Applying Mathematics ideas to Quantum Mechanics

We now expand our understanding of QM theory
and also explain the color/hardness experiments

Quantum state = set of information **from** measurements.

It will be represented by a KET $|....\rangle$ $|banana\rangle\,,|747\rangle\,,|you\rangle$

where ……. = labels indicating what we know(must have measured) about the state

important aspect

Thus

electron - no measurements yet $|undetermined\rangle = |?\rangle$

electron into COLOR box emerges magenta aperture $|magenta\rangle = |m\rangle$

electron into COLOR box emerges green aperture $|green\rangle = |g\rangle$

electron into HARDNESS box emerges hard aperture $|hard\rangle = |h\rangle$

electron into HARDNESS box emerges soft aperture $|soft\rangle = |s\rangle$

Clearly the labels tell us everything we “know” about the state

As we will see later — **Postulate #1 of quantum theory will be — it is all in the labels!**

Continuing, a measuring device is represented by an OPERATOR

Definition: Operators take Kets into Kets $\hat{O}\,|x\rangle = |y\rangle$

Some operator properties from experimental results:

$$\mathcal{P}_g\,|hard\rangle = (|g\rangle\langle g|)\,|h\rangle \;= \langle g\,|\,h\rangle\,|g\rangle \rightarrow |g\rangle$$

All state always **renormalized** to 1

green projection operator

$$\mathcal{P}_m\,|h\rangle = (|m\rangle\langle m|)\,|h\rangle \;= \langle m\,|\,h\rangle\,|m\rangle \rightarrow |m\rangle$$

magenta projection operator

Last step:

All state always normalized to 1

$\langle g\,|\,g\rangle = 1$ by definition

State $|Q\rangle = \langle g\,|\,h\rangle\,|g\rangle$

is not normalized to 1

—> always renormalize to 1 at **each** stage

Projection operators **seem** to represent measurement(pick out states) - we will see!

You will note that I do not tell you WHY I make certain choices, I only tell you HOW the my particular choices work.

That is the difference between Physics and Philosophy.

Physics NEVER answers WHY questions, we leave those for Philosophers, who have never once answered a "why" question!!

Now send a magenta electron into a hardness box

it comes out either hard or soft (50%-50%) (out the appropriate aperture).

A magenta electron was described earlier as a **SUPERPOSITION** of hard and soft electrons,

although we did not, at that time, specify what was meant by the term “superposition”.

Now we represent a **superposition** mathematically by **addition of kets** (simply vector addition) as follows:

$$|m\rangle = \frac{1}{\sqrt{2}}|h\rangle - \frac{1}{\sqrt{2}}|s\rangle$$
$$|g\rangle = \frac{1}{\sqrt{2}}|h\rangle + \frac{1}{\sqrt{2}}|s\rangle$$
$$|h\rangle = \frac{1}{\sqrt{2}}|g\rangle + \frac{1}{\sqrt{2}}|m\rangle$$
$$|s\rangle = \frac{1}{\sqrt{2}}|g\rangle - \frac{1}{\sqrt{2}}|m\rangle$$

This will be Postulate #2 of quantum theory.

The reasons for the particular numerical coefficients will become clear shortly.

This is the way theory works:

Make assumptions(postulates),

i.e., meaning of term “superposition” above.

Develop measurable predictions from the assumptions.

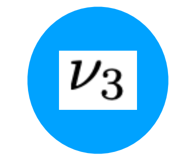
Then test experimentally to check the validity of the postulates.

More about Probability

Suppose we have a box with N numbered balls

set of numbers **on** balls — number of balls **with** given number

$$\{\nu_i\} = \nu_1, \nu_2, \nu_3, \nu_4, ... \quad , \quad \{n_i\} = n_1, n_2, n_3, n_4, ...$$



$$\sum_k n_k = n_1 + n_2 + n_3 + n_4 + ... = N$$ total number

probability that a random picked ball has k[th] number ν_k is $p_k = \frac{n_k}{N}$

$n_k = 0$ —> $p_k = 0$ **impossible**

$n_k = N$ —> $p_k = 1$ **certainty**

$$0 \leq p_k \leq 1 \text{ for all } k \quad , \quad \sum_k p_k = 1$$

above are all common sense ideas

In QM

$$probability(q) = \frac{number\, of\, times\, value\, q\, was\, already\, measured}{total\, number\, of\, measurements\, so\, far}$$

identical systems —> create ensemble —> allows frequency model representing probabilities

Using the standard definition of the average one can also show $\langle f(\nu)\rangle = \sum_k p_k f(\nu_k)$ for any function.

Continuing with QM discussion

Although QM is a probabilistic theory, the fundamental quantity will **not** be probability but instead it is a mathematical object called the **probability amplitude,** which we will define below.

Now if the electron to be measured was in an arbitrary state $|\psi\rangle$,

then probability amplitude that one will measure color = magenta

if one sends this electron into a color box is given by

a new mathematical object (amplitude) **represented** by the Dirac braket symbol

$$\langle property\,to\,be\,measured \,|\, state\,being\,measured\rangle = \langle m \,|\, \psi\rangle$$

—-> complex number = component of ket $|\psi\rangle$ in $|m\rangle$ direction

amount

$$A = probability\,amplitude = \langle m \,|\, \psi\rangle$$

The Dirac Bracket!!!!

$$P = probability = |\,\langle m \,|\, \psi\rangle\,|^2 \;\; = |A|^2$$

= probability to measure magenta if electron in state $|\psi\rangle$

This will be Postulate #3 of quantum theory

Now, we use this idea and see what it actually says, i.e., what it implies………

Send an electron into a color box

and look at the beam emerging from magenta aperture

= magenta electrons - all in state $|m\rangle$ if box is working properly

Experiments can be used to determine the numerical value of amplitudes:

Probability that magenta electron emerges from magenta aperture of color box

$$P = |\langle m \mid m\rangle|^2 = 1 \longrightarrow \langle m \mid m\rangle = 1$$

Probability that magenta electron emerges from green aperture of color box

$$P = |\langle m \mid g\rangle|^2 = 0 \longrightarrow \langle m \mid g\rangle = 0$$

Probability that hard electron emerges from hard aperture of hardness box

$$P = |\langle h \mid h\rangle|^2 = 1 \longrightarrow \langle h \mid h\rangle = 1$$

Probability that hard electron emerges from soft aperture of hardness box

$$P = |\langle h \mid s\rangle|^2 = 0 \longrightarrow \langle h \mid s\rangle = 0$$

Similarly, we have

$$\langle g \mid g\rangle = 1 \qquad \langle s \mid s\rangle = 1 \qquad \langle g \mid m\rangle = 0 \qquad \langle s \mid h\rangle = 0$$

What do these numerical results mean?

Clearly,

kets $\{|m\rangle\,, |g\rangle\}$ or kets $\{|h\rangle\,, |s\rangle\}$ from earlier discussions

are just two different orthonormal bases for all quantum states in world of color/hardness

$\langle m\,|\,m\rangle = 1$ $\langle g\,|\,g\rangle = 1$ $\langle m\,|\,g\rangle = 0$ —> orthonormal set

normalized to 1 orthogonal

Using one of these basis sets to describe world —> using "color" or "hardness" language

Since they are a basis ⟶ why we can write any arbitrary quantum state

as $$|\psi\rangle = a\,|m\rangle + b\,|g\rangle$$

or as $$|\psi\rangle = c\,|h\rangle + d\,|s\rangle$$

Remember I wrote earlier

$$|m\rangle = \frac{1}{\sqrt{2}}\,|h\rangle - \frac{1}{\sqrt{2}}\,|s\rangle \qquad |g\rangle = \frac{1}{\sqrt{2}}\,|h\rangle + \frac{1}{\sqrt{2}}\,|s\rangle$$

$$|h\rangle = \frac{1}{\sqrt{2}}\,|g\rangle + \frac{1}{\sqrt{2}}\,|m\rangle \qquad |s\rangle = \frac{1}{\sqrt{2}}\,|g\rangle - \frac{1}{\sqrt{2}}\,|m\rangle$$

Mysterious property called **SUPERPOSITION** is, as we stated, **just vector addition!**

Does this formalism work?

Does it allow us to say correct things about quantum experiments?

YES! Let me show you.

Consider hard electron sent into color box,

then, using rules we have developed,

the probability that it will be (**emerge as or be measured as**)

a magenta electron is given by(**this is how the algebra works**)

$$|\langle magenta \mid hard\rangle|^2 = \left|\langle magenta| \left[\frac{1}{\sqrt{2}}|green\rangle + \frac{1}{\sqrt{2}}|magenta\rangle\right]\right|^2$$

$$= \left|\frac{1}{\sqrt{2}}\langle magenta \mid green\rangle + \frac{1}{\sqrt{2}}\langle magenta \mid magenta\rangle\right|^2$$

$$= \left|\frac{1}{\sqrt{2}}(0) + \frac{1}{\sqrt{2}}(1)\right|^2 = \frac{1}{2}$$ **which is the correct probability!!**

and probability that it will emerge as green electron given by

$$|\langle green \mid hard\rangle|^2 = \left|\langle green|\left[\frac{1}{\sqrt{2}}|green\rangle + \frac{1}{\sqrt{2}}|magenta\rangle\right]\right|^2$$

$$= \left|\frac{1}{\sqrt{2}}\langle green \mid green\rangle + \frac{1}{\sqrt{2}}\langle green \mid magenta\rangle\right|^2$$

$$= \left|\frac{1}{\sqrt{2}}(1) + \frac{1}{\sqrt{2}}(0)\right|^2 = \frac{1}{2}$$ **which is also correct!!**

Thus, a hard electron has 50-50 chance

of coming out of magenta and green apertures

which agrees with the results of the earlier experiments.

That is why I earlier chose particular numerical coefficients (component values) $\pm\frac{1}{\sqrt{2}}$

So our formalism seems to agree so far with all of the earlier experiments.

For example, look at repeatability experiment.

First, hard electron $|h\rangle$ sent into color box.

Probability will emerge from magenta aperture is

$$|\langle m \mid h\rangle|^2 = \frac{1}{2}$$

Then, electron emerging from magenta aperture sent into another color box.

However, it is **now** a magenta electron and is represented by ket $|m\rangle$.

Probability that it will emerge from magenta aperture of second color box is

$$|\langle m \mid m\rangle|^2 = 1$$ **= repeatability property.**

During experiment,

hard electron(initially in superposition of magenta/green properties), i.e.,

$$|h\rangle = \frac{1}{\sqrt{2}}|g\rangle + \frac{1}{\sqrt{2}}|m\rangle$$

appears as electron in state $|m\rangle$

when it emerges from magenta aperture of first color box.

The so-called **Copenhagen** interpretation of quantum theory says

that during 1st measurement (first color box), the state of electron(hard)

is “**collapsed” or “reduced”** from a superposition of possibilities to a definite

value(magenta) corresponding to value just measured, i.e., to aperture we looked at!

This interpretation seems to say that the measurement **caused** the collapse to occur.

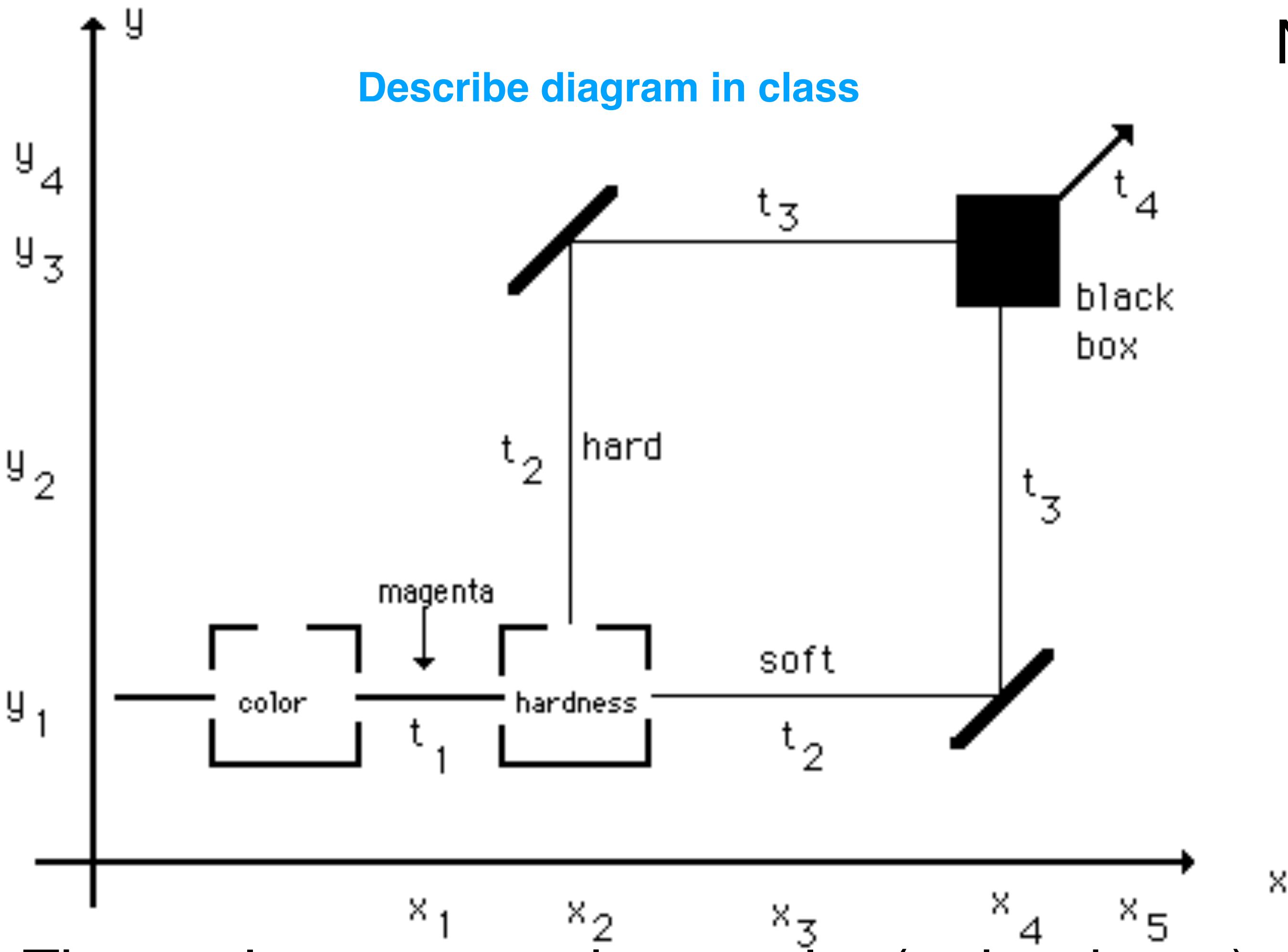
This will be postulate #4 of our 1st version of quantum theory.

Eventually, we will eliminate the need for this “collapse” postulate completely.

We will find that it is already built into the other postulates we define!

Two-Path Experiment now explained using the New Formalism

Added time and position values to diagram.



Need to use states or kets which correspond

to electrons having color or hardness

AND positions.

This is OK because these are what are called "**compatible** properties",

i.e., we will see that it is valid in QM to

say a magenta electron is located

at a particular position

(since it can be created in lab).

Thus, electrons can have color (or hardness) and simultaneously be at definite position.

However, as saw earlier,

an electron CANNOT be hard AND magenta simultaneously.

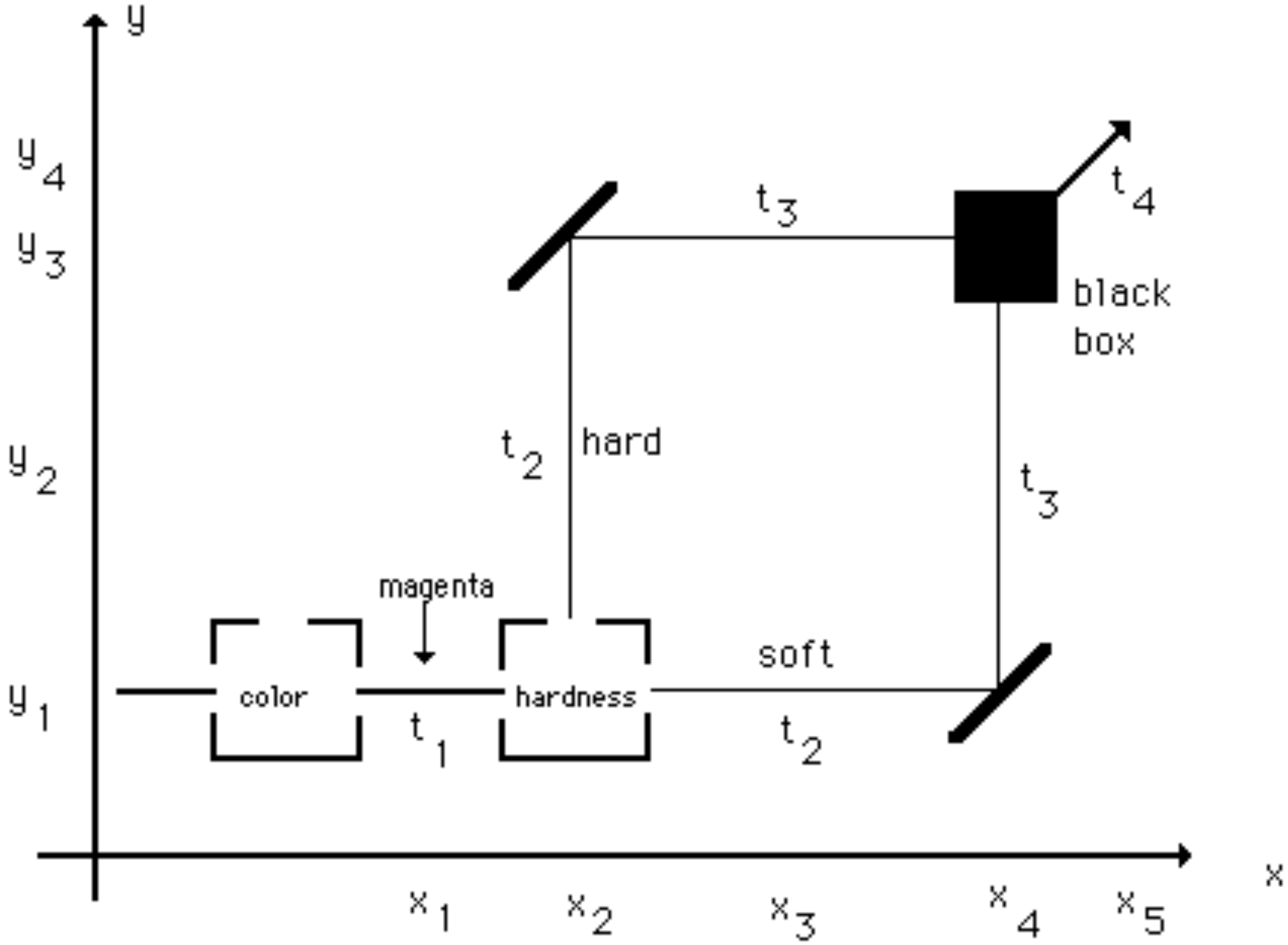
These are incompatible properties, which we will see is related to the Uncertainty Principle.

These states take the form (just **expand the labels** because we **KNOW** more stuff):

$$|color, x, y\rangle = |color\rangle\,|x, y\rangle$$

$$|hardness, x, y\rangle = |hardness\rangle\,|x, y\rangle$$

so the labels still specify all that we "know"



At time t_1,

when particle is about to enter apparatus,

(having just left a color box via

magenta aperture)

the state is

$$|color = magenta, x = x_1, y = y_1\rangle = |m\rangle\,|x_1, y_1\rangle \qquad \text{at } t_1$$

$$= \left(\frac{1}{\sqrt{2}}|h\rangle - \frac{1}{\sqrt{2}}|s\rangle\right)|x_1, y_1\rangle = \frac{1}{\sqrt{2}}|h\rangle\,|x_1, y_1\rangle - \frac{1}{\sqrt{2}}|s\rangle\,|x_1, y_1\rangle$$

(everything just multiplied out)

It is a superposition of a hard electron at (x_1,y_1) and soft electron at (x_1,y_1)!

A magenta electron is always such a superposition at any given position!

Here is how QM will tell us to calculate what happens next.

Consider: If state at time t_1 were just the hard part

$$\frac{1}{\sqrt{2}}|h\rangle|x_1, y_1\rangle$$

and if a hardness box behaves properly,

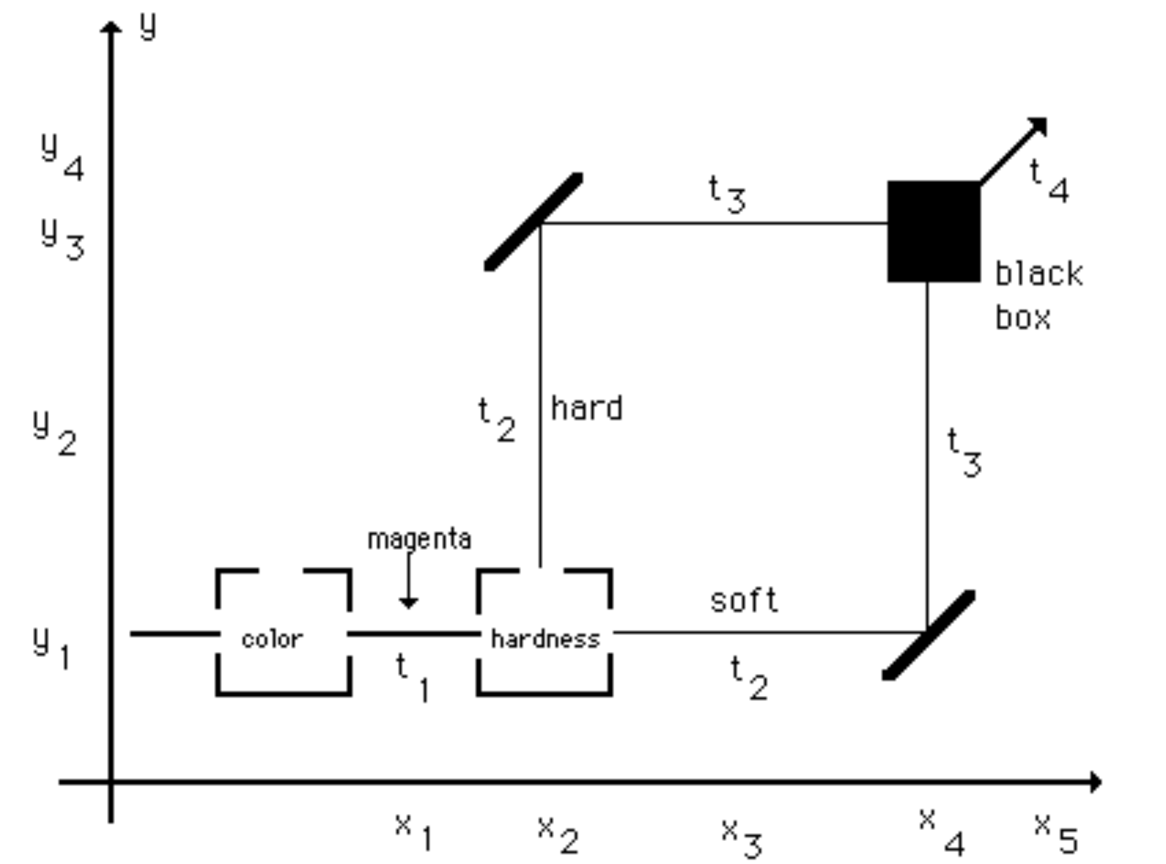
then the state at time t_2 would be

$$\frac{1}{\sqrt{2}}|h\rangle|x_2, y_2\rangle$$

i.e., the electron would have just emerged

through hard aperture

and be on what we will call the **hard path**.



Similarly, if state at time t_1 were just soft part $\frac{1}{\sqrt{2}}|s\rangle|x_1, y_1\rangle$

and if hardness box behaves properly, then state at time t_2 would be $\frac{1}{\sqrt{2}}|s\rangle|x_3, y_1\rangle$

i.e., electron would have just emerged through soft aperture

and be on what we will call the **soft path.**

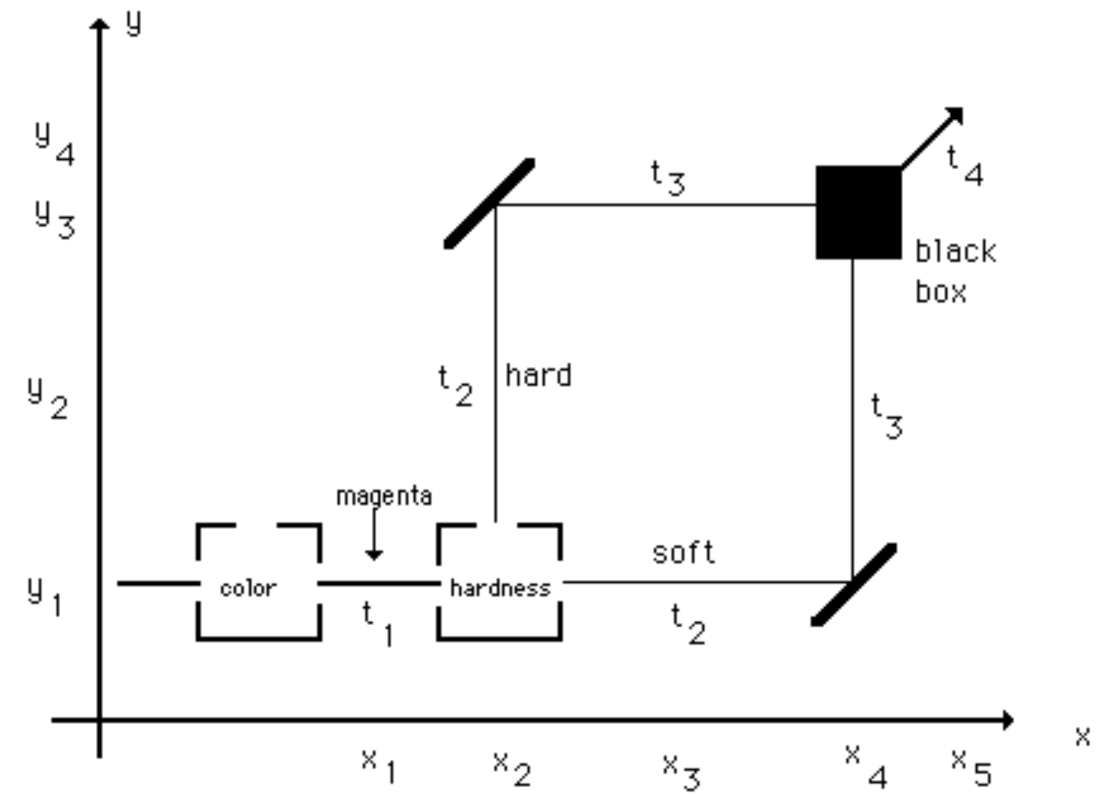
This is what we earlier said the apparatus does to hard/soft electrons!

IMPORTANT: we can only say the electron has two compatible measurable properties at the same time if the respective state separates exactly like this one did.

However, as we said, the state at t_1

is neither just a soft part nor just a hard part,

but the superposition



$$|m\rangle\,|x_1,y_1\rangle = \frac{1}{\sqrt{2}}\,|h\rangle\,|x_1,y_1\rangle - \frac{1}{\sqrt{2}}\,|s\rangle\,|x_1,y_1\rangle$$

when using color box

-> USE color language to describe world —> start in color language(basis)

when using hardness box

-> USE hardness language to describe world —> change to hardness language(basis)

This is VERY Important

Physics will only make sense if you are using the correct language (appropriate to next measurement) when discussing what is happening!

Earlier discussion therefore says

that state at time t_2 must be (if hardness box behaves properly)

$$\frac{1}{\sqrt{2}}\left|h\right\rangle\left|x_2,y_2\right\rangle - \frac{1}{\sqrt{2}}\left|s\right\rangle\left|x_3,y_1\right\rangle$$

|1> **|2>**

i.e., if something happens to state |1> (hard part) and something happens to state |2> (soft part) when sent through separately

then when both are present at same time in a superposition **in words**

SAME things still happen to each part

[Remember we are not measuring anything yet!!]

So we have at time t_2

$$\frac{1}{\sqrt{2}}\left|h\right\rangle\left|x_2,y_2\right\rangle - \frac{1}{\sqrt{2}}\left|s\right\rangle\left|x_3,y_1\right\rangle$$

Now expand

$$\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left|g\right\rangle + \frac{1}{\sqrt{2}}\left|m\right\rangle\right)\left|x_2,y_2\right\rangle - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left|g\right\rangle - \frac{1}{\sqrt{2}}\left|m\right\rangle\right)\left|x_3,y_1\right\rangle$$

$$\longrightarrow \quad \frac{1}{2}\left|g\right\rangle\left(\left|x_2,y_2\right\rangle - \left|x_3,y_1\right\rangle\right) + \frac{1}{2}\left|m\right\rangle\left(\left|x_2,y_2\right\rangle + \left|x_3,y_1\right\rangle\right)$$

at time t_2

This correspond to color electrons at superpositions of two different positions !!!!

i.e., a green electron can be found at two positions if we look(observe/measure)

i.e., there is a nonzero probability to be at two positions

The state

$$\frac{1}{\sqrt{2}}\left|h\right\rangle\left|x_2,y_2\right\rangle-\frac{1}{\sqrt{2}}\left|s\right\rangle\left|x_3,y_1\right\rangle$$

is an **entanglement** between hardness/color properties

and position (coordinate space) properties of electron

(note color and position properties

were separated to start with, i.e., $\left|color=magenta,x=x_1,y=y_1\right\rangle=\left|m\right\rangle\left|x_1,y_1\right\rangle$

i.e., **had definite values**, namely, magenta and (x_1,y_1)).

This new state involves **nonseparable correlations(entanglement)** between hardness/color and coordinate-space properties of electron.

There are no definite hardness or color properties of electron in this state.

Neither do any of its coordinate-space properties (position, momentum, etc)

have any definite values — the hardness box did that!!!

This is a **superposition** of states,

where one part of electron state indicates

probability amplitude for being on hard path

and other of electron state indicates

probability amplitude for being on soft path.

IMPORTANT: Note that we are NOT saying that electron is actually on either path.

REMEMBER: No measurements have been made yet!!!!!

These results are the reason that

we earlier had to make statements about not hard, not soft, not both and not neither.

Those ideas are not only false, they are meaningless in this state!

We have entangled the properties of two **classical states** that **cannot exist** at same time for single electron!

That is the idea of the state that is called an **entangled state**.

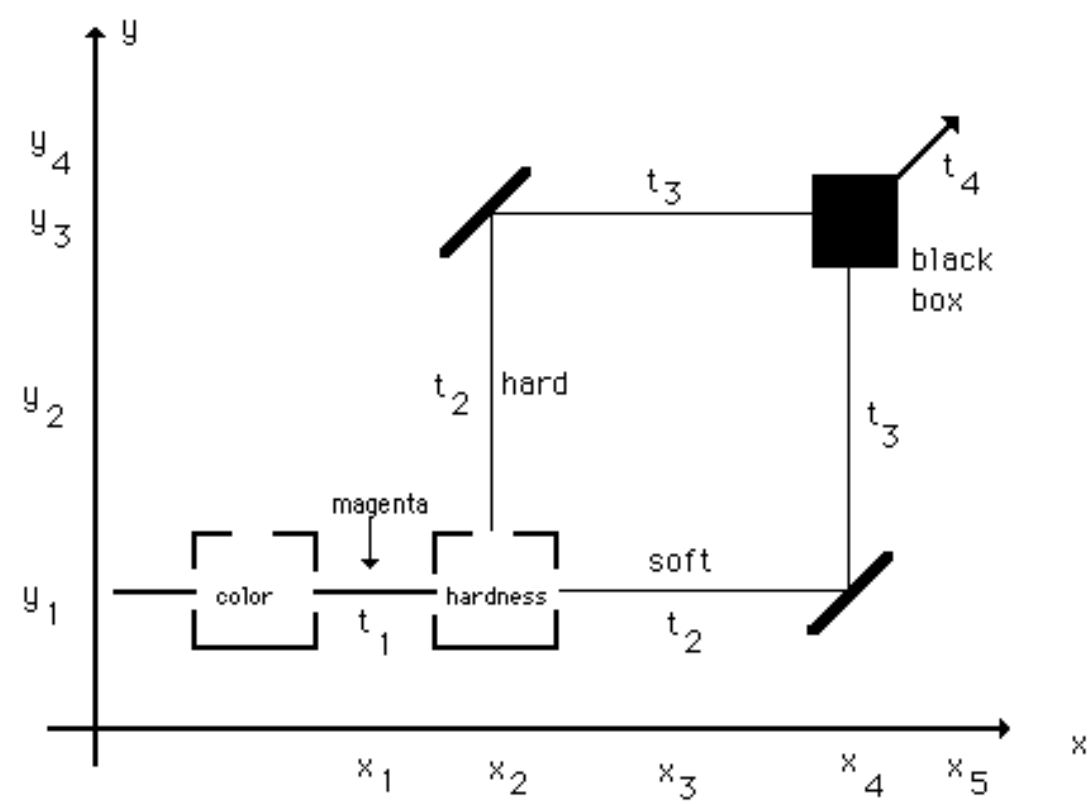
These states will get our thoughts/ideas all entangled later on

when we discuss the Einstein paradox and Bell inequality!!

Continuing on.......

last state of electron above (at t_2)

then leads to state of electron at t_3



$$\frac{1}{\sqrt{2}}\left|h\right\rangle\left|x_3,y_3\right\rangle-\frac{1}{\sqrt{2}}\left|s\right\rangle\left|x_4,y_2\right\rangle$$

and then state at time t_4 is

$$\frac{1}{\sqrt{2}}\left|h\right\rangle\left|x_5,y_4\right\rangle-\frac{1}{\sqrt{2}}\left|s\right\rangle\left|x_5,y_4\right\rangle=\left(\frac{1}{\sqrt{2}}\left|h\right\rangle-\frac{1}{\sqrt{2}}\left|s\right\rangle\right)\left|x_5,y_4\right\rangle$$

$$=\left|m\right\rangle\left|x_5,y_4\right\rangle$$

pull out common factor

$$\frac{1}{\sqrt{2}}|h\rangle|x_5,y_4\rangle - \frac{1}{\sqrt{2}}|s\rangle|x_5,y_4\rangle = \left(\frac{1}{\sqrt{2}}|h\rangle - \frac{1}{\sqrt{2}}|s\rangle\right)|x_5,y_4\rangle = |m\rangle|x_5,y_4\rangle$$

We see

that color state and coordinate-space state have become **separate** again!

Position of electron **once again** has definite value and color **once again** has definite value!!

Remember that we have not bothered(measured) the electron during this experiment.

So fact that hard electron fed into total device will come out hard(follow hard path),

and that soft electron fed into total device will come out soft(follow soft path),

together with our quantum assumptions,

means that a magenta electron fed into total device

will come out (as EXPERIMENT says) magenta!

That is way quantum mechanics works! Again, the formalism agrees with experiment!

What if we **change the experiment** in middle by measuring position of the electron at, say, t_3?

For example, let us insert wall on soft path.

This Implies that if any electron gets through device, it **must** have traveled on hard path.

This is a measurement of position.

Same goes for if inserting wall on hard path.

Then the superposition goes away.

According to the suggested postulates,

during a measurement a collapse **would** occur

and state just after measurement would be either

$$|h\rangle\,|x_3, y_3\rangle \quad or \quad |s\rangle\,|x_4, y_2\rangle$$

block soft path	block hard path

each with probability 1/2 (square component assumption)

Then state at t_4 would be either

$$|h\rangle\,|x_5, y_4\rangle \quad or \quad |s\rangle\,|x_5, y_4\rangle$$

block soft path	block hard path

After this point in time,

only one part of state continues to evolve in time,

while other remains unchanged (due to the wall).

Must emphasize that we collapsed the particle to hard or soft

by measuring its position in an entangled state.

That is way entangled or nonseparable states work

collapsing one property (position in this case)

collapses all other properties (hardness in this case) simultaneously.

The state vector

$$\frac{1}{\sqrt{2}}\left|h\right\rangle\left|x_5,y_4\right\rangle-\frac{1}{\sqrt{2}}\left|s\right\rangle\left|x_3,y_1\right\rangle$$

is **nonseparable**

between hardness/color properties and coordinate-space properties.

This means it is **not associated** with any definite values

for hardness or color or position or momentum.

Note that since we have inserted a wall on only one path(soft),

the remaining electrons in device have **definite hardness**,
which are 50/50 magenta/green.

This agrees with experiments described earlier.

Again, the formalism works!

Now I must say something about how to build a **do-nothing box**.

If electron is in a state $|A\rangle$ and we measure color,

then the probability of measuring magenta is determined as shown below.

Write $|A\rangle$ as superposition of color states

(since color is thing want to measure, we must use color language).

$$|A\rangle = \alpha\,|magenta\rangle + \beta\,|green\rangle$$

Can always be done since color kets are a basis(anything measurable provides a basis).

This says that

$$\langle magenta \mid A\rangle = \alpha\,\langle magenta \mid magenta\rangle + \beta\,\langle magenta \mid green\rangle = \alpha$$

$$\langle green \mid A\rangle = \alpha\,\langle green \mid magenta\rangle + \beta\,\langle green \mid green\rangle = \beta$$

The probability that the measurement indicates the electron is magenta is

$$|\langle magenta \mid A\rangle|^2 = |\alpha|^2$$

Now consider the state

$$-|A\rangle = -\alpha\,|magenta\rangle - \beta\,|green\rangle$$

If we measure the color on this state,

we get same probabilities for every possible outcome as with state $|A\rangle$, i.e.,

$$|\langle magenta \mid A\rangle|^2 = |\alpha|^2 = |-\langle magenta \mid A\rangle|^2$$

So if a state vector $|A\rangle$ changes to state vector $-|A\rangle$

there are NO OBSERVABLE CHANGES

there are NO PROBABILITIES CHANGES

NOTHING HAPPENS to that state as far as quantum theory is concerned!!!.

So any box which changes state of any incoming electron

into –1 times incoming state will be **do-nothing box**,

since it will not change any measurable values,

i.e., will not change any probabilities

of values of any of observables of any electron which passes through it.

Obviously, it will also not effect any electron which passes outside of it.

But the effects of such a box on an electron

which is in a superposition of passing through it and outside of it

may be **quite a different matter**.

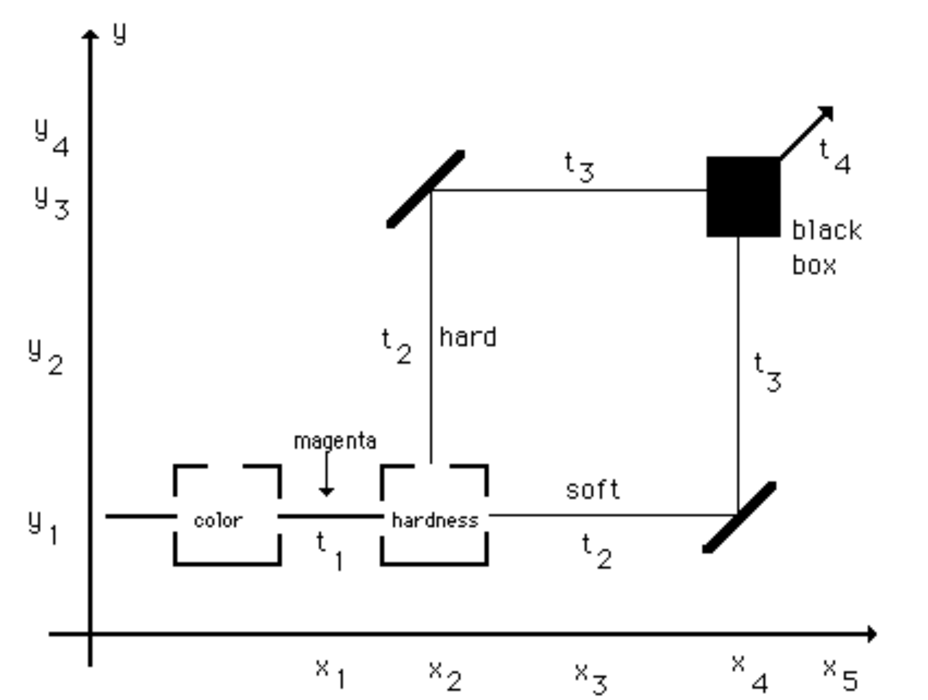
Suppose a do-nothing box is inserted in soft path of two-path device at (x_3,y_1).

Then, if initial input electron magenta(as earlier), state at t_2 will be

$$\frac{1}{\sqrt{2}}\,|h\rangle\,|x_2,y_2\rangle - \frac{1}{\sqrt{2}}\,|s\rangle\,|x_3,y_1\rangle \qquad \text{(same as earlier)}$$

and state at t_3 (after passage through box) will be not be

$$\frac{1}{\sqrt{2}}\,|h\rangle\,|x_3,y_3\rangle - \frac{1}{\sqrt{2}}\,|s\rangle\,|x_4,y_2\rangle \qquad \text{(earlier result)}$$



but will be

$$\frac{1}{\sqrt{2}}|h\rangle|x_3,y_3\rangle+\frac{1}{\sqrt{2}}|s\rangle|x_4,y_2\rangle$$

where sign of 2nd term has been changed by the do-nothing box in the soft path.

If we now follow new state to t_4 as before we find

$$\begin{aligned}\frac{1}{\sqrt{2}}|h\rangle|x_5,y_4\rangle+\frac{1}{\sqrt{2}}|s\rangle|x_5,y_4\rangle &= \left(\frac{1}{\sqrt{2}}|h\rangle+\frac{1}{\sqrt{2}}|s\rangle\right)|x_5,y_4\rangle \\ &= |g\rangle|x_5,y_4\rangle\end{aligned}$$

instead of

$$\begin{aligned}\frac{1}{\sqrt{2}}|h\rangle|x_5,y_4\rangle-\frac{1}{\sqrt{2}}|s\rangle|x_5,y_4\rangle &= \left(\frac{1}{\sqrt{2}}|h\rangle-\frac{1}{\sqrt{2}}|s\rangle\right)|x_5,y_4\rangle \\ &= |m\rangle|x_5,y_4\rangle\end{aligned}$$

So a do-nothing box has changed color of all electrons from magenta to green

(as the experiment said) even though it has no measurable effect

on electrons that passed though it !!!!.

That is way quantum mechanics works!! Hopefully you think it is strange!!

The new mathematical formalism seems to works very well since the theory it provides agrees with experiment, **which is only test required!**

Short digression: Some further mathematical thoughts based on what we have learned and to clean up some loose ends.

Define the operator

$$\hat{G} = |g\rangle \langle g|$$

Properties

$$\hat{G}|g\rangle = |g\rangle \langle g \mid g\rangle = |g\rangle \quad , \quad \hat{G}|m\rangle = |g\rangle \langle g \mid m\rangle = 0$$

renormalization

Expectation value

$$\langle g|\hat{G}|g\rangle = |\langle g \mid g\rangle|^2 = 1 \qquad \text{in green state}$$

$$\langle m|\hat{G}|m\rangle = |\langle g \mid m\rangle|^2 = 0 \qquad \text{in magenta state}$$

These results make sense(physics) if we interpret $\hat{G} = |g\rangle \langle g|$

as operator corresponding to measurement of green property of electrons,

i.e., it represents an observer looking at the output of the green aperture of the color box.

1st result then says, using an earlier result we derived $prob(b_k) = |\langle b_k \mid \psi\rangle|^2$

$$\langle g|\hat{G}|g\rangle = |\langle g \mid g\rangle|^2 = 1$$ in green state

that probability that color of green electron is green is 1 as expected

and

2nd result says if measure probability color of green electron is magenta get 0 as expected.

$$\langle m|\hat{G}|m\rangle = |\langle g \mid m\rangle|^2 = 0$$ in magenta state

Things make sense!

Pushing these strange ideas further.

If we assume, as earlier, that hard electron is superposition of green and magenta electrons,

$$|hard\rangle = |h\rangle = \frac{1}{\sqrt{2}}|g\rangle + \frac{1}{\sqrt{2}}|m\rangle$$

then the expectation value of $\hat{G}$ in hard state is

$$\langle g|\,\hat{\mathbf{G}}\,|h\rangle = \left(\frac{1}{\sqrt{2}}\langle g| + \frac{1}{\sqrt{2}}\langle m|\right)\hat{\mathbf{G}}\left(\frac{1}{\sqrt{2}}|g\rangle + \frac{1}{\sqrt{2}}|m\rangle\right)$$

$$= \frac{1}{2}\langle g|\,\hat{G}\,|g\rangle + \frac{1}{2}\langle g|\,\hat{G}\,|m\rangle + \frac{1}{2}\langle m|\,\hat{G}\,|g\rangle + \frac{1}{2}\langle m|\,\hat{G}\,|m\rangle$$

$$= \frac{1}{2}\langle g|\,\hat{G}\,|g\rangle = \frac{1}{2}$$

i.e., equal parts 0(m) and 1(g) as expected!!!!

Another way of saying this is, using earlier result (for those who like more math)

$$\left\langle \hat{B} \right\rangle = \sum_k b_k prob(b_k) = \sum_k b_k \left|\langle b_k \mid \psi\rangle\right|^2$$

$$\langle h|\,\hat{G}\,|h\rangle = \sum \text{(eigenvalue g)(probability of g in } |h\rangle\text{)}$$

$$= (1)\left|\langle eigenvalue = 1 \mid h\rangle\right|^2 + (0)\left|\langle eigenvalue = 0 \mid h\rangle\right|^2$$

$$= (1)\left|\langle g \mid h\rangle\right|^2 + (0)\left|\langle m \mid h\rangle\right|^2 = (1)\frac{1}{2} + (0)\frac{1}{2} = \frac{1}{2}$$

again makes sense,

i.e., if have beam of hard electrons,

then we measure electron to be green 1/2 of time as observed earlier!

Clearly, this formalism and associated ideas is both neat and very powerful

and certainly seems to have the potential to describe earlier observations.

We will see shortly that this formalism, based on set of postulates,

can completely represent quantum systems and quantum measurements.

Now let us summarize some of ideas have been discussing.

More times we think about it, better we will understand it.

Sometimes theory is just **guessed** (educated guesses based on experiments)

as a set of postulates.

Let us take this approach at this time.

"The mathematician plays a game in which he himself invents the rules while the physicist plays a game in which the rules are provided by nature, but as time goes on it becomes increasingly evident that the rules which the mathematician finds interesting are the same as those which nature has chosen"

— Paul A.M. Dirac

Quantum Mechanics Postulates (using the color/hardness world)

5 parts(ASSUMPTIONS/AXIOMS) to QM algorithm.

(A) 1st Postulate -**Physical States** - **All** physical systems represented by ket vectors normalized to 1.

Called **“state vectors”** → $|\psi\rangle$ where $\langle\psi | \psi\rangle = 1$.

Literally means ALL.

A green or hard electron is represented by a ket.

An atom is represented by a ket.

A banana is represented by a ket.

A car is represented by a ket.

YOU are represented by a ket (albeit a very complex one).

Ket labels consist of everything we “know” i.e., have measured about the system represented by the Ket!

(B) 2nd Postulate - **Measurable Properties = observables**

Remember: for linear operators: $(\hat{A}+\hat{B})\,|\psi\rangle = \hat{A}\,|\psi\rangle + \hat{B}\,|\psi\rangle$

If $\hat{A}\,|\psi\rangle = \alpha\,|\psi\rangle$ then $|\psi\rangle$ is an eigenvector of $\hat{A}$ and α is the corresponding eigenvalue.

If $\hat{A}$ is an observable, then the system represented by the state $|\psi\rangle$ **has** the value α of that observable in that state.

That is, if you perform measurement corresponding to $\hat{A}$ on system in state represented by $|\psi\rangle$, then with **certainty** (probability = 1) you measure value α .

Since eigenvalues of operators representing observables are supposed to be measurable numbers, they must also be real numbers.

This means we can only choose a certain kind of operator to represent observables, namely, HERMITIAN operators that are guaranteed to always have real eigenvalues => added bonus for quantum theory is

Eigenvectors of HERMITIAN operator are always a complete, orthonormal set.

i.e., they comprise a set of mutually orthonormal vectors - basis - **see later that there are important connections to measurement here!**

Example: back to color and hardness.......

Operators $\hat{H}$ and $\hat{C}$ representing observables hardness/color are Hermitian

-> use either set of eigenvectors as basis set

for quantum theory of color and hardness.

Corresponds to choosing a language!!!!

One such basis is then $|hard\rangle = |h\rangle \quad , \quad |soft\rangle = |s\rangle$

where(by definition)

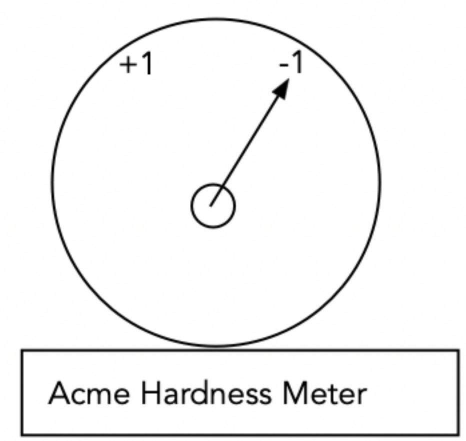
$$\hat{H}\,|h\rangle = |h\rangle \rightarrow eigenvalue = 1 \quad (by\ convention)$$
$$\hat{H}\,|s\rangle = -\,|s\rangle \rightarrow eigenvalue = -1 \quad (by\ convention)$$

± 1 values are arbitrary

$|h\rangle$ is state with measured value of hardness = 1

or on some meter!

$|s\rangle$ is state with measured value of hardness = –1



Since the basis is an orthonormal set -> satisfies

$$\langle h \mid h\rangle = 1 = \langle s \mid s\rangle$$
$$\langle h \mid s\rangle = 0 = \langle s \mid h\rangle$$

Showed earlier, any operator can be written in terms of its eigenvalues and projection operators (spectral decomposition)

$$\hat{H} = (+1)\,|h\rangle\,\langle h| + (-1)\,|s\rangle\,\langle s| = |h\rangle\,\langle h| - |s\rangle\,\langle s|$$

Using $\hat{H} = (+1)|h\rangle\langle h| + (-1)|s\rangle\langle s| = |h\rangle\langle h| - |s\rangle\langle s|$

$$\hat{H}|h\rangle = (|h\rangle\langle h| - |s\rangle\langle s|)|h\rangle = |h\rangle\langle h \mid h\rangle - |s\rangle\langle s \mid h\rangle = |h\rangle(1) - |s\rangle(0) = |h\rangle$$
$$\hat{H}|s\rangle = (|h\rangle\langle h| - |s\rangle\langle s|)|s\rangle = |h\rangle\langle h \mid s\rangle - |s\rangle\langle s \mid s\rangle = |h\rangle(0) - |s\rangle(1) = -|s\rangle$$

as expected

Eigenvector/eigenvalue equations say that hardness operator and hence hardness box acting on state vector

does not change states of definite hardness,

namely, $|h\rangle$ and $|s\rangle$

(overall minus sign does not change any measurable properties of the $|s\rangle$ state)

as required.

Now can write matrices representing these objects(using earlier definitions/rules)

in hardness basis (called **matrix representation**)

$$|h\rangle = \begin{pmatrix} \langle h \mid h\rangle \\ \langle s \mid h\rangle \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad |s\rangle = \begin{pmatrix} \langle h \mid s\rangle \\ \langle s \mid s\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$[H] = \begin{pmatrix} \langle h|\hat{H}|h\rangle & \langle h|\hat{H}|s\rangle \\ \langle s|\hat{H}|h\rangle & \langle s|\hat{H}|s\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(Hermitian!)

where we have used

$$\langle h|\hat{H}|h\rangle = \langle h|(|h\rangle\langle h| - |s\rangle\langle s|)|h\rangle = \langle h \mid h\rangle\langle h \mid h\rangle - \langle h \mid s\rangle\langle s \mid h\rangle = 1 + 0 = 1$$

=1 =1 =0 =0

Similarly, another basis(equivalent) is $|magenta\rangle = |m\rangle \quad , \quad |green\rangle = |g\rangle$

$$\hat{C}\,|g\rangle = |g\rangle \rightarrow eigenvalue = 1 \quad (by\ convention)$$
$$\hat{C}\,|m\rangle = -\,|m\rangle \rightarrow eigenvalue = -1 \quad (by\ convention)$$

$|g\rangle$ is state with measured value of color = 1

and $|m\rangle$ is state with measured value of color = –1.

Operator $\hat{C}$ represents entire color box.

Operator can be written in terms of its eigenvalues and projection operators —> we have

$$\hat{C} = |g\rangle\langle g| - |m\rangle\langle m|$$

They form a basis (orthonormal set) -> satisfies

$$\langle g \mid g\rangle = 1 = \langle m \mid m\rangle$$
$$\langle g \mid m\rangle = 0 = \langle m \mid g\rangle$$

Eigenvector/eigenvalue equations say that color operator

and hence color box acting on state vector does not change states of definite color,

namely, $|g\rangle$ and $|m\rangle$ as required.

Can write matrices representing these objects in any basis

(called matrix representation in that basis)

In color language we have

$$|g\rangle = \begin{pmatrix} \langle g \mid g\rangle \\ \langle g \mid m\rangle \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad |m\rangle = \begin{pmatrix} \langle m \mid g\rangle \\ \langle m \mid m\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$[C] = \begin{pmatrix} \langle g|\hat{C}|g\rangle & \langle g|\hat{C}|m\rangle \\ \langle m|\hat{C}|g\rangle & \langle m|\hat{C}|m\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where we have used $\langle g|\hat{C}|m\rangle = -\langle g \mid m\rangle = 0$

Now write $|g\rangle$ and $|m\rangle$ vectors in terms of $|h\rangle, |s\rangle$ vectors.

Always possible since $|h\rangle, |s\rangle$ is basis and $|g\rangle, |m\rangle$ just other vectors in same space(world).

Just principle of SUPERPOSITION mentioned earlier, Write

$$|g\rangle = a\,|h\rangle + b\,|s\rangle$$
$$|m\rangle = c\,|h\rangle + d\,|s\rangle$$

Normalization: states must be normalized to 1(assume a, b, c, d are real for simplicity)

$$\langle g \mid g\rangle = 1 = (a\,\langle h| + b\,\langle s|)\,(a\,|h\rangle + b\,|s\rangle) = a^2 + b^2$$
$$\langle m \mid m\rangle = 1 = (c\,\langle h| + d\,\langle s|)\,(c\,|h\rangle + d\,|s\rangle) = c^2 + d^2$$

Orthogonality: the states must be orthogonal (they are a basis)

$$\langle g \mid m\rangle = 0 = (a\,\langle h| + b\,\langle s|)\,(c\,|h\rangle + d\,|s\rangle) = ac + bd$$

One possible solution to equations (the color states) is

$$a = b = c = -d = \frac{1}{\sqrt{2}}$$

$$|g\rangle = \frac{1}{\sqrt{2}}|h\rangle + \frac{1}{\sqrt{2}}|s\rangle \quad , \quad |m\rangle = \frac{1}{\sqrt{2}}|h\rangle - \frac{1}{\sqrt{2}}|s\rangle$$

Similarly, can express $|h\rangle$,$|s\rangle$ vectors in terms $|g\rangle$, $|m\rangle$ basis to get

$$|h\rangle = \frac{1}{\sqrt{2}}|g\rangle + \frac{1}{\sqrt{2}}|m\rangle \quad , \quad |s\rangle = \frac{1}{\sqrt{2}}|g\rangle - \frac{1}{\sqrt{2}}|m\rangle$$

Sums/differences of vectors = **superpositions** of physical states.

States of definite color = **superpositions** of hardness states.

States of definite hardness = **superpositions** of color states.

Hardness/color operators are **incompatible** observables

in the sense that states of definite hardness (eigenvectors of hardness operator)

apparently have no definite color value (not eigenvectors of color operator)

and vice versa.

Since color and hardness are **incompatible —-> operators do not commute**.

Can see by determining matrix for $\hat{H}$ in color basis and then computing commutator. We have

$$[H] = \begin{pmatrix} \langle g|\hat{H}|g\rangle & \langle g|\hat{H}|m\rangle \\ \langle m|\hat{H}|g\rangle & \langle m|\hat{H}|m\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where we have used

to do algebra with operators one must write them in same basis

$$\langle g|\hat{H}|m\rangle = \left(\frac{1}{\sqrt{2}}\langle h| + \frac{1}{\sqrt{2}}\langle s|\right)\hat{H}\left(\frac{1}{\sqrt{2}}|h\rangle - \frac{1}{\sqrt{2}}|s\rangle\right)$$

$$= \frac{1}{2}\left(\langle h|\hat{H}|h\rangle - \langle h|\hat{H}|s\rangle + \langle s|\hat{H}|h\rangle - \langle s|\hat{H}|s\rangle\right)$$

$$= \frac{1}{2}\left(\langle h \mid h\rangle + \langle h \mid s\rangle + \langle s \mid h\rangle + \langle s \mid s\rangle\right) = \frac{1}{2}(1+0+0+1) = 1$$ and so on

Then

$$\left[\hat{C},\hat{H}\right] = \hat{C}\hat{H} - \hat{H}\hat{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= 2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \neq 0$$

These descriptions of color/hardness = 2-dimensional vector space.

Now for the **final part** of the 2nd postulate:

if system is in state $|\psi\rangle$ and measure observable $\hat{B}$ (and $|\psi\rangle$ **NOT** an eigenstate of $\hat{B}$),

then the **ONLY possible results** of measurement are the eigenvalues of $\hat{B}$

that is, the set $\{b_k\}$

Read that statement again! It is a truly amazing statement!!!!!

Nothing else will ever be measured but the eigenvalues!!!!!

(C) 3rd Postulate: Dynamics of state vectors

There exist "**deterministic**" laws (same as classical world rules)

about how a state vector of any system changes with time.

Every state vector representing real physical system must have length = 1

—> changes of state vectors dictated by dynamical laws

(called **Schrodinger** equation or the **time-development** equation)

are changes of "**direction**" (and never of "**length**") of ket vectors.

We define a "time evolution or development" operator $\hat{U}$

that governs how state vector changes in time by the relationship

$$|A, t + \Delta t\rangle = \hat{U}(\Delta t)\,|A, t\rangle$$

—> state vector at time t + Δt given

by time evolution operator $\hat{U}$ operating on state vector at time t.

In general, ket labels(which contain whatever we know(have measured) about state)

are the **only thing that changes**

Time evolution operator = unitary operator(because length does not change) $\hat{U}$ which means

if $\hat{U}\hat{U}^{-1} = \hat{I}$ or $\hat{U}^{-1}$ is the inverse of $\hat{U}$ then the Hermitian conjugate $\hat{U}^{\dagger} = \hat{U}^{-1}$

Time evolution operator is related to energy operator $\hat{U}(t) = e^{i\hat{E}t/\hbar}$ more later

(D) 4th Postulate - Connection with Experiment/Measurements

Have said so far :

particular physical state whose state vector is an eigenvector, with eigenvalue α ,

of the operator associated with a measurable property

will **"have"** value α for that property

and that a measurement of that property, carried out on the system

which happens to be in that state,

will produce result α with certainty(probability = 1)

and that if system is **not** in an eigenvector of the observable being measured,

then one can **only** measure one of its corresponding **eigenvalues**.

Need much more than that to deal with real-world experiments.

What if we measure a certain property of physical system

at a moment when the state vector of system

does not happen to be an eigenvector of that measurement property operator?

(this is the case most of the time).

i.e., what if we measure the color of a hard electron(remember it is a superposition of being green and being magenta)?

What happens then?

All our earlier assumptions/postulates are no help here. We need a **new** assumption/postulate.

Suppose we have system in state $|\psi\rangle$,

and carry out measurement of value of property (observable) B

associated with operator $\hat{B}$.

Assume eigenvectors of $\hat{B}$ are states $|b_i\rangle$ which means that

$$\hat{B}\,|b_i\rangle = b_i\,|b_i\rangle \quad , \quad i = 1, 2, 3, 4, ...$$

where the b_i are the corresponding eigenvalues.

Quantum theory —> outcome of measurement is strictly a matter of "probability".

Quantum theory **stipulates** that the probability that

outcome of measurement of $\hat{B}$ on state $|\psi\rangle$ (not an eigenvector)

will yield result b_i

(remember only possible results are eigenvalues of $\hat{B}$ no matter what state we are in),

is equal to

$$|\langle b_i \mid \psi\rangle|^2$$

or absolute-square of **corresponding** ket vector component!

This postulate means the following:

(a) Probability as so defined is always ≤ 1 (must be to make sense),

which results from requirement that allowable states have length = 1.

That was reason for earlier imposing the normalization requirement.

(b) If $|\psi\rangle = |b_i\rangle$ (= eigenvector),

then probability to measure b_i is

$$\text{probability} = |\langle b_i \mid b_i\rangle|^2 = 1$$

and for any other eigenvalue b_k $k \neq i$

$$\text{probability} = |\langle b_k \mid b_i\rangle|^2 = 0 \quad , \quad k \neq i$$

(c) Probability that green electron is found to be soft during hardness measurement = 1/2.

State being measured is

$$|g\rangle = \frac{1}{\sqrt{2}}|h\rangle + \frac{1}{\sqrt{2}}|s\rangle$$

$$prob(soft|green) = |\langle s \,|\, g\rangle|^2 = \left|\langle s| \left(\frac{1}{\sqrt{2}}|h\rangle + \frac{1}{\sqrt{2}}|s\rangle\right)\right|^2$$

$$= \left|\frac{1}{\sqrt{2}}(\langle s \,|\, h\rangle + \langle s \,|\, s\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}(0+1)\right|^2 = \frac{1}{2}$$

(d) Similarly, probability that green electron is hard during hardness measurement = 1/2.

Probability that hard electron green during color measurement = 1/2.

Probability that hard electron is magenta during a color measurement = 1/2 and so on.

The new formalism(4 Postulates) correctly predicts

all experimental results for hardness and color measurements(experiments).

It is important to realize that we cannot say anything definite about color if system in hardness state and vice versa.

Can then only make probability statements.

BEFORE we measure the color of electron, the electron DOES NOT have a color,

according to quantum theory!

Our information about electron color is only set of probabilities before measurement!!!!

But **all of your experience** says

that objects **have** values for measured quantities

before they are measured,

i.e., your experience tells you

that electron **has** color even if we do not measure it.

That is your view (standard classical view) about what is real and what is not real.

Quantum theory says you are **wrong** in both cases!!

If you assume otherwise, then QM would not work...**but it does!!!**

Eventually we will devise real experiments to show that quantum theory is correct

and your classical views about reality are incorrect!! Think about what I just said!

Must also **devise theory** to show

why it seems to be this way for electrons

but does not seem to be true for macroscopic objects.

Finally, let me state the last and most controversial postulate.

(**E) Collapse - Measurements are always repeatable. Seems like innocuous statement!**

Once a measurement is carried out and a result obtained for some observable,

the state of system must be such as to guarantee (probability = 1)

that if **same** measurement is repeated,

the exact **same** result is obtained.

Since systems evolve in time,

this can only be true if the 2nd measurement follows the 1st **instantaneously**

or within such a small time that the system **does not** have a chance to evolve.

What does this mean about state vector of a measured (after measurement) system?

One view - something **must happen** to state vector when measurement occurs.

If measurement of observable $\hat{O}$ carried out on system S,

and if outcome of measurement is value o_q (one of its eigenvalues)

then, whatever state vector of S was before measurement of $\hat{O}$

The **only way to guarantee** (probability = 1)

that another measurement of $\hat{O}$ will give the same result

is that state vector of system S after the measurement

necessarily must now be the eigenvector of $\hat{O}$ with eigenvalue o_q .

This must be true according to Postulate #2 and **all postulates must be consistent**.

Thus, in **this view**,

the effect of measuring any observable

must necessarily be to **"change"** state vector of measured system,

to "**COLLAPSE"** it, i.e.,

to make it **"JUMP"** from whatever state it was in prior to measurement

into an eigenvector of the just measured observable operator.

This is called **collapse of state vector or reduction of state vector.**

It says that state vector changes(**discontinuously**)

during measurement **from** representing a range of possibilities

(superposition of all possible states)

to **definite** state or only one possible outcome.

Which particular eigenvector it gets changed into

is determined by the outcome of measurement

and cannot be known until then!!!!!!.

It cannot be predicted!!!!

Remember we only can predict probabilities!

Since the outcome(earlier assumption) is matter of probability,

it is **at this point and at no other point** in the discussion,

that an element of **"pure chance"** enters into time evolution of the state vector

and **determinism goes out window**.

The postulates are correct!!

The time evolution of system is **continuous and deterministic** between measurements,

and **discontinuous and probabilistic**(random) during measurements.

The last(5th) postulate is **very controversial.**

We will discuss this controversy in detail later.

Those are principles(postulates) of quantum theory.

They are **most precise** mechanism

for predicting outcomes of experiments on physical systems ever devised.

No exceptions to them have ever been discovered (108 years).

NOBODY expects any.

Now let us use these principles to **study QM**.

We will **see** how quantum theory makes predictions

and **how** the strange results of various experiments are **accounted for** by the theory.

So now we enter a strange world

where quantum systems behave in rather mysterious and non-intuitive ways.

Remember that any behavior we predict and observe

will just be a **consequence** of the 5 postulates just stated.

That is how theoretical physics works!