega^2 = v^2/r$ that exists during circular motion.

The fifth and last term is a totally new quantity that arises from the motion of the particle in the rotating coordinate system.

It is called the **Coriolis** force.

Note that the Coriolis force does indeed arise from the motion of the particle because the force is proportional to v_r and hence vanishes if there is no motion.

What is the physical meaning of these quantities?

It is important to realize that the centrifugal and Coriolis forces are not forces in the usual sense of the word; they have been introduced in an artificial manner as a result of our arbitrary requirement that we be able to write equation(12.2)

$$\vec{F}_{eff} = m\vec{a}^{rot}$$

i.e., that we be able to write an equation resembling Newton's 2nd law that is valid in the rotating (noninertial) reference frame.

In order to do this we must have

$$\vec{F}_{eff} = \vec{F} + \text{ noninertial terms} \tag{12.23}$$

where the "noninertial terms" are identified as the centrifugal and Coriolis "forces".

Thus, for example, if a body rotates about a fixed force center, the only real force on the body is the force of attraction toward the force center (and gives rise to centripetal acceleration).

An observer moving with the rotating body, however, measures the central force and also notes that the body does not fall toward the force center.

To reconcile this result with the requirement that the net force on the body vanish, the observer must postulate an additional force - the centrifugal force.

But the "requirement" is artificial; it arises solely from an attempt to extend the form of Newton's 2nd law to a noninertial system, and this can be done only by introducing a fictitious "correction forces".

The same can be said for the Coriolis force; this "force" arises when an attempt is made to describe motion relative to the rotating body.

Motion Relative to the Earth

The motion of the Earth with respect to an inertial frame (the fixed stars) is dominated by the Earth's rotation about its own axis.

If we place a fixed inertial frame $x'y'z'$ at the center of the earth and the moving reference frame xyz on the surface of the Earth, then we can describe the motion of a moving object close to the surface of the Earth as shown in the figure below.

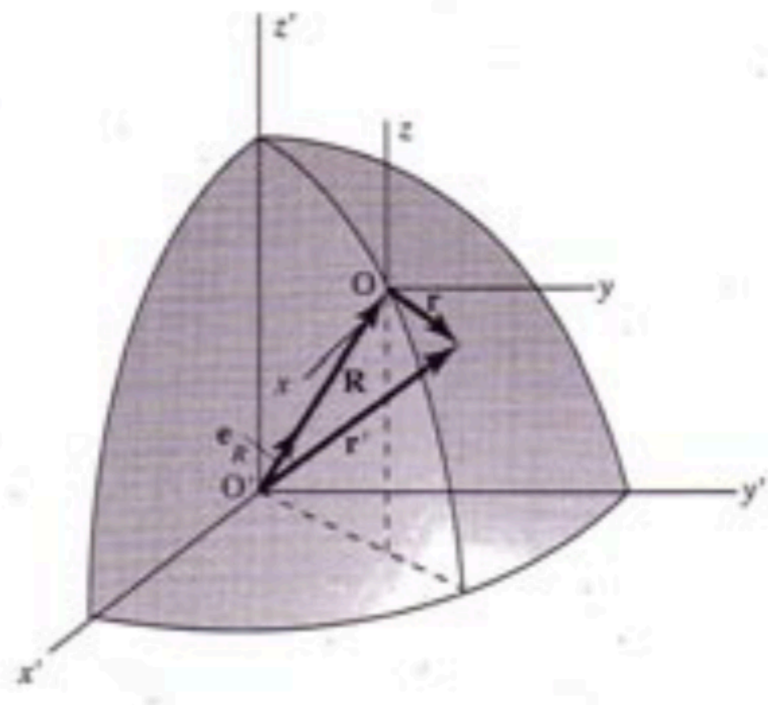

Figure 325:

We can then apply equation(12.22) to the dynamical motion.

Let us write the forces as measured in the fixed inertial system as

$$\vec{F} = \vec{F}_{other} + \vec{F}_{grav} = \vec{F}_{other} + m\vec{g}_0 \tag{12.24}$$

where $\vec{F}_{other}$ represents the sum of all external forces other than gravitation and $m\vec{g}_0$ represents the gravitational attraction to the Earth.

In this case

$$\vec{g}_0 = -G\frac{M_E}{R_E^2}\hat{e}_R = \text{ gravitation field vector} \tag{12.25}$$

where M_E = mass of the Earth, R_E = radius of the Earth, and the unit vector $\hat{e}_R$ is a unit vector unlog the direction $\vec{R}$ as shown in the figure above.

We are assuming the Earth is spherical and has a uniform density and that

$\vec{R}$ originates from the center of the Earth.

The effective force $\vec{F}_{eff}$ as measured in the moving system placed on the surface of the Earth becomes

$$\vec{F}_{eff} = \vec{F}_{other} + m\vec{g}_0 - m\ddot{\vec{R}}_f - m\dot{\vec{\omega}} \times \vec{r} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m\vec{\omega} \times \vec{v}_r \tag{12.26}$$

We let the Earth's angular velocity $\vec{\omega}$ be along the inertial system's x' direction ($\hat{e}'_z$).

The magnitude is $\omega = 7.3 \times 10^{-5}$ rad/sec.

We assume to a good approximation that the value of $\vec{\omega}$ is constant in time and thus the term $m\dot{\vec{\omega}} \times \vec{r}$ can be neglected.

According to equation(12.9) we can write

$$\ddot{\vec{R}}_f = \vec{\omega} \times \dot{\vec{R}}_f = \vec{\omega} \times (\vec{\omega} \times \vec{R}) \tag{12.27}$$

Equation(12.26) then becomes

$$\begin{aligned} \vec{F}_{eff} &= \vec{F}_{other} + m\vec{g}_0 - m\vec{\omega} \times (\vec{\omega} \times \vec{R}) - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m\vec{\omega} \times \vec{v}_r \\ &= \vec{F}_{other} + m\vec{g}_0) - m\vec{\omega} \times (\vec{\omega} \times (\vec{r} + \vec{R})) - 2m\vec{\omega} \times \vec{v}_r \end{aligned} \tag{12.28}$$

The second and third terms (divided by m) are what we experience and measure on the surface of the Earth as the effective $\vec{g}$ (actual measured acceleration due to gravity).

Its value is

$$\vec{g} = \vec{g}_0 - m\vec{\omega} \times (\vec{\omega} \times (\vec{r} + \vec{R})) \tag{12.29}$$

The second term in (12.29) is the centrifugal force.

Because we are limiting our attention to motion near the surface of the Earth, we have $r \ll R$ and therefore to a good approximation

$$\vec{g} = \vec{g}_0 - m\vec{\omega} \times (\vec{\omega} \times \vec{R}) \tag{12.30}$$

Equation(12.28) then becomes

$$\vec{F}_{eff} = \vec{F}_{other} + m\vec{g} - 2m\vec{\omega} \times \vec{v}_r \tag{12.31}$$

This is the appropriate equation to use to discuss the motion of objects close to the surface of the Earth.

To get a picture of the effective $\vec{g}$ consider the diagram below.

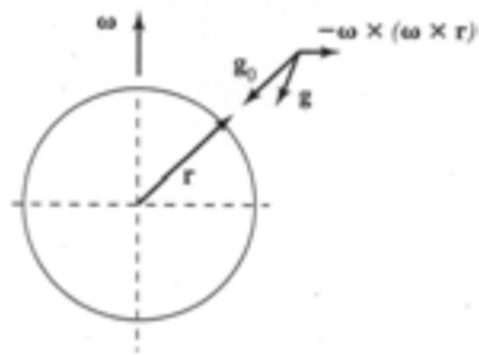

Figure 326:

where we have assumed the z direction is upwards.

$\vec{g}_0$ is along a radial line (called the "true" vertical).

The centrifugal term and the effective $\vec{g}$ are along the directions shown (deviation from the radial line is greatly exaggerated).

A mass on a string will hang in the direction of the effective $\vec{g}$ (called the "local" vertical).

Coriolis Force Effects

The angular velocity vector $\vec{\omega}$, which represents the Earth's rotation about its axis, is directed in a northerly direction.

Therefore, in the Northern Hemisphere, $\vec{\omega}$ has a component ω_z directed **outward** along the local vertical.

If a particle is projected in a horizontal plane (in the local coordinate system at the surface of the Earth) with a velocity $\vec{v}_r$, then the Coriolis force $-2m\vec{\omega} \times \vec{v}_r$, has a component in the plane of magnitude $2m\omega_z v_r$ directed toward the right of the particle's motion as in the figure below

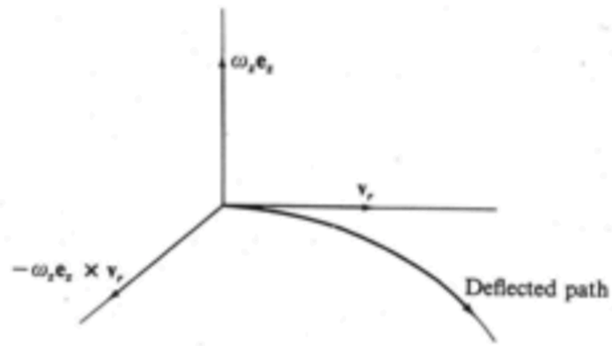

Figure 327:

and a deflection from the original direction of motion results Because the magnitude of the horizontal component of the Coriolis force is proportional to the vertical component of $\vec{\omega}$, the portion of the Coriolis force producing deflections depends on the latitude, being a maximum at the North Pole and zero at the equator.

In the Southern Hemisphere, the component ω_z is directed **inward** along the local vertical, and hence, all deflections are in the opposite sense from those in the Northern Hemisphere.

During a naval engagement near the Falkland Islands early in World War I, the British gunners were surprised to see their accurately aimed salvos falling 100 yards to the left of the German ships.

The designers of the sighting mechanisms were well aware of the Coriolis deflection and had carefully taken this into account, but they apparently were under the impression that ALL sea battles took place near 50° N latitude (England) and never near 50° S latitude (Falkland Islands).

The British shots, therefore, fell at a distance from the targets equal to twice the Coriolis deflection.

The most noticeable effect of the Coriolis force is that on the air masses.

As air flows from high-pressure regions to low-pressure regions, the Coriolis force deflects the air toward the right in the Northern Hemisphere, producing cyclonic motion as shown in the figure below.

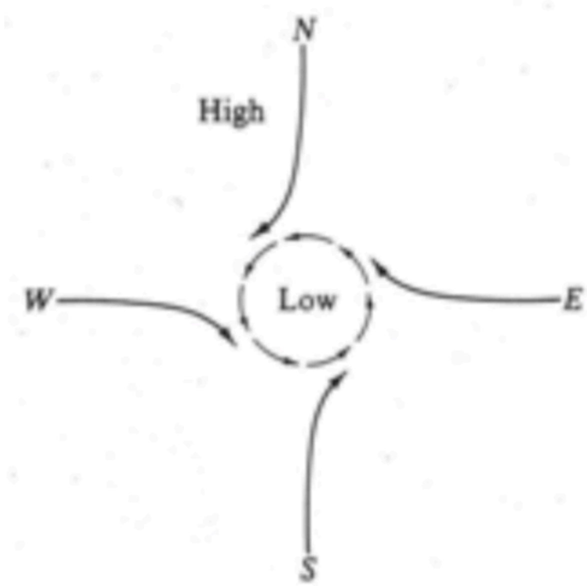

Figure 328:

The air rotates with high pressure on the right and with low pressure on the left.

The high pressure prevents the Coriolis force from deflecting the air masses farther to the right, resulting in a counterclockwise flow of air.

In temperate regions, the airflow does not tend to be along pressure gradients, but along the pressure isobars due to the Coriolis force and the associated centrifugal force of the rotation.

Let us now determine the horizontal deflection from the local vertical caused by the Coriolis force acting on a particle falling freely in the Earth's gravitational field from a height h above the Earth's surface.

We use equation(12.31) with $\vec{F}_{other} = 0$. If we set $\vec{F}_{eff} = m\vec{a}_f$ we can solve for the acceleration of the particle in the rotating coordinate system fixed on the Earth.

$$\vec{a}_r = \vec{g} = 2\vec{\omega} \times \vec{v}_r \tag{12.32}$$

The acceleration due to gravity $\vec{g}$ is the effective one along the local vertical.

We choose the z-axis directed vertically outward (along $-\vec{g}$) from the surface of the Earth.

With this definition of $\hat{e}_z$, we complete the construction of a right-handed coordinate system by specifying that $\hat{e}_x$ be in the southerly direction and that $\hat{e}_y$ be in the easterly direction as shown in the figure below.

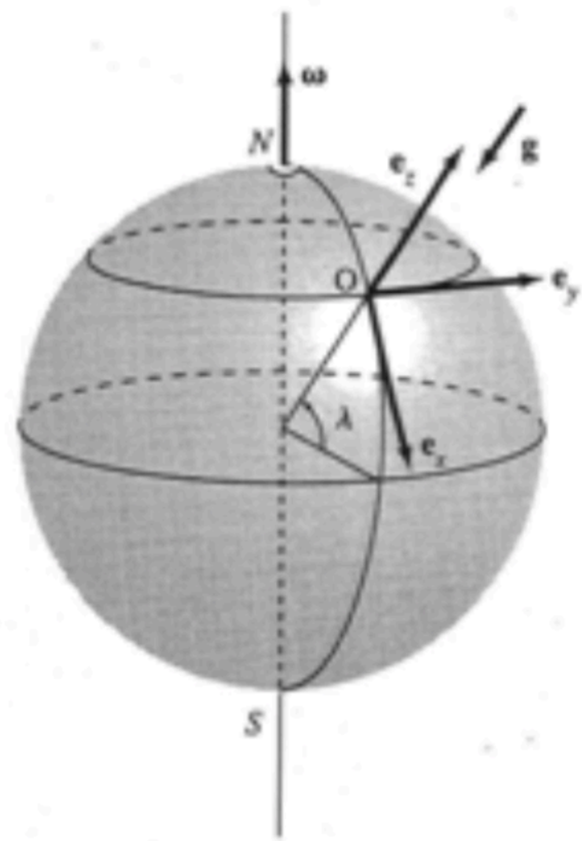

Figure 329:

We also assume that h is sufficiently small that g remains constant during the process.

Because we have chosen the origin O of the rotating coordinate system to lie in the Northern Hemisphere, we have

$$\omega_x = -\omega\cos\lambda \ , \ \omega_y = 0 \ , \ \omega_z = \omega\sin\lambda \tag{12.33}$$

where the angle λ specifies the latitude as shown in the figure.

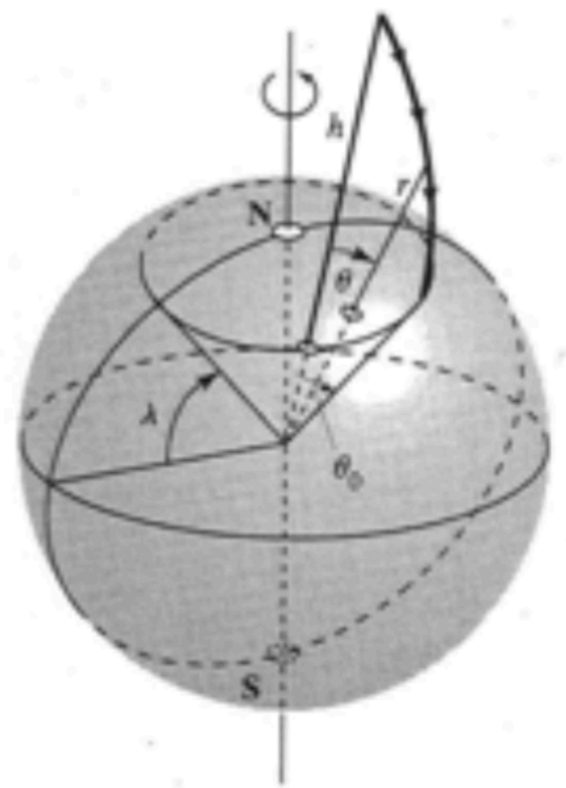

Figure 330:

Although the Coriolis force produces small velocity components in $\hat{e}_y$ and $\hat{e}_x$ directions, we can certainly neglect $\dot{x}$ and $\dot{y}$ compared to $\dot{z}$, the vertical velocity.

Then, approximately, we have

$$\dot{x} \approx 0 \ , \ \dot{y} \approx 0 \ , \ \dot{z} = -gt \tag{12.34}$$

where we have obtained $\dot{z} = -gt$ by assuming a fall from rest.

Therefore, we have

$$\begin{aligned}\vec{\omega} \times \vec{v}_r &= (-\omega \cos\lambda \hat{e}_x + \omega \sin\lambda \hat{e}_z) \times (-gt\hat{e}_z) \\ &= -(\omega g t \cos\lambda)\hat{e}_x \times \hat{e}_z = -(\omega g t \cos\lambda)\hat{e}_y \end{aligned} \tag{12.35}$$

Now the components of $\vec{g}$ are

$$g_x = 0 \ , \ \ g_y = 0 \ , \ \ g_z = -g \tag{12.36}$$

so the equations of motion for the components of $\vec{a}_r$ (neglecting terms containing ω^2 which is very small) become

$$(\vec{a}_r)_x = \ddot{x} \approx 0 \tag{12.37}$$

$$(\vec{a}_r)_y = \ddot{y} \approx 2\omega g t \cos\lambda \tag{12.38}$$

$$(\vec{a}_r)_z = \ddot{z} \approx -g \tag{12.39}$$

Thus, the effect of the Coriolis force is to produce an acceleration in the $\hat{e}_y$ or easterly direction.

Integrating $\ddot{y}$ twice we have

$$y(t) = \frac{1}{3}\omega g t^3 \cos\lambda \tag{12.40}$$

assuming $y(0) = \dot{y}(0) = 0$ at $t = 0$.

The integration of $\ddot{z} \approx -g$ gives the familiar result for the distance of fall

$$z(t) = h - \frac{1}{2}gt^2 \tag{12.41}$$

using $z(0) = h$, $\dot{z}(0) = 0$.

The time of fall is $t \approx \sqrt{2hg}$.

Hence the result for the eastward deflection d of a particle dropped from rest at a height h at a northern latitude λ is

$$d = \frac{1}{3}\omega\sqrt{\frac{8h^3}{g}}\cos\lambda \tag{12.42}$$

An object dropped from a height of 100 meters at a latitude of 45° is deflected approximately 1.55 cm to the east (neglecting air resistance)!

Foucault Pendulum

The effect of the Coriolis force on the motion of a pendulum produces a precession or rotation with time of the plane of oscillation.

This is called a Foucault pendulum.

To describe this effect, let us select a set of coordinate axes with origin at the equilibrium point of the pendulum and the z-axis along the local vertical.

We are only interested in the rotation of the plane of oscillation, that is, we want to consider the motion of the pendulum bob in the $x-y$ plane (the horizontal plane).

We therefore limit the motion to oscillation of very small amplitude, with the horizontal excursions small compared to the length of the pendulum.

Under these conditions $\dot{z}$ is small compared with $\dot{x}$ and $\dot{y}$ and can be neglected.

Considering the figure below

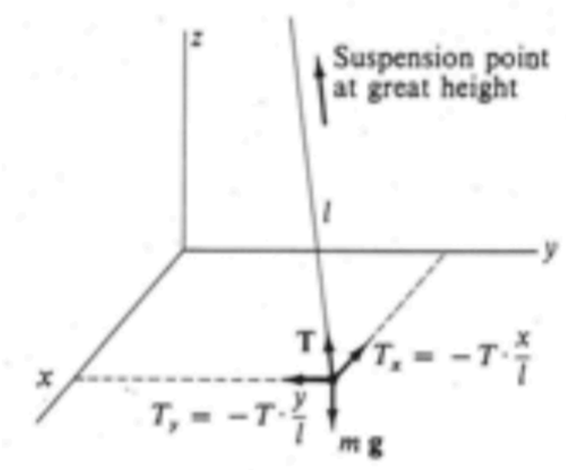

Figure 331:

the equation of motion is

$$\vec{a}_r = \vec{g} + \frac{\vec{T}}{m} - 2\vec{\omega} \times \vec{v}_r$$

We therefore have, approximately,

$$T_x = -T\frac{x}{\ell} \;\; , \;\; T_y = -T\frac{y}{\ell} \;\; , \;\; T_z \approx T$$

AS before,

$$g_x = 0 \;\; , \;\; g_y = 0 \;\; , \;\; g_z = -g$$

and

$$\omega_x = -\omega\cos\lambda \;\; , \;\; \omega_y = 0 \;\; , \;\; \omega_z = \omega\sin\lambda$$

with

$$(\vec{v}_r)_x = \dot{x} \;\; , \;\; (\vec{v}_r)_y = \dot{y} \;\; , \;\; (\vec{v}_r)_z = \dot{x} \approx 0$$

Therefore, we have

$$\begin{aligned}\vec{\omega} \times \vec{v}_r &= (-\omega\cos\lambda\hat{e}_x + \omega\sin\lambda\hat{e}_z) \times (\dot{x}\hat{e}_x + \dot{y}\hat{e}_y) \\ &= -\dot{y}\omega\sin\lambda\hat{e}_x + \dot{x}\omega\sin\lambda\hat{e}_y - \dot{y}\omega\cos\lambda\hat{e}_z\end{aligned}$$

Thus, the equations of interest are

$$\begin{aligned}\ddot{x} &= -T\frac{x}{m\ell} + 2\dot{y}\omega\sin\lambda \\ \ddot{y} &= -T\frac{y}{m\ell} - 2\dot{x}\omega\sin\lambda\end{aligned}$$

For small displacements, $T \approx mg$.

Defining $\alpha^2 = T/m\ell \approx g/\ell$, and writing $\omega_z = \omega\sin\lambda$ we get

$$\begin{aligned}\ddot{x} + \alpha^2 x &= 2\omega_z\dot{y} \\ \ddot{y} + \alpha^2 y &= -2\omega_z\dot{x}\end{aligned}$$

These are **coupled equations** in that the x-equation contains y terms and vice versa.

You will learn how to solve these equations in a more advanced course.

Here I will just quote the solution

$$x(t) = x_0(t)\cos\omega_z t + y_0(t)\sin\omega_z t$$
$$y(t) = -x_0(t)\sin\omega_z t + y_0(t)\cos\omega_z t \tag{12.43}$$

where

$$x_0(t) + iy_0(t) = Ae^{i\beta t} + Be^{-i\beta t} \quad , \quad \beta = \sqrt{\omega_z^2 + \alpha^2}$$

This solution correspond to the pendulum oscillating in a plane which is rotating with angular velocity $\omega_z = \omega\sin\lambda$.

At the Franklin Institute in Philadelphia where $\lambda = 40°$ so that

$$\omega_z = \omega\sin\lambda = -0.642\omega$$

which means that the plane of oscillation of a Foucault pendulum rotates through 2π in

$$T = \frac{1}{0.642}T_0 = 37.38 \text{ hours}$$

Observation of this rotation is direct evidence that the Earth is rotating!

Some other images of Foucault pendula, the paths of motion are shown below.

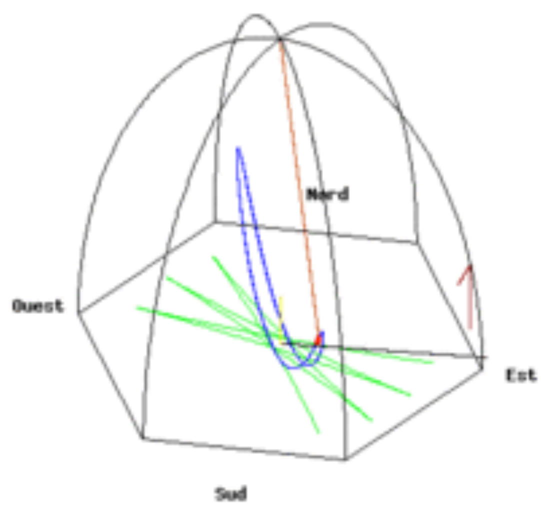

Figure 332:

Figure 333:

Figure 334:

13. Special Relativity

It has been experimentally observed that when a source of light and a detector of light are moving relative to each other with a speed v the wavelength of the observed light changes with the relative speed.

The experimental result is given by the formula

$$\lambda_{observed} = k(v)\lambda_0 \tag{13.1}$$

where

$$(v) = \sqrt{\frac{c+v}{c-v}} \ , \ \lambda_0 = \text{ observed wavelength when } v - 0 \ , \ c = \text{ speed of light} \tag{13.2}$$
$$v > 0 \rightarrow \text{ source and observer moving away from each other}$$

This is the famous **galactic red shift** observed by astronomers for light received on the earth from distant galaxies moving away from the earth.

The wavelength of the light is related to frequency and the period by the formula

$$\lambda f = c = \frac{\lambda}{T} \quad f = \text{ frequency } , T = \text{ period} \tag{13.3}$$

We can therefore write in place of Equation(13.1)

$$T_{observed} = k(v)T_0 \tag{13.4}$$

We have implicitly assume the result of another experiment in writing this formula, namely,

$$c = \text{ speed of light } = \text{ constant for all observers.}$$

Using these results as our theoretical assumptions, we can now derive special relativity.

Consider the experiments represented by the spacetime diagrams below.

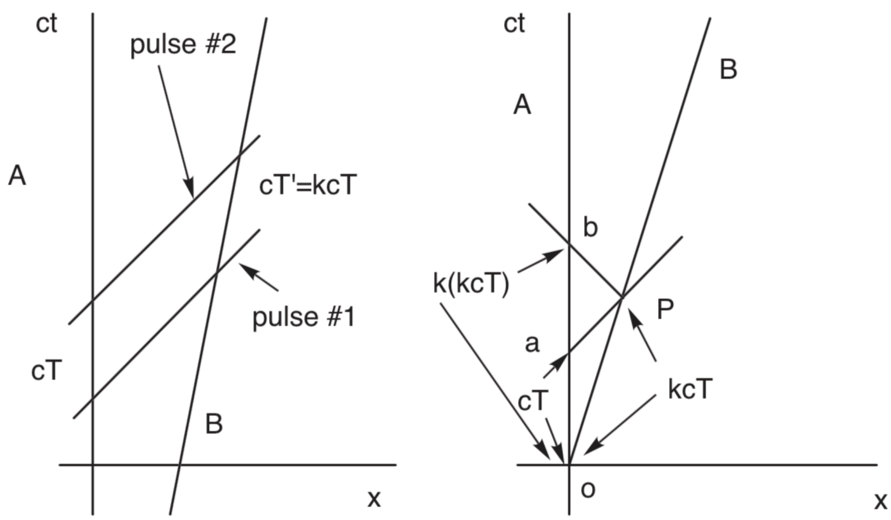

Figure 335:

In each case, observers A and B are moving away from each other with speed v.

In both of these cases, we are assuming the light being sent out consists of a series of pulses separated by a time T_0 in the frame of the source.

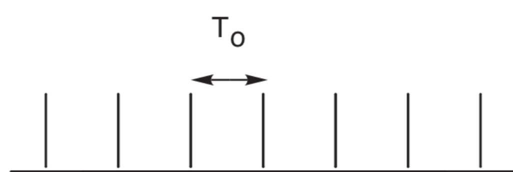

Figure 336:

In the first experiment we see two pulse sent out by A separated by T and received by B separated by kT.

In the second experiment, we see two pulse sent out by A (first when then are at same spacetime point) and received by B separated by kT and then sent back to A and received separated by $k(kT)$.

Remember that **relativity is relative**.

The simple geometry as shown below

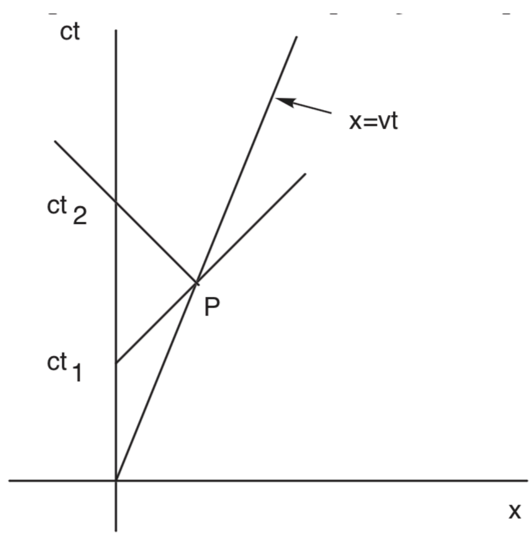

Figure 337:

illustrates the so-called "radar" method for determining the location of a point(event) in space and time(spacetime). The diagram says that

$$x_P = c\frac{t_2 - t_1}{2} \quad , \quad ct_P = c\frac{t_2 + t_1}{2} \tag{13.5}$$

The set of assumptions with associated ideas and methods is all we need to derive everything.

Consider the experiment below where the same two light lines (worldlines) allow both A and B to locate the same event P in spacetime.

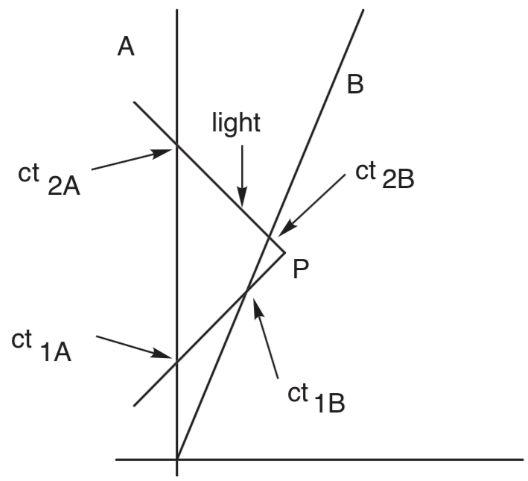

Figure 338:

Observer A says that

$$x_P = c\frac{t_{2A} - t_{1A}}{2} \quad , \quad ct_P = c\frac{t_{2A} + t_{1A}}{2} \tag{13.6}$$

and observer B says that

$$x'_P = c\frac{t_{2B} - t_{1B}}{2} \quad , \quad ct'_P = c\frac{t_{2B} + t_{1B}}{2} \tag{13.7}$$

It is then clear (using $\Delta x = c\Delta t$ for a light ray) that

$$c(t_{2A} - t_P) = x_P = c(t_P - t_{1A}) \quad \text{and} \quad c(t_{2B} - t'_P) = x'_P = c(t'_P - t_{1B}) \tag{13.8}$$

or

$$ct_{2A} = ct_P + x_P \quad , \quad ct_{2B} = ct'_P + x'_P \tag{13.9}$$

$$ct_{21} = ct_P - x_P \quad , \quad ct_{1B} = ct'_P - x'_P \tag{13.10}$$

Our earlier results(Equation13.4) now imply that

$$ct_{1B} = kct_{1A} \quad \text{and} \quad ct_{2A} + kct_{2B} \tag{13.11}$$

or

$$ct'_P + x'_P = \frac{ct_P - x_P}{k} \tag{13.12}$$

$$ct'_P - x'_P = k(ct_P - x_p) \tag{13.13}$$

Algebra then gives(dropping the subscript P since there is nothing special about that spacetime point)

$$ct' = \gamma(ct - \beta x) \quad \text{and} \quad x' = \gamma(x - \beta ct) \tag{13.14}$$

where

$$\beta = \frac{v}{c} \quad , \quad \gamma = \sqrt{\frac{1}{1 = \beta^2}} \tag{13.15}$$

These are the **Lorentz transformations** which relate observation by two observers moving uniformly relative to each other in their common $x - x'$ direction.

Now consider the experiment below involving three observers and two light rays.

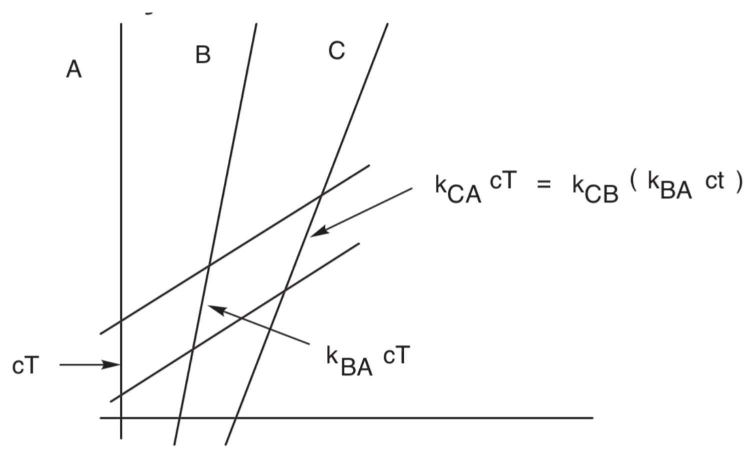

Figure 339:

We have

$$k_{BA} = \sqrt{\frac{c + v_{BA}}{c - v_{BA}}} \quad , \quad k_{CA} = \sqrt{\frac{c + v_{CA}}{c - v_{CA}}} \quad , \quad k_{CB} = \sqrt{\frac{c + v_{CB}}{c - v_{CB}}} \tag{13.16}$$

and

$$k_{CA} = k_{CB} k_{BA} \tag{13.17}$$

or

$$v_{CA} = \frac{v_{CB} + v_{BA}}{1 + \frac{v_{CB} v_{BA}}{c^2}} \tag{13.18}$$

which is the **relativistic velocity addition formula**.

For relative motion in the x-direction (as above) the y and z coordinates are unchanged, i.e., $y' = y$ and $z' = z$.

In this new picture, space and time(which are separate entities in Newtonian dynamics) merge into a 4-dimensional continuum.

The most important variables in any theory are those that are unchanged for different observers.

The speed of light is such an invariant.

Another invariant is the so-called spacetime interval, which is constructed as follows.

Observers A and B can independently measure the spacetime coordinates for two events

$$\begin{aligned}&\text{Observer } A : (ct_{A1}, x_{A1}, y_{A1}, z_{A1}) \text{ and } (ct_{A2}, x_{A2}, y_{A2}, z_{A2})\\&\text{Observer } B : (ct_{B1}, x_{B1}, y_{B1}, z_{B1}) \text{ and } (ct_{B2}, x_{B2}, y_{B2}, z_{B2})\end{aligned}$$

The Lorentz transformations relate these coordinates by

$$ct_{Bi} = \gamma(ct_{Ai} - \beta x_{Ai}) \ , \ x_{Bi} = \gamma(x_{Ai} - \beta ct_{Ai}) \ , \ y_{Bi} = y_{Ai} \ , \ z_{Bi} = z_{Ai} \ , \ i = 1, 2 \tag{13.19}$$

It is then easy to show using the Lorentz transformations that the corresponding spacetime intervals for each observer

$$(\Delta s_A)^2 = c^2(t_{A2} - t_{A1})^2 - (x_{A2} - x_{A1})^2 - (y_{A2} - y_{A1})^2 - (z_{A2} - z_{A1})^2 \tag{13.20}$$

$$(\Delta s_B)^2 = c^2(t_{B2} - t_{B1})^2 - (x_{B2} - x_{B1})^2 - (y_{B2} - y_{B1})^2 - (z_{B2} - z_{B1})^2 \tag{13.21}$$

are invariant, i.e.,

$$(\Delta s_A)^2 = (\Delta s_B)^2 \tag{13.22}$$

An alternative derivation of special relativity that illustrates the powerful methods of theoretical physics.

We have the following **postulates:**

(1) All the laws of nature (not just mechanics) must be the same for all inertial observers moving with constant velocity relative to each other.

NOTE: If we were to write "All the laws of nature must be the same for all observers" for any pair of frames(could be accelerating), then we could derive General Relativity-much harder.

This is the **Principle of Relativity** and restricts the **form** of the laws in each frame.

(2) The speed of light is an invariant.

(3) The motion of a particle observed to be linear in one inertial frame must be linear in all inertial frames.

The implies that the Lorentz transformations must be linear.

Our imposition of the red shift experiment in the first derivation is equivalent to this postulate.

We now do a **thought experiment**.

We consider two inertial frames K and K' moving relative to each other with speed v.

At the instant that the two origins coincide, we set both clocks to zero, i.e., their worldlines cross at the event $(x = 0, ct = 0)$, $(x' = 0, ct' = 0)$ and a light pulse is emitted.

The equations that describe the propagation of the light pulse must be of the same form in each frame (Postulate 1).

We have, if both observers describe the light worldline:

$$c^2t^2 - x^2 - y^2 - z^2 = s^2 = 0 \quad \text{in } K \tag{13.23}$$

$$c^2t'^2 - x'^2 - y'^2 - z'^2 = s'^2 = 0 \quad \text{in } K' \tag{13.24}$$

We have explicitly used the second postulate at this point.

These equations state that the vanishing of the spacetime interval between two events in any inertial frame implies the vanishing of the interval between the same two events in any other inertial frame.

However, we want to prove a more powerful statement, namely, that

$$s^2 = s'^2$$

in general!

We now use the third postulate.

A general theorem from the mathematics of quadratic forms or a lot of messy algebra then says that the quadratic forms s^2 and s'^2 can be connected, at most, by a proportionality factor

$$s'^2 = k(x, y, z, t, \vec{v})s^2$$

Now all physical theories assume that for a free particle

(1) the laws of motion are independent of the choice of origin for the coordinate system

(2) the laws of motion are independent of the orientation of the coordinate system

(3) its velocity during any time interval is the same

These are rules that correspond to the statement spacetime is **homogeneous**.

This implies that the proportionality factor can only depend o$\vec{v}$, i.e.,

$$s'^2 = k(\vec{v})s^2$$

Physicists also assume that space is **isotropic**, which means we cannot have a dependence on the direction of $\vec{v}$. Thus, we have

$$s'^2 = k(v)s^2$$

where v is the magnitude of $\vec{v}$.

Now, if we transform from K' back to K we must have the result

$$s^2 = k(v)s'^2$$

since $-\vec{v}$ has the same magnitude as $\vec{v}$.

Putting these two results together we have $k^2 = 1 \to k = \pm 1$.

k is a constant, but which one?

If we let $v \to 0$, then the systems K and K' become identical and hence $k(0) = 1$ and thus $k = +1$ for all v.

We have thus proved that

$$s^2 = s'^2 \tag{13.25}$$

in general.

Once we have invariance of the spacetime interval and the linearity of the transformation equations between frames it is straightforward to derive the Lorentz transformations and all the other results follow.

Given the Lorentz transformations, we can derive velocity addition formulas in all three directions (remember relative motion is in common $x - x'$ direction).

Suppose we have two observers K and K' and they are observing a moving particle.

The velocity of the particle in each frame is given by

$$u_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \ , \ u_y = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} \ , \ u_z = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} \tag{13.26}$$

$$u'_x = \lim_{\Delta t' \to 0} \frac{\Delta x'}{\Delta t'} \ , \ u'_y = \lim_{\Delta t' \to 0} \frac{\Delta y'}{\Delta t'} \ , \ u'_z = \lim_{\Delta t' \to 0} \frac{\Delta z'}{\Delta t'} \tag{13.27}$$

The Lorentz transformation give

$$\Delta x' = \gamma(\Delta x - v\Delta t) \ , \ \Delta y' = \Delta y \ , \ \Delta z' = \Delta z \ , \ \Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) \tag{13.28}$$

Substitution gives

$$u'_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}} \ , \ u'_y = \frac{u_y}{1 - \frac{vu_x}{c^2}} \ , \ u'_z = \frac{u_z}{1 - \frac{vu_x}{c^2}} \tag{13.29}$$

To see that this agrees with our earlier result we make the associations $K = B$, $K' = A$, Particle $= C$.

We then get (as before)

$$u'_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}} = v_{CA} = \frac{v_{CB} + v_{BA}}{1 + \frac{v_{CB}v_{BA}}{c^2}} \tag{13.30}$$

As we have seen earlier, in order to progress beyond kinematics we must introduce momentum and energy and forces.

In Newtonian physics we had $\vec{p} = m_0\vec{v}$.

We start by figuring out how the relationship between momentum and velocity is modified by special relativity.

We assume the momentum and energy are **conserved** in each frame (they are **not invariants** however) and we assume that the relationship between velocity and momentum is given by

$$\vec{p} = m_0\alpha(\vec{v})\vec{v} \tag{13.31}$$

where for agreement with the Newtonian result for small velocities we must have

$$\lim_{v\to 0} \alpha(\vec{v}) = 1 \tag{13.32}$$

Now consider a glancing elastic collision(only y-velocities) of the form

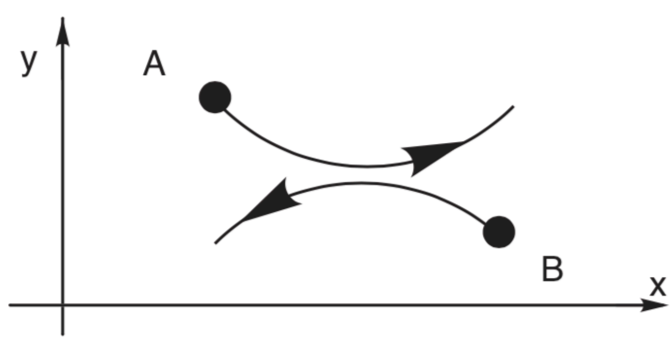

Figure 340:

In A's frame moving along the x-axis with A, A looks like

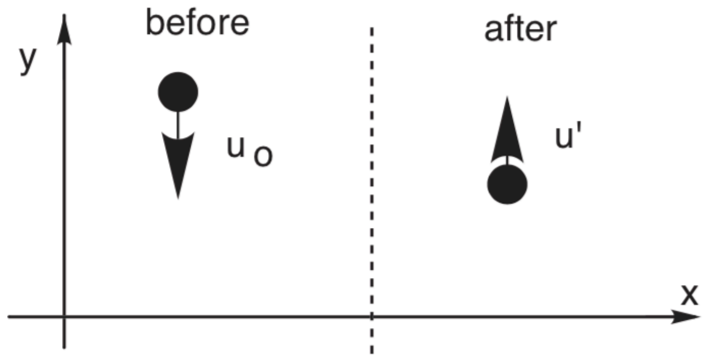

Figure 341:

and in B's frame moving along the x-axis with B, B looks like

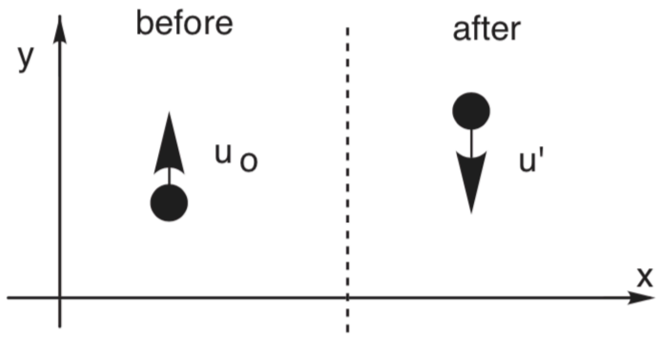

Figure 342:

These two frames are assumed to move with relative velocity v along the x-axis.

We have assumed above that the collision is completely symmetrical.

Each particle has the same y speed u_0 in its own frame before the collision and u' after the collision.

The complete collision looks like the figure below

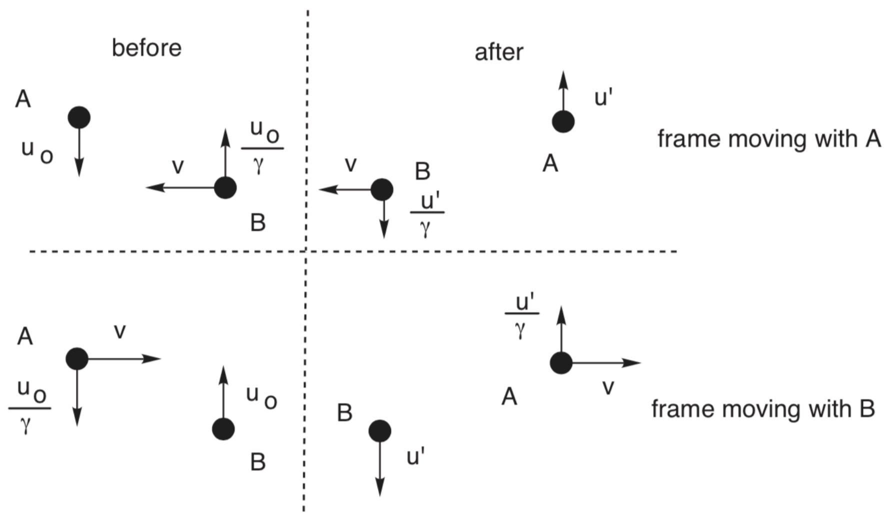

Figure 343:

After the collisions the y-velocities have reversed their directions as shown.

The situation remains symmetrical.

The y-velocities of the other particle in each case follow from the velocity addition formula for motion perpendicular to the relative motion of the frames.

If the speed of A and B in their own frames is u_y and the relative velocity is v, then the y-velocity of the other particle is

$$u'_y = \frac{u_y}{1 - \frac{V(0)}{c^2}} \tag{13.33}$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

In the frame moving with A, the X-momentum is due entirely to B.

Before the collision B's speed is

$$w = \left(v^2 + \left(\frac{u_0}{\gamma}\right)^2\right)^{1/2} \tag{13.34}$$

and after the collision it is

$$w' = \left(v^2 + \left(\frac{u'}{\gamma}\right)^2\right)^{1/2} \tag{13.35}$$

Conservation of x-momentum (as seen in the A frame) gives

$$m_0\alpha(w)v = m_0\alpha(w')v \tag{13.36}$$

$$w = w' \rightarrow u' = u_0 \tag{13.37}$$

Note that the factor α is a function of the total velocity magnitude and not just the x-velocity. Conservation of y-momentum (as seen in the A frame) gives

$$-m_0\alpha(u_0)u_0 + m_0\alpha(w)\frac{u_0}{\gamma} = m_0\alpha(u_0)u_0 - m_0\alpha(w)\frac{u_0}{\gamma} \tag{13.38}$$

$$\alpha(w) = \gamma\alpha(u_0) \tag{13.39}$$

Now in the limit $u_0 \rightarrow 0$ we have $\alpha(u_0) \rightarrow 1$ and $w \rightarrow v$ which says that

$$\alpha(v) = \gamma(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{13.40}$$

Therefore the relativistically correct form of the momentum is

$$\vec{p} = m_0\gamma(v)\vec{v} \tag{13.41}$$

This result been confirmed by experiment in the following way.

The force felt by a charged particle in electric and magnetic fields is given by the Lorentz force law

$$\vec{F} = q\left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right) \tag{13.42}$$

where $\vec{v}$ is the particle velocity.

Consider the experimental setup below:

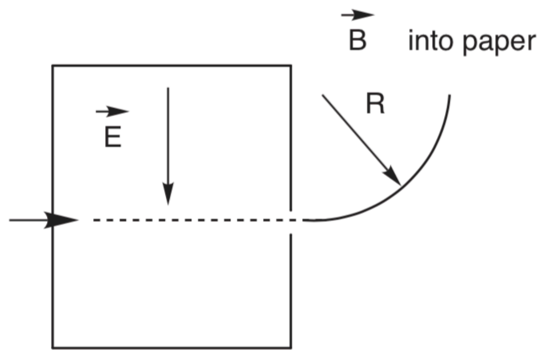

Figure 344:

In the box region the electric and magnetic fields are adjusted so that $\vec{F} = 0$ for a particle moving along the dotted line with a definite velocity.

The electric force always points downward and the magnetic force is always perpendicular to the velocity direction(upward in the box for a particle moving along the dotted line).

This means that particles with a particular velocity, namely,

$$q\left(-E + \frac{v}{c}B\right) = 0 \rightarrow \frac{v}{c} = \frac{E}{B} \tag{13.43}$$

pass undeflected though the box.

The box is a velocity selector.

Outside the box there is no electric field so the particle moves on a circular path (force always perpendicular to the velocity).

In plane polar coordinates we have

$$\vec{v} = v\hat{\theta} \qquad v = \text{ constant}$$

$$\vec{p} = m_0\gamma(v)\vec{v} = m_0\gamma(v)v\hat{\theta}$$

$$\begin{aligned}\frac{d\vec{p}}{dt} &= \vec{F} = m_0\gamma(v)v\frac{d\hat{\theta}}{dt} = -m_0\gamma(v)v\dot{\theta}\hat{r} = -\frac{m_0\gamma(v)v^2}{R}\hat{r} \\ &= -\frac{pv}{R}\hat{r} = \frac{q}{c}v\hat{\theta}\times\vec{B} = -\frac{q}{c}vB\hat{r}\end{aligned} \tag{13.44}$$

$$\rightarrow R = \frac{pc}{qB} \tag{13.45}$$

We have used the fact that the magnitude of the velocity is constant as it moves on the circle.

So measuring the radius corresponds to measuring the relativistic momentum.

Thus, in the **same experiment** we can measure both the velocity and momentum.

A plot of the experimental results looks like

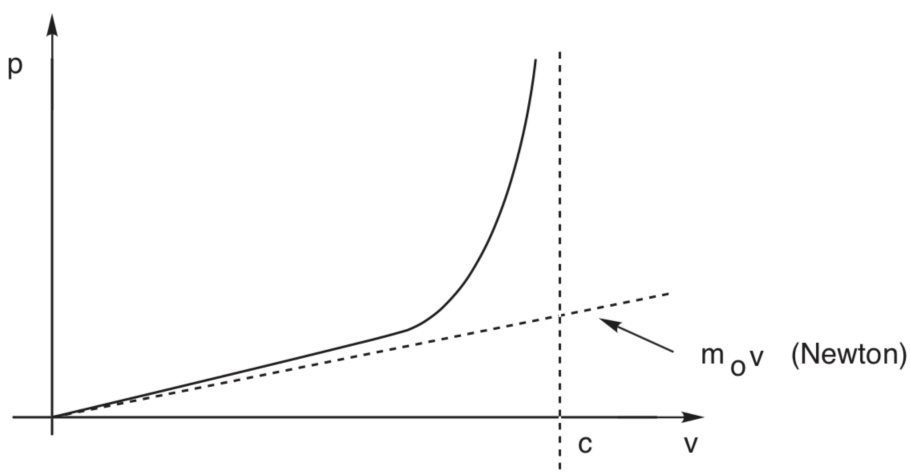

Figure 345:

confirming the relativistic result.

In general, when the velocity is changing both its magnitude and direction we have

$$\vec{F} = \frac{d\vec{p}}{dt} - \frac{dm_0\gamma(v)\vec{v}}{dt} = m_0\frac{d}{dt}\left(\frac{\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right) \tag{13.46}$$

Rectilinear Motion

$$\vec{F} = m_0\left(\frac{\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = m_0\frac{dv}{dt}\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right) + m_0 v\frac{-\frac{1}{2}\left(-2\frac{v}{c^2}\right)\frac{dv}{dt}}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}$$
$$= m_0\frac{dv}{dt}\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} = m_0\gamma^3\frac{dv}{dt} \tag{13.47}$$

Newton's law is modified by the factor γ^3 which has a dramatic effect as $v \to c$

Now suppose that we have a constant force F = constant.

We can then integrate the equation as follows:

$$F dt = m_0\gamma^3(v) dv$$
$$Ft = m_0\int_0^v \gamma^3(v) dv$$

Now

$$\frac{d}{dv}(\gamma v) = \gamma + v\frac{d\gamma}{dv}$$
$$\frac{d\gamma}{dv} = \frac{d}{dv}\left(1-\frac{v^2}{c^2}\right)^{1/2} = \frac{\frac{v}{c^2}}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} = \gamma^3\frac{v}{c^2}$$
$$\frac{d}{dv}(\gamma v) = \gamma + \gamma^3\frac{v^2}{c^2} = \gamma^3\left(\frac{1}{\gamma^2} + \frac{v^2}{c^2}\right) = \gamma^3$$

Therefore,

$$Ft = m_0\int_0^v d(\gamma v) = m_0\gamma v = m_0\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}$$
$$F^2t^2 = m_0^2\frac{v^2}{1-\frac{v^2}{c^2}} \to v^2 = \frac{\left(\frac{Ft}{m_0}\right)^2}{1+\left(\frac{Ft}{m_0 c}\right)^2}$$
$$\to v = \frac{dx}{dt} = \frac{F}{m_0 c}\frac{ct}{\sqrt{1+\left(\frac{F}{m_0 c}\right)^2 t^2}}$$

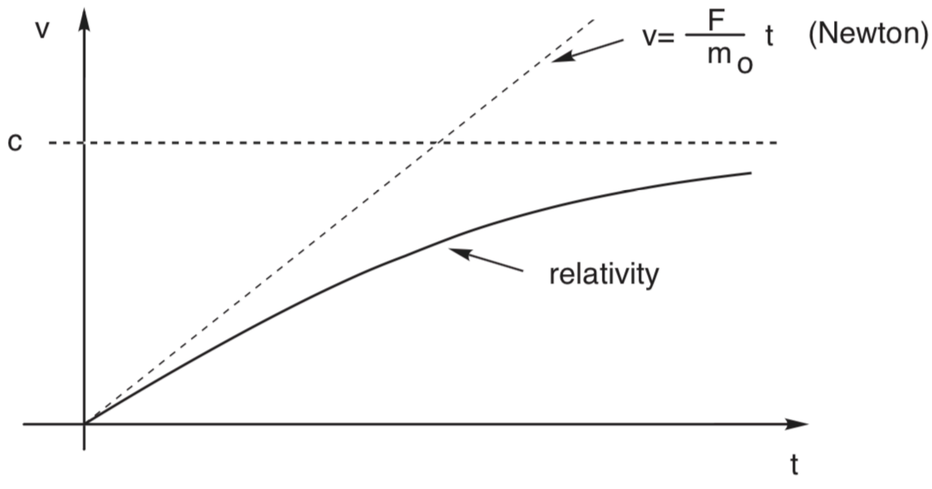

Figure 346:

A plot of v versus t is shown below.
It is clear that no matter how long a constant force is applied we still have $v < c$.

Continuing the integration

$$dx = \frac{F}{m_0 c}\frac{ct}{\sqrt{1+\left(\frac{F}{m_0 c}\right)^2 t^2}}dt$$

$$\begin{aligned} x &= \frac{F}{m_0}\int_0^t \frac{F}{m_0}\frac{t}{\sqrt{1+\left(\frac{F}{m_0 c}\right)^2 t^2}}dt = \frac{F}{m_0}\left(\frac{m_0 c}{F}\right)^2 \times \int_0^{Ft/m_0 c}\frac{u}{\sqrt{1+u^2}}du \\ &= \frac{m_0 c^2}{F} \times \int_0^{Ft/m_0 c} d(\sqrt{1+u^2}) = \frac{m_0 c^2}{F}\left(\sqrt{1+\left(\frac{F}{m_0 c}\right)^2 t^2} - 1\right) \end{aligned} \tag{13.48}$$

A plot of x versus t is shown below:

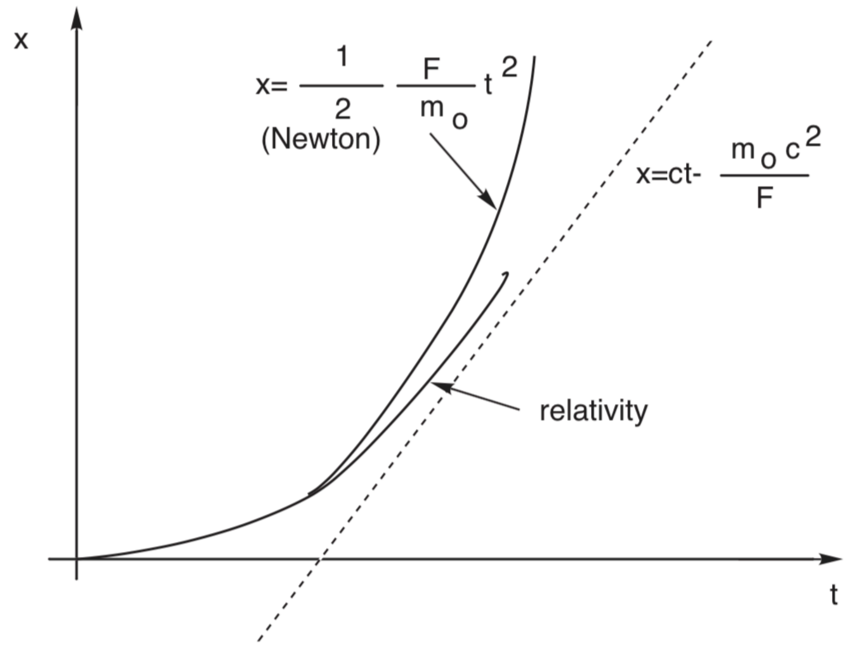

Figure 347:

Whenever, one does complex calculation you should check your results by calculating **limits** where the answer is known.

Letting $t \to 0$ we get

$$\sqrt{1 + \left(\frac{F}{m_0 c}\right)^2 t^2} \approx 1 + \frac{1}{2}\left(\frac{F}{m_0 c}\right)^2 t^2$$

$$v \approx \frac{F}{m_0} t \text{ as expected}$$

$$a \approx \frac{F}{m_0} \text{ as expected}$$

$$x \approx \frac{1}{2}\frac{F}{m_0} t^2 \text{ as expected}$$

Letting $t \to \infty$ we get

$$v \to c \quad \text{and} \quad x \to ct \text{ as expected}$$

Now what about the energy?

What is the relativistically correct form of the energy of a particle?

One way to generalize the concept of energy is to use the old definition of kinetic energy in conjunction with the relativistically correct definition of momentum.

We proceed as follows.

The definition of kinetic energy is

$$\Delta K = K - K_0 = \int_{\vec{r}_0}^{\vec{r}} \vec{F} \cdot d\vec{r} = \int_{\vec{r}_0}^{\vec{r}} \frac{d\vec{p}}{dt} \cdot d\vec{r} \tag{13.49}$$

We found that

$$\vec{p} = m_0 \gamma(v) \vec{v}$$

where

$$\gamma(v) = (1 - \beta^2)^{-1/2} \quad , \quad \beta = \frac{v}{c}$$

Therefore we have

$$K - K_0 = \int_{\vec{r}_0}^{\vec{r}} \frac{d}{dt}(m_0 \gamma(v) \vec{v}) \cdot \vec{v} dt = m_0 \int_0^v \vec{v} \cdot d(\gamma(v) \vec{v}) \tag{13.50}$$

Since the kinetic energy is zero when the velocity is zero we finally have

$$K = m_0 \int_0^v \vec{v} \cdot d(\gamma(v) \vec{v}) \tag{13.51}$$

Now since

$$d(\gamma v^2) = d(\gamma \vec{v} \cdot \vec{v}) = \vec{v} \cdot d(\gamma \vec{v}) + \gamma \vec{v} \cdot d\vec{v} \tag{13.52}$$

we can write

$$\begin{aligned} K &= m_0 \int_0^v (d(\gamma v^2) - \gamma \vec{v} \cdot d\vec{v}) = m_0 \int_0^v d(\gamma v^2) - \frac{1}{2} m_0 \int_0^v \gamma d(v^2) \\ &= m_0 \gamma v^2 - \frac{1}{2} m_0 c^2 \int_0^{v^2/c^2} \frac{du}{\sqrt{1-u}} = m_0 \gamma v^2 + m_0 c^2 \left(\frac{1}{\gamma} - 1 \right) \\ &= m_0 c^2 \left(\gamma \beta^2 + \frac{1}{\gamma} \right) - m_0 c^2 = m_0 c^2 (\gamma - 1) \end{aligned} \tag{13.53}$$

where we let $u = v^2/c^2$ in one step above.

The first thing we should do is check that this makes sense.

What is the low velocity limit of this expression?

Using

$$\gamma = (1 - \beta^2)^{-1/2} \to 1 + \frac{1}{2}\beta^2 = 1 + \frac{1}{2}\frac{v^2}{c^2}$$

we have

$$K = m_0 c^2(\gamma - 1) \to m_0 c^2 \frac{1}{2}\frac{v^2}{c^2} = \frac{1}{2} m_0 v^2$$

as expected.

If we rearrange this result we have

$$\begin{aligned} \gamma m_0 c^2 &= K + m_0 c^2 && (13.54)\\ &= \text{Energy(motion)+Energy(rest)} && (13.55)\\ &= \text{Total energy } = E && (13.56) \end{aligned}$$

The total energy is conserved.

What is the connection to momentum?

Some algebra gives the following results:

$$\frac{pc}{E} = \frac{\gamma m_0 v c}{\gamma m_0 c^2} = \frac{v}{c} = \beta \tag{13.57}$$

and

$$\left(\frac{E}{c}\right)^2 - \vec{p}^2 = m_0^2 c^2 = \text{ invariant} \tag{13.58}$$

Digression to 4-Vectors

What is an ordinary vector in 3-dimensional space?

Consider a vector $\vec{A}$ representing some physical variable.

Using cartesian unit vectors, we can write

$$\vec{A} = \sum_i A_i \hat{e}_i \tag{13.59}$$

The components of the vector A_i, $i = 1, 2, 3$ are its representation in a given coordinate system.

We must choose a coordinate system in order to define the unit vectors.

The coordinate system is not an essential part of the physics however.

We can just as well use any other coordinate system to define the unit vectors and the vector $\vec{A}$.

In particular, we consider another coordinate system with the same origin, but rotated from the first system.

In another coordinate system we would write

$$\vec{A} = \sum_i A'_i \hat{e}'_i \tag{13.60}$$

Note that the vector $\vec{A}$ has not changed; only its representation (components) in the new system has changed.

We relate the two representations (components) as follows:

$$\sum_i A_i \hat{e}_i = \sum_i A'_i \hat{e}'_i$$

$$\hat{e}'_j \cdot \sum_i A_i \hat{e}_i = \hat{e}'_j \cdot \sum_i A'_i \hat{e}'_i = \sum_i A'_i \hat{e}'_j \cdot \hat{e}'_i = \sum_i A'_i \delta_{ij} = A'_j$$

or

$$A'_j = \sum_i A_i (\hat{e}'_j \cdot \hat{e}_i) \tag{13.61}$$

The coefficients $(\hat{e}'_j \cdot \hat{e}_i)$ are the numbers that are determined by a given rotation; they are independent of the vector $\vec{A}$.

We now **define** a vector.

> **A vector in 3 dimensions is a set of 3 numbers components) which transform under a rotation of the coordinate system according to**
>
> $$A'_j = \sum_i A_i (\hat{e}'_j \cdot \hat{e}_i)$$

Any quantity which is unchanged by a coordinate transformation is called an **invariant** of the transformation.

Since the principle of relativity requires that the results of physical theories (physical laws) be independent of the choice of coordinate system (must be inertial however), all physical laws must involve **only** invariants.

The dot product of two vectors is a scalar.

Scalars are numbers that are independent of our choice of coordinate system.

This gives us a method for constructing invariants.

We can show that the dot product produces an invariant as follows:

$$\begin{aligned} \vec{A}' \cdot \vec{B}' &= \left(\sum_i A_i \hat{e}_i\right) \cdot \left(\sum_j B_j \hat{e}_j\right) = \sum_{i,j} A_i B_j \hat{e}_i \cdot \hat{e}_j \\ &= \sum_{i,j} A_i B_j \delta_{ij} = \sum_i A_i B_i = \vec{A} \cdot \vec{B} \end{aligned}$$

In particular, the norm or length-squared of a vector, $A^2 = A?A$, is a scalar invariant. We now define a rotation.

> **A rotation is any transformation which leaves** $r^2 = \vec{r} \cdot \vec{r} = x^2 + y^2 + z^2$ **invariant**

In Minkowski 4-dimensional spacetime we define vectors in a different manner.

Both the ordinary space 3-dimensional and the Minkowski 4-dimensional vector definitions are special cases of a more general definition.

The ordinary 3-dimensional definition corresponds to Euclidean geometry.

In Minkowski 4-dimensional spacetime we write the spacetime 4-vector in this way

$$\overleftarrow{s} = (ct, x, y, z) \tag{13.62}$$

and the scalar product of the vector with itself (its norm) as

$$\overleftarrow{s} \cdot \overleftarrow{s} = c^2 t^2 - x^2 - y^2 - z^2 = ds^2 \tag{13.63}$$

This is a scalar invariant under Lorentz transformations (it is the **spacetime interval**).

In fact, any set of 4 numbers

$$\overleftarrow{A} = (A_0, A_1, A_2, A_3) \tag{13.64}$$

represents a Minkowski 4-vector if its norm

$$\overleftarrow{A} \cdot \overleftarrow{A} = A_0^2 - A_1^2 - A_2^2 - A_3^2 \tag{13.65}$$

is a scalar invariant.

In addition if a set of 4 numbers is a 4-vector, then the components transform between frames via the Lorentz transformations as

$$A'_0 = \gamma(A_0 - \beta A_1) \tag{13.66}$$

$$A'_1 = \gamma(A_1 - \beta A_0) \tag{13.67}$$

$$A'_2 = A_2 \tag{13.68}$$

$$A'_3 = A_3 \tag{13.69}$$

for relative motion along the 1-axis.

It is in this sense that spatial and time variables are **not distinct entities** but are simply **different components** of the same vector and transform into each other under Lorentz transformations.

This corresponds to a non-Euclidean geometry - Minkowski geometry.

Another 4-vector is

$$d\overleftarrow{s} = (cdt, dx, dy, dz) \tag{13.70}$$

since it is the difference of two 4-vectors.

Hence, its norm

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \tag{13.71}$$

is a Lorentz invariant.

A related quantity of great importance is

$$d\tau^2 = \frac{ds^2}{c^2} \tag{13.72}$$

It is also an invariant since dividing an invariant by another invariant produces another invariant.

In particular,

$$d\tau^2 = dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2) \tag{13.73}$$

is an invariant.

Consider a displacement $d\overleftarrow{s}$ between two event on the worldline of a moving particle.

In the rest frame of the particle, $dx = dy = dz = 0$ and hence

$$d\tau = dt \tag{13.74}$$

i.e., in the particle rest frame (the events are separated only by time).

$d\tau$ is the time interval between the two events measured in the rest frame and is thus the **proper time**.

It is a Lorentz invariant.

Time Dilation (the easy way)

Consider and observer at rest in x', y', z', t' system.

In this system the proper time between two events is

$$d\tau = dt'$$

In the x, y, z, t system moving with velocity $\vec{v}$ relative to the first frame, the time interval between the same two events is given by

$$d\tau^2 = dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2) = dt'^2$$

since $d\tau$ is an invariant or its value is the same in all frames.

We therefore have

$$\left(\frac{dt'}{dt}\right)^2 = 1 - \frac{1}{c^2}\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right)$$

$$1 - \frac{v^2}{c^2} = \frac{1}{\gamma^2}$$

Therefore,

$$dt = \gamma dt' \tag{13.75}$$

which is the time dilation formula.

We do not need to introduce hypothetical experiments or discussions of simultaneity to obtain this result as is done in the usual derivation.

That is an example of the power of using 4- vectors.

Other 4-Vectors

Using the 4-vector

$$d\overleftarrow{s} = (cdt, dx, dy, dz) \tag{13.76}$$

and dividing by the Lorentz invariant $d\tau$ automatically yields another 4-vector

$$\frac{d\overleftarrow{s}}{d\tau} = \left(c\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right) = \overleftarrow{u} = \text{ 4-vector velocity} \tag{13.77}$$

Its norm is an invariant so it can be calculated in any frame.

We pick the rest frame where

$$\overleftarrow{u} = (c, 0, 0, 0) \to u^2 = c^2 = \text{ invariant} \tag{13.78}$$

For a moving particle where the x, y, z, t system moves with velocity $-\vec{v}$ relative to the rest frame of the particle we have

$$dt = \gamma d\tau$$

and thus

$$\overleftarrow{u} = \left(c\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right) = \gamma(c, \vec{v})$$

Since the rest mass m_0 is a Lorentz invariant, $m_0\overleftarrow{u}$ is a 4-vector with dimensions of momentum.

We define the 4-momentum as

$$\overleftarrow{\rho} = m_0\overleftarrow{u} = m_0\gamma(c, \vec{v}) = \left(\frac{E}{c}, \vec{p}\right) \tag{13.79}$$

We already saw that

$$\rho^2 = \left(\frac{E}{c}\right)^2 - \vec{p}^2 = m_0^2c^2 = \text{ invariant} \tag{13.80}$$

Since the variables E and $\vec{p}$ are components of a 4-vector they must obey the Lorentz transformations.

Therefore we have

$$\frac{E'}{c} = \gamma\left(\frac{E}{c} - \beta p_x\right) \tag{13.81}$$

$$p'_x = \gamma\left(p_x - \beta\frac{E}{c}\right) \tag{13.82}$$

$$p'_y = p_y \tag{13.83}$$

$$p'_z = p_z \tag{13.84}$$

You will use these relations when you study Electromagnetism to prove that a magnetic field is observed in frames moving relative to fixed charged particles whereas only electric fields are observed in the rest frame of the charged particles.

Magnetic fields are a consequence of special relativity!!

Finally we confirm our identification of the energy.

We define the 4-vector Minkowski force as

$$\overleftarrow{\phi} = \frac{d\overleftarrow{\rho}}{d\tau} = \left(\frac{d\gamma m_0 c}{d\tau}, \frac{d\vec{p}}{d\tau}\right) \tag{13.85}$$

If dt is the time interval in the observer?s frame corresponding to the interval of proper time $d\tau$, then $dt = \gamma d\tau$ and we get

$$\overleftarrow{\phi} = \gamma\left(\frac{d\gamma m_0 c}{dt}, \vec{F}\right) \quad , \quad \vec{F} = \frac{d\vec{p}}{dt} \tag{13.86}$$

With this construction, the 4-momentum is conserved (constant) when the 4-force is zero.

This corresponds to energy and momentum conservation.

If the 4-force is zero in one frame then it is zero in all frames and hence if energy and momentum are conserved in one frame they are conserved in all frames.

In Newtonian physics

$$\vec{F} \cdot \vec{v} = \frac{dE}{dt} \tag{13.87}$$

where E = total energy.

Let us look at the corresponding quantity in 4 dimensions

$$\overleftarrow{\phi} \cdot \overleftarrow{u} = \gamma \left(\frac{d\gamma m_0 c}{dt}, \vec{F} \right) \cdot \gamma(c, \vec{v}) = \gamma^2 \left[\frac{d\gamma m_0 c^2}{dt} - \vec{F} \cdot \vec{v} \right] \tag{13.88}$$

Now the scalar product is an invariant and thus we can evaluate it in the rest frame of the particle.

In this frame $vecF \cdot \vec{v} = 0$ since $\vec{v} = 0$.

We also have

$$\frac{d\gamma m_0 c^2}{dt} = \gamma m_0 v \left(\frac{dv}{dt} \right) = 0 \tag{13.89}$$

since $v = 0$

Therefore,

$$\overleftarrow{\phi} \cdot \overleftarrow{u} = 0 = \gamma^2 \left[\frac{d\gamma m_0 c^2}{dt} - \vec{F} \cdot \vec{v} \right] \tag{13.90}$$

$$\vec{F} \cdot \vec{v} = \frac{d\gamma m_0 c^2}{dt} \rightarrow E = \gamma m_0^2 \tag{13.91}$$

as we indicated earlier.

In this sense the momentum and energy variables are **not distinct entities** but are simply **different components** of the same vector and transform into each other under Lorentz transformations.

A Generalization

The scalar product can be generalized to any number of dimensions and any kind of geometry.

We have

$$\mathbf{A} \cdot \mathbf{B} = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} A_i B_j \tag{13.92}$$

where g_{ij} is the so-called metric object or metric tensor.

We can represent it by a matrix.

In ordinary 3-dimensional space we have

$$[g] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow g_{ij} = \delta_{ij} \tag{13.93}$$

and hence

$$\mathbf{A} \cdot \mathbf{B} = \sum_{i=1}^{n} A_i B_i = vecA \cdot \vec{B} \tag{13.94}$$

In Minkowski 4-space we have

$$[g] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{13.95}$$

and hence

$$\mathbf{A} \cdot \mathbf{B} = \overleftarrow{A} \cdot \overleftarrow{B} \tag{13.96}$$

In the theory of gravitation (general relativity) we have

$$[g] = \begin{pmatrix} g_{00}(x,y,z,t) & g_{01}(x,y,z,t) & g_{02}(x,y,z,t) & g_{03}(x,y,z,t) \\ g_{10}(x,y,z,t) & g_{11}(x,y,z,t) & g_{12}(x,y,z,t) & g_{13}(x,y,z,t) \\ g_{20}(x,y,z,t) & g_{21}(x,y,z,t) & g_{22}(x,y,z,t) & g_{23}(x,y,z,t) \\ g_{30}(x,y,z,t) & g_{31}(x,y,z,t) & g_{32}(x,y,z,t) & g_{33}(x,y,z,t) \end{pmatrix} \tag{13.97}$$

and clearly the world is considerably more complicated (a very advance course).

Now back to special relativity.

Let us return to the relations

$$\left(\frac{E}{c}\right)^2 - \vec{p}^2 = m_0^2 c^2$$

and

$$\frac{pc}{E} = \frac{v}{c} = \beta$$

Notice that if $v = c$, then $E = pc$ and $m_0 = 0$.

Therefore, particles with zero rest mass exist.

They always move with the speed of light.

Even though the have no mass they do have energy and momentum!

An example of such a particle is the photon, the particle of light.

Radiation Pressure

When light(photons) which carries momentum and energy reflects off of a surface it transfers momentum and energy to the surface.

Since a change in momentum corresponds to a force and a force on a surface area orresponds to pressure, light exerts radiation pressure on any reflecting surface.

If we have normal incidence on the surface, then the total change in the photon momentum is

$$\Delta p = 2p = 2\frac{E}{c}$$

If there are n photons per unit area are per second, then the total momentum change per second per unit area is

$$\text{pressire } = 2n\frac{E}{c} = 2\frac{I}{c}$$

where $I = nE$ is the intensity of the light (the power per unit area).

The average intensity of sunlight falling on the earth surface is $\approx 1000\, W/m^2 = 1000\, J/m^2 \cdot sec$.

The radiation pressure on a mirror is then

$$\text{pressure } = 2\frac{I}{c} = 7 \times 10^{-4}\, N/m^2$$

This is verysmall (atmospheric pressure is $10^6 N/m^2$).

On a cosmic scale, however, this radiation pressure is large; it helps to keep stars from collapsing under their own gravitational forces.

How big must the sail of a light-sail starship be to work effectively?

Suppose that the sail material has the property

$$\text{mass per } m^2 = \rho\, kg$$

and that the ship has a mass of M kg.

A crude calculation goes like this

$$\text{pressure at distance } r \text{ from sun } = 7 \times 10^{-4} \left(\frac{r_{earth}}{r}\right)^2 N/m^2$$

$$\text{force on sail} = \text{pressure x area } = 7 \times 10^{-4} \left(\frac{r_{earth}}{r}\right)^2 A$$

$$[acceleration] = a = \frac{\text{force}}{\text{total mass}} = \frac{7 \times 10^{-4} \left(\frac{r_{earth}}{r}\right)^2 A}{M + \rho A}$$

Suppose we have a sail with a maximum area = USA = $10^{13}\, m^2$.

For $r = n \times r_{earth}$, $\rho = 10^{-8}$, $M = 10^5$, $\alpha =$ fraction of area used , we get

$$a = \frac{7 \times 10^{-4} \left(\frac{1}{n}\right)^2 10^{13}\alpha}{10^5 + 10^5\alpha} = 7 \times 10^{-4} \left(\frac{\alpha}{1+\alpha}\right)\left(\frac{1}{n}\right)^2 m/sec^2 \;\; , \;\; 0 \le \alpha \le 1 \;\; , \;\; n \ge 1$$

The acceleration will drop below $0.0001g$ for $\alpha = 1$ when $n = 20000$ or we are at a distance of 20000 earth radii or about 2×10^{12} miles from the sun.

This is about 0.03 light-year.

Depending on what we did earlier we could have a sizable speed by this point.

The Doppler Effect

Sound and the Acoustic Doppler Effect

Sound travels through a medium such as air with a speed w.

This speed is determined by the properties of the medium and is independent of the motion of the source.

We consider a source of sound that is moving with velocity v through the medium towards an observer at rest.

We assume for simplicity that the observer (detector) lies along the line of motion of the source.

As shown in the diagram

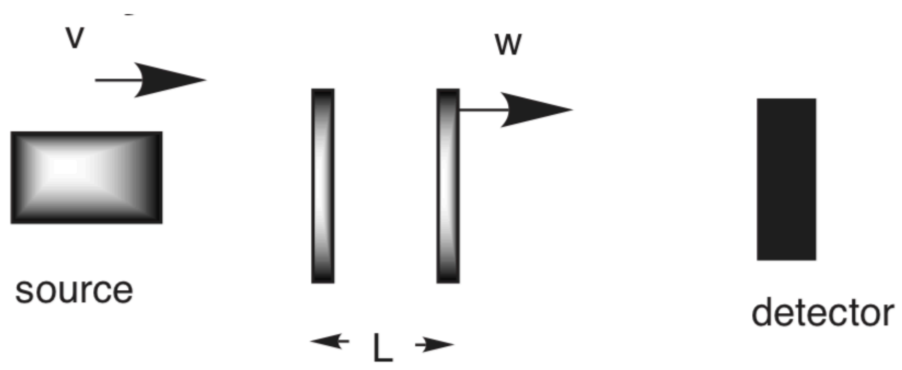

Figure 348:

we represent the sound wave as a regular series of pulses.

These pulses are separated in time by an amount $\tau_0 = \frac{1}{\nu_0}$ where ν_0 is the frequency of the sound from the source.

In a time T the sound travels a distance wT, and if the pulses are separated by a distance L, the number reaching the detector is $\frac{wT}{L}$.

The rate at which pulses arrive is $\frac{w}{L}$ = frequency of sound at the detector $= \nu_D$.

To determine L, we consider a pulse emitted at $t = 0$ and a second pulse emitted at $t = \tau_0$.

During the interval τ_0 the first pulse travels a distance $w\tau_0$ in the medium and the source travels a distance $v\tau_0$.

As shown in the figure below

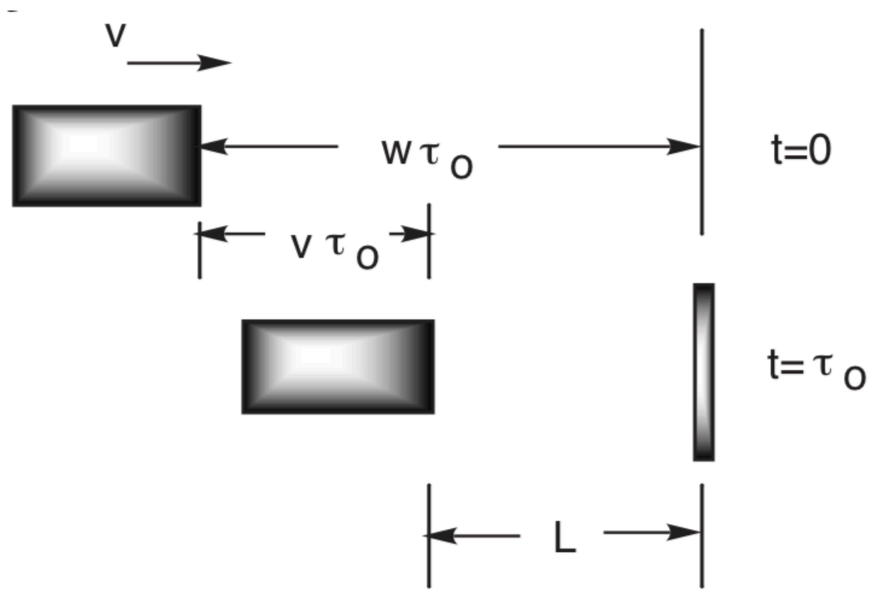

Figure 349:

the distance between the pulses is given by

$$L = w\tau_0 - v\tau_0 = (w - v)\tau_0 = \frac{w - v}{\nu_0}$$

and

$$\nu_D = \nu_0 \frac{1}{1 - \frac{v}{w}} \qquad \text{for a moving source}$$

For an approaching source, $v > 0$ and thus $\nu_D > \nu_0$.

For a receding source, $v < 0$ and thus $\nu_D < \nu_0$.

If the source is at rest and the detector is moving (as shown below)

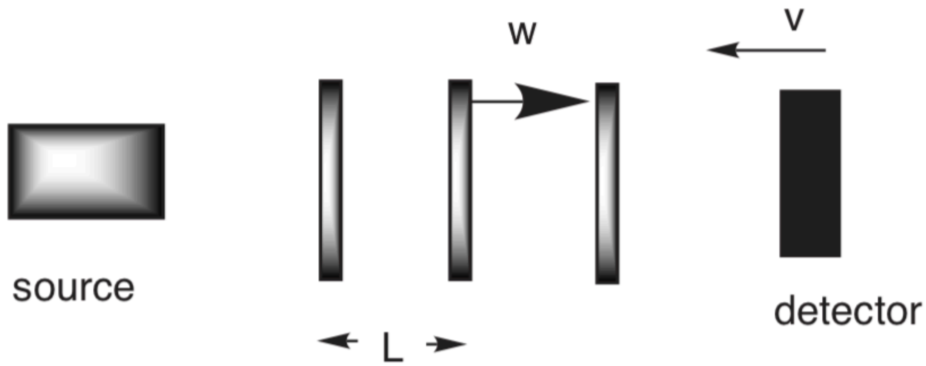

Figure 350:

the situation is different.

The speed of the pulses relative to the detector is w_v.

The rate at which the pulses arrive is

$$\nu_D = \frac{w+v}{L}$$

Since the source is at rest,

$$L = w\tau_0 = \frac{w}{\nu_0}$$

and thus

$$\nu_D = \nu_0 \left(1 + \frac{v}{w}\right) \qquad \text{for a moving detector}$$

The two results are **not symmetric**.

They are approximately the same for small $\frac{v}{w}$ since

$$\frac{1}{1-\frac{v}{w}} \approx 1 + \frac{v}{w}$$

in that case.

If we know, then we can tell whether it is the source or the detector that is moving!!

This is so because the speed of sound is not a universal constant, but only has a definite value relative to the medium where it is propagating.

Light and the Relativistic Doppler Effect

Suppose a light source flashes with period $\tau_0 = \frac{1}{\nu_0}$ in its rest frame and that the source is moving towards the observer(detector) with velocity v.

Due to time dilation, the period in the detector rest frame is $\tau = \gamma\tau_0$.

Since the speed of light is a universal constant, the pulses arrive at the detector with speed c.

As shown in the diagram below

Figure 351:

the frequency of the pulses is $\nu_D = \frac{c}{L}$, where L is the pulse separation in the detector frame.

Since the source is moving towards the detector we have (see diagram below)

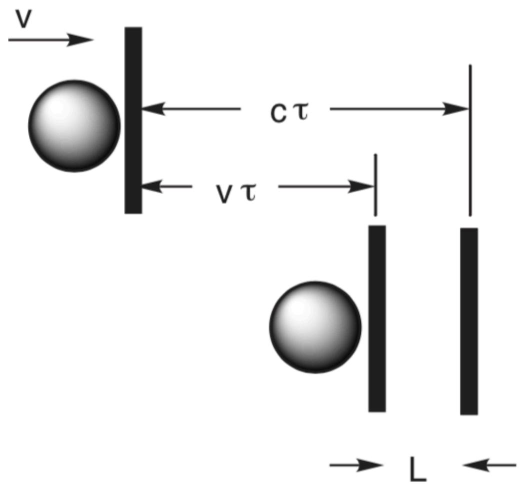

Figure 352:

we have

$$L = c\tau - v\tau = (c-v)\tau = (c-v)\gamma\tau_0 = \gamma\frac{c-v}{\nu_0}$$

and

$$\nu_D = \nu_0\frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c}} = \nu_0\sqrt{\frac{c+v}{c-v}}$$

Here ν_D is the frequency in the detector frame and v is the relative velocity of the source and the detector.

It does not matter which one is actually moving!!

This result is just the red shift formula we started with earlier, as expected.

The Power of 4-Vectors

We now illustrate the power of 4-vectors.

We consider a photon with energy $E = h\nu$ and momentum

$$p = \frac{E}{c} = \frac{h\nu}{c} = \frac{h}{\lambda}$$

traveling in the $x-y$ plane at an angle ϕ with the x-axis.

The 3-momentum is

$$\vec{p} = \frac{h\nu}{c}(\cos\phi, \sin\phi, 0)$$

and the energy-momentum 4-vector is

$$\overleftarrow{p} = \frac{h\nu}{c}(1, \cos\phi, \sin\phi, 0)$$

In another system moving relative to the first with velocity v along the common $x - x'$ axis the energy-momentum 4-vector is

$$\overleftarrow{p}' = \frac{h\nu'}{c}(1, \cos\phi', \sin\phi', 0)$$

Now these two frames are related by the Lorentz transformation such that

$$\frac{E'}{c} = \frac{h\nu'}{c} = \gamma\left(\frac{E}{c} - \beta p_x\right) = \gamma\left(\frac{h\nu}{c} - \beta\frac{h\nu}{c}\cos\phi\right)$$

this says that

$$\nu' = \gamma\nu(1 - \beta\cos\phi)$$

or

$$\nu = \frac{\nu'}{\gamma}\frac{1}{1 - \beta\cos\phi} = \nu'\frac{\sqrt{1-\beta^2}}{1 - \beta\cos\phi}$$

This is identical to our earlier result for $\phi = 0$ and $\nu_D = \nu$ and $\nu' = \nu_0$.

Our derivation gives an additional result however.

In a direction perpendicular to the line of relative motion, $\phi = \pi/2$ we get

$$\nu_D = \nu_0\sqrt{1-\beta^2}$$

which is called the **transverse Doppler effect** and is due to time dilation.

High Energy Particle Physics

The center of mass or zero momentum system we discussed earlier is very useful when discussing high energy particle reactions.

We consider a collision between two particles with rest masses m_1 and m_2.

We assume that particle 1 is moving with velocity $\vec{u}$ in the laboratory system and that particle 2 is at rest.

We have the energy-momentum 4-vectors

$$\overleftarrow{p}_1 = \left(\frac{E_1}{c}, p_1, 0, 0\right) \quad \text{and} \quad \overleftarrow{p}_2 = \left(\frac{E_2}{c}, 0, 0, 0\right) \tag{13.98}$$

and the total energy-momentum

$$\overleftarrow{P} = \overleftarrow{p}_1 + \overleftarrow{p}_2 = \left(\frac{E_1 + E_2}{c}, p_1, 0, 0\right) \tag{13.99}$$

In a new frame moving along the x-axis with speed V we have

$$P_1' = \Gamma\left(p_1 - \beta\frac{E_1 + E_2}{c}\right) \;\; , \;\; P_2' = 0 \;\; , \;\; P_3' = 0 \tag{13.100}$$

where

$$\Gamma = \left(1 - \frac{V^2}{c^2}\right)^{-1/2} \tag{13.101}$$

In the center of mass system, $\vec{P}_1' = 0$.

This says that

$$V_{CM} = \frac{p_1 c^2}{E_1 + E_2} \tag{13.102}$$

The energy available for physical processes such as the production of new particles or inelastic events is the total energy in the center of mass system, E'.

In the center of mass system the energy-momentum 4-vector is

$$\left(\frac{E'}{c}, 0, 0, 0\right) \tag{13.103}$$

We can find E' by using the fact that the norm of the energy-momentum 4-vector is invariant

$$\left(\frac{E'}{c}\right)^2 = \left(\frac{E_1 + E_2}{c}\right)^2 - p_1^2 \tag{13.104}$$

or

$$\begin{aligned} E^2 &= E_1^2 + E_2^2 + 2E_1E_2 - p_1^2c^2 = E_1^2 + E_2^2 + 2E_1E_2 - (E_1^2 - m_1^2c^4) \\ &= m_1^2c^2 + 2E_1E_2 + E_2^2 \end{aligned} \tag{13.105}$$

We have

$$E_1 = \gamma m_1 c^2 \quad \text{and} \quad E_2 = m_2 c^2 \quad , \quad \gamma = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} \tag{13.106}$$

Therefore

$$E = (\gamma m_1 + m_2)c^2 = \text{ total energy in laboratory system} \tag{13.107}$$

and

$$E' = (m_1^2 + m_2^2 + 2\gamma m_1 m_2)^{1/2} c^2 \tag{13.108}$$

The fraction of energy available for physical processes is

$$\frac{E'}{E} = \frac{(m_1^2 + m_2^2 + 2\gamma m_1 m_2)^{1/2}}{\gamma m_1 + m_2} \tag{13.109}$$

For the special case $m_1 = m_2 = m$ we have

$$\frac{E'}{E} = \sqrt{\frac{2}{1+\gamma}} \tag{13.110}$$

At low velocity or low energy of the incident particle (the one that is moving), we have

$$\gamma \approx 1 \rightarrow \frac{E'}{E} = 1 \rightarrow \text{ all energy available} \tag{13.111}$$

In this case, most of the energy is rest energy and kinetic energy is unimportant.

In the high speed or high energy limit we have

$$\frac{E'}{E} = \sqrt{\frac{2}{1 + \frac{E_1}{mc^2}}} \rightarrow \sqrt{\frac{2mc^2}{E_1}} \tag{13.112}$$

Thus, the useful fraction of energy decreases as $E_1^{-1/2}$.

For example, in a 300 GeV accelerator (1 GeV = 10^9 eV = $10^9 \times 1.6 \times 10^{-12}$ J = 1.6×10^{-3} J) an accelerated proton ($mc^2 \approx 1$GeV) colliding with a hydrogen target (protons) has

$$\frac{E'}{E} = \sqrt{\frac{2}{300}} = 0.082 \tag{13.113}$$

or only 25 GeV is available for reactions!!!

We will show how to fix this up shortly.

Let us look at **production reactions** in another way.

Suppose that we have two particles that interact with each other(one is at rest - the target) and produce N final particles.

The high energy available from the incident particle is converted into mass of newly created particles.

We ask the question: What is the minimum energy needed by the incident particle in order to produce the final state of N particles?

In the initial state we have

$$\left(\frac{E_{inc}}{c}, p_{inc}, 0, 0\right) + (m_{target}c, 0, 0, 0) = \left(\frac{E_{inc}}{c} + m_{target}c, p_{inc}, 0, 0\right) \quad (13.114)$$

$$E_{inc}^2 p_{inc}^2 c62 + m^2 c^4 \quad (13.115)$$

In the final state we have

$$\left(\frac{\sum_{i=1}^N E_i}{c} m \sum_{i=1}^N \vec{p}_i\right) \quad \text{where} \quad E_i^2 = p_i^2 c^2 + m_i^2 c^4 \;\;,\;\; i = 1, 2, 3, 4, ..., N \quad (13.116)$$

Now, the norm of the energy-momentum 4-vector is invariant in time and across different frames.

Therefore

norm in laboratory before = norm in center of mass after

This gives

$$\left(\frac{E_{inc}}{c} + m_{target}c\right)^2 - p_1^2 = \left(\frac{\sum_{i=1}^N E_{i,CM}}{c}\right)^2 - \left(\sum_{i=1}^N \vec{p}_{i,CM}\right)^2 \quad (13.117)$$

By definition, however,

$$\sum_{i=1}^N \vec{p}_{i,CM} = 0 \quad (13.118)$$

After some algebra we have

$$E_{inc} = \frac{\left(\sum_{i=1}^{N} E_{i,CM}\right)^2 - (m_{inc}c^2)^2 - (m_{target}c^2)^2}{2m_{target}c^2} \tag{13.119}$$

This is a minimum when

$$\sum_{i=1}^{N} E_{i,CM} \tag{13.120}$$

is a minimum or when

$$\sum_{i=1}^{N} E_{i,CM} = \sum_{i=1}^{N} m_i c^2 \tag{13.121}$$

or all the particles are at rest in the center of mass system after the collision (what are they doing in the laboratory system).

Therefore the minimum energy needed by the incident particle (this is called the **threshold energy**) is

$$E_{inc,threshold} = \frac{\left(\sum_{i=1}^{N} m_i c^2\right)^2 - (m_{inc}c^2)^2 - (m_{target}c^2)^2}{2m_{target}c^2} \tag{13.122}$$

For example, consider the reaction $p + p \to p + p + \pi + \pi + \pi$ where a proton is incident on another proton producing two protons and three π mesons.

The threshold energy is

$$E_{p,threshold} = \frac{(2m_p + 3m_\pi)^2 - 2m_p^2}{2m_p} c^2 = \left(m + p + 6m_\pi + \frac{9}{2}\frac{m_\pi^2}{m_p}\right) c^2 \tag{13.123}$$

Clearly, this is a very non-intuitive answer!!!

Now let us consider the difference between a particle accelerator where one particle is accelerated and collides with a second particle at rest (as above = laboratory system) and two particle accelerators where each particle is accelerated in the same way (colliding beams = center of mass system).

We have

Single Accelerator

$$\left(\frac{E_{total\,lab}}{c}, \vec{p}_{total\,cm}\right) = \left(\frac{E_1 + m_2 c^2}{c}, \vec{p}_1\right) \quad , \quad E_1^2 = p_1^2 c^2 + m_1^2 c^4 \tag{13.124}$$

Colliding Beams

$$\left(\frac{E_{total\,cm}}{c}, \vec{p}_{total\,lab}\right) = \left(\frac{2E}{c}, 0\right) \quad , \quad E = \text{ energy of each particle} \tag{13.125}$$

In the first case the accelerator must produce energy E_1 second case each accelerator must produce energy E.

The two accelerators are equivalent (same energy available for physical processes) if

$$\left(\frac{E_1 + m_2 c^2}{c}, \vec{p}_1\right)^2 = \left(\frac{2E}{c}, 0\right)^2 \tag{13.126}$$

Algebra gives the result

$$E = \frac{1}{2}\sqrt{m_1^2 c^4 + m_2^2 c^4 + 2m_2 c^2 E_1} \tag{13.127}$$

If we consider the case of very high energy accelerators where $E_1 \gg m_i c^2$ we have

$$E = \frac{1}{2}\sqrt{2m_2 c^2 E_1} \tag{13.128}$$

Suppose we want to build a single 10 TeV accelerator (1 TeV = 10^3 GeV) so that $E_1 = 10^4$ GeV.

This is very difficult to design and requires the development of significant new equipment ($$$$$$$).

If instead we build two smaller accelerators and use them in the colliding beams configuration, then we get the same available energy with

$$E = \frac{1}{2}\sqrt{2E_1} = \sqrt{5000} = 71\,Gev \tag{13.129}$$

which we already know how to build.

In fact, if we use an old single accelerator of this size that already exists, we then only have to build one small new accelerator ($$).

High Energy Collisions

Earlier we discussed low energy collisions between particles using conservation of energy and momentum.

Let us look at the same processes at high energy.

We consider a collision in which the incident particle has zero rest mass (photon) and the target particle is at rest.

If the target particle is an electron, then this if the so-called **Compton Effect**.

The process looks like

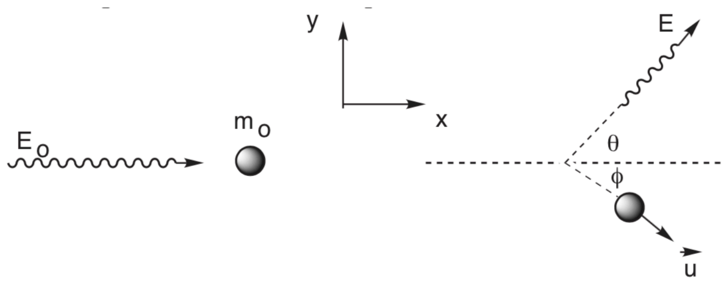

Figure 353:

The photon momentum is $\frac{E_0}{c}$.

After the collision the photon is scattered through an angle θ with energy E and the electron recoils at an angle ϕ with velocity $\vec{u}$.

The final electron energy is

$$E_e = \gamma(u) m_0 c^2 = \frac{m_0 c^2}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Conservation of energy gives $E_0 + m_0 c^2 = E + E_e$.

Conservation of momentum gives (x and y directions)

$$\frac{E_0}{c} = \frac{E}{c}\cos\theta + p\cos\phi$$

$$0 = \frac{E}{c}\sin\theta - p\sin\phi$$

where

$$\vec{p} = \gamma m_0 \vec{u} \quad \text{or} \quad E_e^2 = p^2 c^2 + m_0^2 c^4$$

We want to eliminate reference to the electron and find the new photon energy (that is what is detected in the experiment).

$$\frac{E_0}{c} = \frac{E}{c}\cos\theta + p\cos\phi \to p\cos\phi = \frac{E_0}{c} - \frac{E}{c}\cos\theta$$

or

$$p^2\cos^2\phi = \left(\frac{E_0}{c} - \frac{E}{c}\cos\theta\right)^2$$

$$0 = \frac{E}{c}\sin\theta - p\sin\phi \to p\sin\phi = \frac{E}{c}\sin\theta \to p^2\sin^2\phi = \frac{E^2}{c^2}\sin^2\theta$$

Adding these equations we get

$$p^2c^2 = E_e^2 - m_0^2c^4 = E_0^2 - 2E_0E_e\cos\theta + E^2$$

Using the energy conservation equation we have (after algebra)

$$E = \frac{E_0}{1 + \left(\frac{E_0}{m_0c^2}\right)(1 - \cos\theta)}$$

The first thing to note is that $E > 0$.

This means that a free electron cannot absorb a photon completely; there will always be a scattered photon of some energy.

If we convert to wavelengths using

$$E = h\nu = h\frac{c}{\lambda}$$

we get

$$\lambda - \lambda_0 = \frac{h}{m_0c}(1 - \cos\theta)$$

The shift in wavelength at a given angle is independent of the incident photon energy.

Doppler effect as a Collision with Photons

We consider an atom with a rest mass of M_0.

If it is held stationary and the atom emits a photon of energy $h\nu_0$, then its rest mass must change (it is losing energy) $M'c^2 = M_0c^2 - h\nu_0$.

Suppose that it is moving as shown in the top part of the figure below and then emits a photon as shown in the bottom part of the figure.

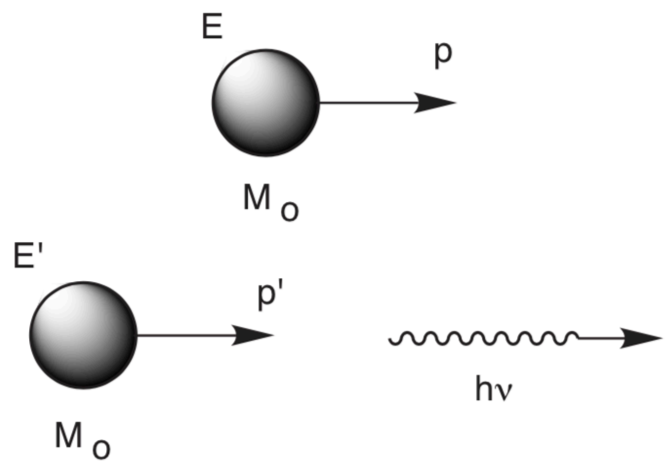

Figure 354:

Before photon emission we have

$$E = \gamma M_0 c^2 = \frac{M_0 c^2}{\sqrt{1 - \frac{u^2}{c^2}}} \quad , \quad p = \gamma M_0 u = \frac{M_0 u}{\sqrt{1 - \frac{u^2}{c^2}}}$$

After emission, the atom has energy E' and momentum p' while the emitted photon has $E_\nu = h\nu = p_\nu c$.

Conservation of energy and momentum says that

$$E = E' + h\nu$$

$$p = p' + \frac{h\nu}{c}$$

Therefore

$$E'^2 - p'^2c^2 = (E - h\nu)^2 - (pc - h\nu)^2 = M_0'^2c^4 = (M_0c^2 - h\nu_0)^2$$

Some algebra gives

$$\nu = \nu_0 \frac{2M_0c^2 - h\nu_0}{2(E - pc)}$$

But

$$E - pc = \frac{M_0c^2}{\sqrt{1 - \frac{u^2}{c^2}}}\left(1 - \frac{u}{c}\right) = M_0c^2\sqrt{\frac{c-u}{c+u}}$$

Thus,

$$\nu = \nu_0 \frac{2M_0c^2 - h\nu_0}{2M_0c^2\sqrt{\frac{c-u}{c+u}}} = \nu_0\left(1 - \frac{h\nu_0}{2M_0c^2}\right)\sqrt{\frac{c-u}{c+u}}$$

The term $\frac{h\nu_0}{2M_0c^2}$ represents a decrease in the photon energy due to the recoil of the atom.

For massive atoms, this is negligible and thus

$$\nu = \nu_0\sqrt{\frac{c+u}{c-u}}$$

which is the standard Doppler formula.

We note for the future that we have shown that two completely different pictures of light, wave and particle, lead exactly to the same prediction for the shift in the frequency of radiation from a moving source.

The Mass of a Photon

Pulsars are collapsed stars that emit regular bursts of energy at repetition frequencies from 30 to 0.1 Hz. The are collapsed stars with intense magnetic fields that are rotating rapidly.

The pulsar in the Crab nebula has the a frequency of 30 Hz and pulses in the optical and x-ray regions, as well as at radio frequencies.

The pulses are extremely sharp and their arrival times can be measured to an accuracy of microseconds or better.

Experimentally, the radiation from the pulsar at all different wavelengths seems to arrive simultaneously (or all within the experimental resolving time).

Let us use this data to set a limit on the rest mass of the photon.

It takes light about 5000 years to arrive at the earth from the Crab nebula.

Suppose that signals at two different frequencies travel with a small difference in speed, Δv, and thus arrive at slightly different times, T and $T + \Delta T$.

Since $T = \frac{L}{v}$, where L = distance to the Crab nebula, we have

$$v = \frac{L}{T} \to \Delta v = -\frac{L}{T^2}\Delta T \to \frac{\Delta v}{v} = -\frac{\Delta T}{T}$$

No such velocity difference has been observed, but by estimating the sensitivity of the experiment we can set an upper limit on the quantity Δv.

ΔT can be measured to an accuracy of about 2×10^{-3} sec and using $T = 5 \times 10^3$ years $= 1.5 \times 10(11)$ sec we have

$$\left|\frac{\Delta v}{c}\right| = \left|\frac{\Delta T}{T}\right| < \frac{2 \times 10^{-3}}{1.5 \times 10^{11}} \approx 10^{-14}$$

where we have assumed that $v \approx c$.

Now we translate this limit on δv into a limit on the possible rest mass of the photon.

If the photon had a nonzero rest mass, the velocity of light would be different from c.

If we let m_p represent the rest mass of the photon, then we would have $E = \gamma m_p c^2$.

If we assume that the photon energy frequency relation $E = h\nu$ is still valid, then we have

$$(h\nu)^2 = (m_p c^2)^2 \frac{1}{1 - \frac{v^2}{c^2}} \to \frac{v^2}{c^2} = 1 - \frac{\nu_0^2}{\nu^2}$$

where $h\nu_0 = m_p c^2$. ν_0 plays the role of a characteristic frequency for the photon.

$h\nu_0$ is the rest energy of the photon.

If $\nu_0 = 0$, then we have $v = c$.

Otherwise the velocity of light depends on frequency.

$$\frac{\nu_1^2}{c^2} - \frac{\nu_2^2}{c^2} = \nu_0^2\left(\frac{1}{\nu_2^2} - \frac{1}{\nu_1^2}\right)$$

$$\frac{1}{c^2}(\nu_1^2 - \nu_2^2) = \frac{1}{c^2}(v_1 - v_2)(v_1 + v_2) = \nu_0^2\left(\frac{1}{\nu_2^2} - \frac{1}{\nu_1^2}\right)$$

$$\frac{1}{c^2}\Delta v(2c) = 2\frac{\Delta v}{c} = \nu_0^2\left(\frac{1}{\nu_2^2} - \frac{1}{\nu_1^2}\right)$$

For observations made in the optical regions we can use

$$\nu_1 = 8 \times 10^{14}\,Hz \text{ (blue)} \quad \text{and} \quad \nu_2 = 5 \times 10^{14}\,Hz \text{ (red)}$$

Then we have

$$2 \times 10^{-14} > 2\frac{\Delta v}{c} = \nu_0^2\left(\frac{1}{\nu_2^2} - \frac{1}{\nu_1^2}\right) = \frac{1}{10^{28}}\left(\frac{1}{5^2} - \frac{1}{8^2}\right)$$

$$\nu_0 < 10^7\,Hz$$

This gives and upper limit to the photon rest mass of

$$m_{upper} = \frac{h\nu_0}{c^2} < 10^{-40}\,kg$$