Feynman's path to Schrödinger (and various other things)

Introduction

This article explores Feynman's path integral formulation of quantum mechanics, a powerful alternative to the conventional operator evolution approach.

We begin by briefly surveying different dynamical approaches in physics, contrasting the path integral with classical variational principles like Maupertuis' and Hamilton's principles of least action.

Our discussion is anchored by the double-slit experiment, which profoundly influenced Feynman's development of these ideas.

This experiment clearly demonstrates that the total probability of an event is not simply the sum of the probabilities of distinct alternative paths.

To explain the observed wave-like interference pattern, Feynman introduced the concept of probability amplitude for every possible way an event can occur.

His postulates state that the total amplitude is the sum of amplitudes for all alternative methods, and the observed probability is the absolute square of this total amplitude.

The core of the path integral formulation is that the total amplitude (kernel) for a particle's transition between two spacetime points is the sum (or integral) of the amplitudes for all possible paths connecting them.

Each path contributes equally in magnitude, but its phase is determined by the classical action along that path, divided by the reduced Planck constant, $\hbar$, which is interpreted as the quantum of action.

In the classical limit, where the action is significantly larger than $\hbar$, contributions from paths deviating considerably from the classical trajectory cancel due to rapid phase oscillations.

This elegantly illustrates how classical dynamics emerges from this quantum framework.

We will also delve into the role of the wave function and derive the Schrödinger equation directly from the path integral formulation.

This derivation underscores Feynman's success in bridging his Lagrangian-based approach with the established Hamiltonian formalism.

Different approaches to dynamics

In physics, two different ways of understanding dynamics exist.

The first and most widely used method in quantum mechanics is operator or state evolution in Hilbert spaces.

We find a Hilbert space, define some states and operators, and find the way in which these operators or states evolve over time.

The second and less common method, at least until Feynman introduced his path integral formulation, is based on the variational principle, which consist in finding natural quantities whose extrema give the laws of motion.

The variational principle first entered classical mechanics in the 18th century, when French mathematician P.L.M. Maupertuis proposed that a physical system follows the path such that that the sum of the products of momentum and displacement taken over time was minimized.

That is

$$\delta S_0 = \delta \int \mathbf{p} \cdot d\mathbf{q} = 0, \tag{1}$$

where S_0 has come to be known as the *abbreviated action*.

With some simple manipulations, using $d\mathbf{q} = \mathbf{v} \cdot dt$, we arrive to:

$$S_0 = \int 2T \, dt, \tag{2}$$

with $T = \mathbf{p} \cdot \mathbf{v}/2$ the kinetic energy of the system.

So, *Maupertuis' principle* is equivalent to:

$$\int T \, dt = 0. \tag{3}$$

This marked an early formulation of a least action principle.

Such principles assign a numerical value, the *action*, to every possible path between two points.

The true dynamics of the system under study are then found by identifying the path that minimizes this action.

A significant limitation of this early version, however, was its restriction to trajectories of constant energy, making it applicable only to systems influenced by conservative forces.

Following Maupertuis' work, foundational advancements in variational calculus by L. Euler and J. L. Lagrange paved the way for the definitive principle of least action, which W. R. Hamilton introduced in the 19th century. *Hamilton's principle*, as it is commonly known, is written in terms of the action functional S, that now involves the Lagrangian L:

$$(\delta S)_{\Delta t} = 0, \text{ where } S[\mathbf{q}] = \int_{t_1}^{t_2} L(\dot{\mathbf{q}}(t), \mathbf{q}(t), t)\ dt. \tag{4}$$

The notation $\Delta t = t_2 - t_1$ indicates that we are constraining the paths to those that start at time t_1 and end at time t_2.

The Lagrangian L is defined as $L = T - V$, where T is the kinetic energy of the system and V is its potential energy at each point in the path.

Note that, from (4), we can see that Maupertuis' principle is just a special case of Hamilton's principle.

Taking the total energy of the system $E = T + V$ and considering it constant (since we have only conservative forces), we have, starting from Hamilton's principle:

$$\delta S = \delta \int L\, dt = \delta \int (T - V)\ dt = \delta \int (T - E + T)\ dt$$

$$= \delta \int 2T\ dt - \delta \int \cancelto{0}{E}\ dt = \delta S_0. \tag{5}$$

So, considering conservative forces, Hamilton's action reduces to Maupertuis' action.

The double-slit experiment

The purpose of the following sections is to give an overview of Feynman's path integral formulation of quantum mechanics, following Feynman and Hibbs book.

Before proceeding, it will be useful to introduce the double-slit experiment.

This conceptual experiment was a primary motivator for Feynman in developing his theory and also illuminates the thinking behind its postulates or axioms.

Consider the setup of Figure 1.

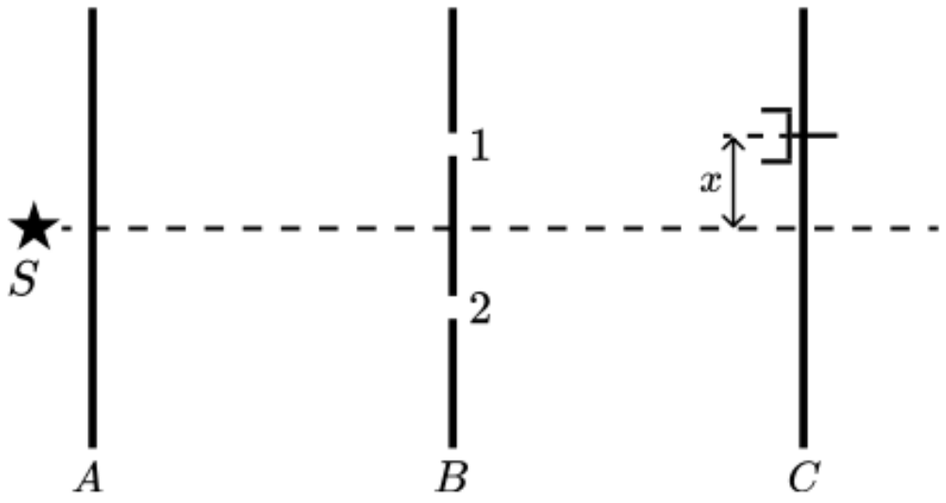

Figure 1. Double-slit experiment set-up.

At A, we have a source of electrons S, all of which have the same energy but come out in random directions to impinge on a screen B.

The screen B has two holes 1 and 2, through which electrons can pass.

Some distance behind the screen B, there is a plane C on which an electron detector may be placed at different distances x from the center of the screen.

If the electron source is capable of emitting single electrons and the detector is sensitive enough, we find that the current at the detector is not continuous, but shows clear peaks corresponding to the arrival of each single electron.

In between the arrival of each electron, nothing will be detected.

Considering now that we have evenly distributed detectors so that they cover the whole of plane C, then at the arrival of each electron exactly one of the detectors will show a peak in current.

There will never be half-detections or anything of the sort, which is the reason that we consider electrons as particles.

With these considerations in mind, we can say that the experimental set-up of Figure 1 is able to detect the passage of single particles (electrons in this case) travelling from S to the point x through a hole in screen B.

By moving the detector to different values of x, we can measure for each position the probability P that the electron passes from S to x.

For example, if the source emits one electron per second and we run the experiment for 1 minute, detecting 1 electron at position x in that time interval, the probability P of the electron going from S to x will be equal to 1/60.

If we now were to plot the probabilities measured as a function of x, we would be surprised to find the distribution of Figure 2a, as opposed to the distribution that we would expect from a classical experiment with macroscopic balls, for instance, as is shown in Figure 2b.

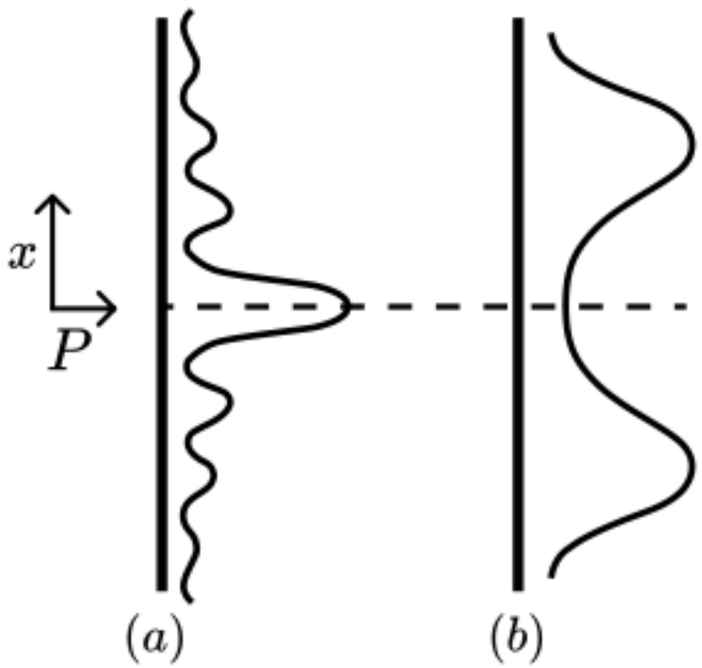

Figure 2. Probability P(x) of the electrons going from the source S to the screen C as a function of the arriving position x. (a) shows the real measured probability, while (b) shows the probability that we would normally expect in an analogous classical experiment (each peak is caused by one of the two holes in screen B).

What is the meaning of such result?

At first, we might suppose the following:

(i) Each electron going from S to x must either do so through hole 1 or through hole 2.

Consequently:

(ii) The probability of arriving at x from S should be the sum of two parts, each corresponding to the chance of arrival through 1 and through 2.

Therefore, we could repeat the experiment with only hole 1 open, and then with only hole 2.

We could then sum the two distributions to obtain the probability of arriving at x.

However, such a sum would give precisely the probability distribution from Figure 2b, which is not what we measure when both holes are open.

Therefore, (ii) must be false.

That is, the probability P of arriving at x from S is not the sum of the probability of arrival through hole 1 plus that of arrival through hole 2.

Feynman's postulates

What is then P?

To answer this question, we can look again at Figure 2a and notice that the pattern is precisely the one to be expected if the experiment was to be done using waves instead of electrons.

By analogy with waves[5], whose amplitudes are best represented using complex numbers, Feynman states the following postulates or axioms:

> **Postulate 1:** "There is a quantity called a *probability amplitude* associated with every method whereby an event in nature can take place".
>
> **Postulate 2:** "We can associate an amplitude with the overall event by adding together the amplitudes of each alternative method".

Postulate 3: "The absolute square of the overall amplitude [is] the probability that the event will happen".

In other words, for the double-slit experiment, there are two complex numbers ϕ_1 and ϕ_2 such that

$$P = |\, \phi_1 + \phi_2 \,|^2, \tag{6}$$

and the probabilities P_1 and P_2 of going through hole 1 and hole 2, respectively, are:

$$P_1 = |\, \phi_1 \,|^2, \qquad P_2 = |\, \phi_2 \,|^2 . \tag{7}$$

Thus, if we are able to compute ϕ_1 and ϕ_2, we can compute the absolute square of $\phi_1 + \phi_2$ and interpret it as the probability that a particle from S will arrive at x.

The resulting distribution in x is the one in Figure 2a.

Which hole does the electron go through?

Let us now provide some insight into the nature of what we are observing in the double-slit experiment.

According to (6) and (7), it is in general not true that $P = P_1 + P_2$.

Then, it is not true that the particle goes through either hole $_1$ or hole $_2$ when both holes are open.

We could give many possible interpretations for what is actually happening with the electron in this experiment.

Is the electron going through both holes at once?

Is it following a complex trajectory where it goes first through one hole and then through the other?

To test these hypotheses, Feynman proposed yet another experiment.

Since electrons scatter light, we can place a photon source at each of the holes 1 and 2.

In this way, the passage of an electron through any one of the holes will cause the scattering of the respective photon beam.

In doing this experiment, one finds that, when an electron passes from S to the screen C, light is scattered in either hole 1 or hole 2, but never in both places.

It seems that the electron does indeed go through either hole 1 or hole 2.

What is more, if we now use the detectors in screen C to find the distribution of P in x, we find the curve of Figure 2b!

What is happening?

The "measurement" of the electron positions at the holes using the photon beams is causing the distribution of P as a function of x to change to $P = P_1 + P_2$ (Figure 2a), whereas not doing that measurement gives $P \neq P_1 + P_2$ (Figure 2b).

However, a weaker source would simply mean less photons being emitted, and thus some electrons would be missed and not scatter any photons.

The electrons that did collide would give Figure 2b, while those that did not collide would give Figure 2a.

The resulting pattern in screen C would then be some weighted average of both distributions.

However, each electron would always either fully scatter a photon or not at all.

Another thought would be to use photons of longer wavelengths to cause smaller disturbances to the electrons.

However, there is always a limit to this, since "a source of light of wavelength λ cannot be located in space with precision greater than λ.

We thus see that any physical agency designed to determine through which hole the electron passes must produce, lest we have a paradox, enough disturbance to alter the distribution" from Figure 2a to Figure 2b.

This idea was first stated by Heisenberg in his uncertainty principle.

In summary of this last discussion, we can add a fourth axiom to Feynman's postulates:

> **Postulate 4:** "If we observe the system [...] to be in one particular state, we exclude the possibility that it can be in any other state, and the amplitudes associated with the excluded states can no longer be added in as alternatives in computing the overall amplitude"

Furthermore, whether we look at the result of the measurement or not is irrelevant.

The operation of the measurement equipment is enough to disturb the probability amplitude of the system (we could obviously look at the result later).

The path integral formulation

Since the total amplitude is the sum of the amplitude of the alternatives, there are many possible ways of evaluating it, depending on the different classes we split the alternatives into.

In one dimension, for instance, we can consider all the alternative motions of a particle from a position x_a at time t_a to a position x_b at time t_b (we will simply say "from a to b").

Each of those alternatives will be given by the position x_a as a function of the time t_a, restricted to the conditions $x(t_a) = x_a$ and $x(t_b) = x_b$.

Each possible *path* would be associated to an amplitude, and the sum (or integral) of the amplitudes for all possible paths would give the total amplitude of the motion.

Such a sum (or integral) is commonly known as the *kernel* and written in as K(b,a) for our particular case of one dimensional motion

Note that, in classical mechanics, there is only a single possible trajectory between a and b, which we will call the classical trajectory x(t), following Feynman's notation.

Such a trajectory is an extremum of the action in (4), which can be written more simply in our one dimensional case as

$$S = \int_{t_a}^{t_b} L(\dot{x}, x, t)\, dt. \tag{8}$$

Fixing the endpoints of x and imposing that the first order change in S is zero, so $S[x + \delta x] - S[x] = 0$, we arrive at the classical Euler-Lagrange equation of motion for one dimension:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0, \tag{9}$$

which gives the unique path followed by a classical particle with Lagrangian L.

The key difference in quantum mechanics is that it is not just this path of extreme action that contributes to the amplitude, but actually all the paths contribute to the total amplitude, or kernel, K(b,a).

Their contribution is equal in magnitude, but different in phase.

The phase contribution for a particular path is given by the action S for that particular path divided by $\hbar$, which we can now interpret as the quantum of action, with units $J \cdot s$.

Thus, we can write the last postulate of Feynman's theory, also in his own words:

Postulate 5: The different alternatives "contribute equal amounts to the total amplitude, but contribute at different phases. The phase of the contribution from a given path is the action S for that path in units of the quantum of action $\hbar$".

That is, the probability of going from a to b is given by $P = K(b,a)^2$, where the kernel K(b,a) is the sum of the amplitudes corresponding to all paths x(t) from a to b:

$$K(b,a) = \sum_{\substack{\text{all paths } x(t) \\ \text{from } a \text{ to } b}} \phi\,[x(t)]\,, \tag{10}$$

and:

$$\phi\,[x(t)] = c \cdot \exp\left(i\,\frac{S[x(t)]}{\hbar}\right), \tag{11}$$

The constant c is a normalization factor that needs to be carefully chosen.

Note that the classical limit corresponds to systems with large dimensions, masses, times, etc., so that S is also very large in comparison to $\hbar$.

In other words, classical systems are those involving huge numbers of action quanta, meaning that $S/\hbar$ is a very large phase.

Small deviations δx in the path will cause small changes in S at the classical scale.

However, these changes will be huge in the scale of $\hbar$.

The rapid oscillations associated with δx imply that when one path contributes positively, an infinitesimally close path will contribute equally but with the opposite sign.

For these cancellations not to occur, $_S$ must remain constant (or change slowly) with respect to δx .

But this is precisely the condition that defines x, since it is an extremum of S.

Therefore, the only place where the variation of S with x is zero, at least in the first order, is in the neighborhood of x, where $S[x + \delta x] = S[x]$.

Thus, for classical systems, the only path that really contributes to the amplitude of a certain motion is the classical path x.

In this way, the classical laws of motion arise from the quantum laws.

The path integral

The sum over all paths in (10) has been treated qualitatively up until now, with the actual mathematical details thrown under the carpet.

Now, a more precise definition of "sum over all paths" is required.

Now we follow the derivation in the Feynman and Hibbs book.

Here, the key idea is to divide the time interval $t_b - t_a$ into N infinitesimal time intervals of length ε, giving a set of time values ti spaced a distance ε apart.

To each of these time values t_i, a position x_i can be assigned.

A path theconsists of joining the points x_i for i between 1 and N-1.

The end points $x_0 = x_a$, $x_N = x_b$, $t_0 = t_a$ and $t_N = t_b$ are fixed and $N_\varepsilon = t_b - t_a$.

The sum over all paths now consists of integrating over all possible choices of the positions x_i and taking the limit $N \to \infty$.

The notation used by Feynman to indicate the path integral from (10) is:

$$K(b,a) = \int_a^b e^{iS[b,a]/\hbar} \mathscr{D}x(t), \tag{12}$$

somewhere S[b,a] denotes the action of each different path x(t) from a to b as we integrate.

This expression needs to be properly normalized in each case.

For any path x(t) between a and b, we may choose $t_c \in (t_a, t_b)$.

Then, the action along the path x(t) between a and b going through $x_c = x(t_c)$ at time t_c can be expressed as

$$S[b,a] = S[b,c] + S[c,a]. \tag{13}$$

Then we can rewrite the kernel as

$$K(b,a) = \int \exp\left\{ \frac{i}{\hbar}(S[b,c] + S[c,a]) \right\} \mathscr{D}[x(t)], \tag{14}$$

where $\mathcal{D}$x(t) now indicates integration over all paths going from a to c, integration over all paths going from c to b, and integration over all possible intermediate positions x_c.

This can be rewritten as:

$$K(b,a) = \int_{x_c} K(b,c)K(c,a)dx_c. \tag{15}$$

A simple extension can be made for an arbitrary number of intermediate positions.

Considering a path from a to b, split into N path segments with intermediate positions $x(t_i) = x_i$ for i = 1, 2, ..., N−1, we obtain:

$$\begin{aligned} K(b,a) = \int_{x_1} \int_{x_2} \cdots \int_{x_{N-1}} & K(b,N-1)\,K(N-1,N-2) \cdots \\ & \cdots K(i+1,i) \cdots K(1,a)\, dx_1\, dx_2 \cdots dx_{N-1}. \end{aligned} \tag{16}$$

Thus, we reach the following rule:

Rule 1: "Amplitudes for events occurring in succession in time multiply"

If we split the time interval $[t_a, t_b]$into infinitesimally separated time points, with $\Delta t = t_{i+1} - t_i \to \varepsilon$, then, for sufficiently small ε, it holds:

$$\begin{aligned} S[i+1,i] &= \int_{t_i}^{t_{i+1}} L(\dot{x},x,t)\ dt \\ &= \varepsilon \cdot L\left(\frac{x_{i+1}-x_i}{\varepsilon}, \frac{x_{i+1}+x_i}{2}, \frac{t_{i+1}+t_i}{2}\right), \end{aligned} \tag{17}$$

which is just the length of the time interval multiplied by the average value of the Lagrangian in that interval, and is correct to the first order in ε.

With this, we can write the kernel for each infinitesimal interval as

$$K(i+1,i) = \frac{1}{A}\exp\left[\frac{i\varepsilon}{\hbar}\cdot L\left(\frac{x_{i+1}-x_i}{\varepsilon}, \frac{x_{i+1}+x_i}{2}, \frac{t_{i+1}+t_i}{2}\right)\right], \tag{18}$$

which is correct to first order in ε.

The 1/A is just a normalization constant.

Applying **Rule 1**, we obtain:

$$\phi[x(t)] = \lim_{\varepsilon \to 0} \prod_{i=0}^{N-1} K(i+1, i), \tag{19}$$

which is the amplitude of a complete path.

The wave function

In previous sections, we have derived the amplitude for a certain motion from a point a to a point b.

However, it is often useful to define this amplitude without consideration of the motion required in getting to the final destination (x,t).

Let this amplitude be $\psi(x,t)$, and the corresponding probability $|\psi(x,t)|^2$.

We may also call it a *wave function*.

The difference to the other amplitudes is simply a matter of notation reinforcing the idea that in this case we are not interested in the previous motion of the particle.

In fact, the kernel K(b, a) is also wave function.

While it is true that it includes some extra information (namely, that the particle comes from (x_a, t_a)), it is exactly the amplitude to get to (x_b, t_b).

Since the wave function is an amplitude, it must satisfy the rules for combination of amplitudes for events occurring in succession.

This means it must satisfy the equation

$$\psi(x_b,t_b) = \int_{x_c} K(b,c)\psi(x_c,t_c)\,dx_c. \tag{20}$$

That is, the total amplitude to arrive at (x_b,t_b) is the sum, over all possible values of x_c, of the total amplitude to arrive at the point (x_c,t_c) multiplied by the amplitude of going from c to b.

This allows us to express the effects of all the past history of the particle in a single function ψ.

One need only know ψ at a particular time to be able to compute the amplitude of arriving at any future point.

Schrödinger's equation

In practice, there are many cases in which evaluating the path integral is very difficult, and not at all practical.

For those cases, there exists the possibility of reducing the path integrals to differential equations.

In fact, the Hamiltonian formalism of quantum mechanics works precisely with one such differential equation called the Schrödinger equation.

The triumph of Feynman was in his ability to show that the Schrödinger equation could be derived from his path integral formulation.

In what follows, we will replicate this derivation, building on what we already know about Feynman's path integrals.

Let us consider two times $t_1 = t$ and $t_2 = t + \varepsilon$ which are infinitesimally separated, $t_2 - t_1 = \varepsilon$.

Consider the particle is at y at t_1 and at x at t_2.

We can write (20) using (18) as

$$\psi(x, t+\varepsilon) = \frac{1}{A}\int_{\mathbb{R}} \exp\left[\frac{i\varepsilon}{\hbar} L\left(\frac{x-y}{\varepsilon}, \frac{x+y}{2}, t+\frac{\varepsilon}{2}\right)\right]\psi(y,t)\,dy. \tag{21}$$

We consider now the particular case of a particle in one dimension moving in a potential

The Lagrangian is then:

$$L = \frac{1}{2}m\dot{x}^2 - V(x,t). \tag{22}$$

Substituting in (21). we find:

$$\psi(x, t+\varepsilon) = \frac{1}{A}\int_{\mathbb{R}} \left\{\exp\left[\frac{im}{\hbar}\frac{(x-y)^2}{2\varepsilon}\right]\right\} \left\{\exp\left[-\frac{i\varepsilon}{\hbar}V\left(\frac{x+y}{2}, t\right)\right]\right\}\psi(y,t)\,dy. \tag{23}$$

Having $(x-y)^2/\varepsilon$ in the first term, we will only find relevant contributions of this integral when $y \to x$.

This is because, as we discussed for the classical limit of the path integral, large values of $x-y$ will cause fast oscillations in the phase of the exponential term, leading to net cancellations.

For this reason we may define $y = x + \eta$ with η being small enough.

Then:

$$\psi(x,t+\varepsilon) = \frac{1}{A}\int_{\mathbb{R}} \left\{ \exp\left[\frac{im\eta^2}{2\hbar\varepsilon}\right]\right\} \times \left\{\exp\left[-\frac{i\varepsilon}{\hbar}V\left(\frac{x+\eta}{2},t\right)\right]\right\}\psi(x+\eta,t)\,d\eta. \tag{24}$$

When η is of the order $\varepsilon\hbar/m$, the first exponential terms changes by the order of 1 radian.

It is then that we shall find the relevant contributions to the integral.

Also, we can let $\varepsilon V((x + \eta)/2, t) \to \varepsilon V(x, t)$.

We now expand to first order in ε and second in η, leaving:

$$\psi(x,t)+\varepsilon\frac{\partial\psi}{\partial t} = \frac{1}{A}\int_{\mathbb{R}} e^{\frac{im\eta^2}{2\hbar\varepsilon}}\left[1-\frac{i\varepsilon}{\hbar}V(x,t)\right] \times\left[\psi(x,t)+\eta\frac{\partial\psi}{\partial x}+\eta^2\frac{1}{2}\frac{\partial^2\psi}{\partial x^2}\right]d\eta. \tag{25}$$

We can separate the right hand side of this equation into three terms with the following integrals:

$$\frac{1}{A}\int_{\mathbb{R}} e^{im\eta^2/2\hbar\varepsilon} d\eta = \frac{1}{A}\left(\frac{2\pi i\hbar\varepsilon}{m}\right)^{1/2}, \tag{26}$$

$$\frac{1}{A}\int_{\mathbb{R}} e^{im\eta^2/2\hbar\varepsilon} \eta\, d\eta = 0, \tag{27}$$

$$\frac{1}{A}\int_{\mathbb{R}} e^{im\eta^2/2\hbar\varepsilon} \eta^2\, d\eta = \frac{i\hbar\varepsilon}{m}. \tag{28}$$

In order for both sides to agree at the limit $\varepsilon \to 0$, we must impose $A = (2\pi i\hbar\varepsilon/m)^{1/2}$.

This is a common method of obtaining this constant, also in more complex problems.

With this, we reach:

$$\psi + \varepsilon\frac{\partial\psi}{\partial t} = \psi - \frac{i\varepsilon}{\hbar}V\psi - \frac{\hbar\varepsilon}{2im}\frac{\partial^2\psi}{\partial x^2}. \tag{29}$$

Subtracting ψ from both sides:

$$\varepsilon \frac{\partial \psi}{\partial t} = -\frac{i\varepsilon}{\hbar} V\psi - \frac{\hbar\varepsilon}{2im} \frac{\partial^2 \psi}{\partial x^2}, \tag{30}$$

and dividing by ε:

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar} V\psi - \frac{\hbar}{2im} \frac{\partial^2 \psi}{\partial x^2}. \tag{31}$$

Finally, multiplying both sides by -$\hbar$/i, we get the well-known time-dependent equation for a particle moving in one dimension:

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x,t)\psi(x,t). \tag{32}$$

In operator form, this is:

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi. \tag{33}$$

3D Schrödinger Equation

One can generalize the previous procedure using three dimensions to get the three dimensional Schrödinger equation.

We provide a summary of the steps, with the Lagrangian, action and propagator defined in three dimensions:

$$L(\vec{x}, \dot{\vec{x}}, t) = \frac{1}{2} m \dot{\vec{x}}^2 - V(\vec{x}, t), \tag{34}$$

$$S[\vec{x}(t)] = \int_{t_a}^{t_b} \left(\frac{1}{2} m \dot{\vec{x}}^2 - V(\vec{x}(t), t) \right) dt, \tag{35}$$

$$K(\vec{x}_b, t_b; \vec{x}_a, t_a) = \int \mathscr{D}[\vec{x}(t)]\, e^{\frac{i}{\hbar} S[\vec{x}(t)]}. \tag{36}$$

Considering that the particle is at y at time t and it is at x after an infinitesimal time ε, the wave function at a time t + ε, is:

$$\psi(\vec{x}, t+\varepsilon) = \int_{\mathbb{R}^3} K(\vec{x}, t+\varepsilon; \vec{y}, t)\, \psi(\vec{y}, t)\, d\vec{y}. \tag{37}$$

Applying the same approximations used for the one dimensional case:

$$K(\vec{x}, t+\varepsilon; \vec{y}, t) \approx \frac{1}{A} \exp\left[\frac{i\varepsilon}{\hbar} \left(\frac{m}{2} \left(\frac{\vec{x}-\vec{y}}{\varepsilon} \right)^2 - V\left(\frac{\vec{x}+\vec{y}}{2}, t \right) \right) \right]. \tag{38}$$

We plug (38) into (37) and, using $\vec{y} = \vec{x} + \vec{r}$ with $||\vec{r}||$ being small enough, we expand $\psi(\vec{y}, t)$around point $\vec{x}$ and the exponential term around $\vec{r}$.

We also expand $\psi(\vec{x}, t+\epsilon)$ in ε.

Keeping terms up to second order in $\vec{r}$ and up to first order in ε, we get:

$$\psi(\vec{x},t)+\varepsilon\frac{\partial\psi}{\partial t}=\frac{1}{A}\int_{\mathbb{R}^3}e^{\frac{im\vec{r}^2}{2\hbar\varepsilon}}\left[1-\frac{i\varepsilon}{\hbar}V(\vec{x},t)\right]\times\left[\psi(\vec{x},t)+\vec{r}\,\nabla\psi+\vec{r}^2\,\frac{1}{2}\nabla^2\psi\right]d\vec{r}. \tag{39}$$

Finally, doing similar integrals to the ones in the one dimensional case, we obtain the three dimensional Schrödinger equation:

$$i\hbar\frac{\partial\psi(\vec{x},t)}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{x},t)+V(\vec{x},t)\psi(\vec{x},t). \tag{40}$$

Conservation of probability

A central requirement of quantum mechanics is the conservation of total probability.

The integral of the probability density over all space must remain constant in time.

This ensures physical consistency, e.g. that a particle described by the wave function is always somewhere in space.

In the operator formulation of quantum mechanics, this concept follows from the Hermitian nature of the Hamiltonian and unitary time evolution.

When it comes to path integral formulations, conservation of probability manifests in the structure of the kernel and how it relates to the probability given by the wave function.

In this section we examine how the concept of conservation of probability arises in Feynman's formalism, and we show the necessary condition to be satisfied by the kernel so that this is the case.

The Hamiltonian operator H given by

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V \tag{41}$$

is Hermitian, as any other Hamiltonian in quantum mechanics.

Then, it must satisfy (using the one dimensional case)

$$\int (H\psi)^* \psi \, dx = \int \psi^* (H\psi) \, dx. \tag{42}$$

As ψ satisfies the Schrödinger equation in (33), we have:

$$\begin{aligned} \int (H\psi)^* \psi \, dx &= \int \left(i\hbar \frac{\partial \psi}{\partial t} \right)^* \psi \, dx \\ &= -i\hbar \int \frac{\partial \psi^*}{\partial t} \psi \, dx, \end{aligned} \tag{43}$$

$$\int \psi^*(H\psi)\,dx = \int \psi^* \left(i\hbar \frac{\partial \psi}{\partial t} \right) dx$$
$$= i\hbar \int \psi^* \frac{\partial \psi}{\partial t}\,dx. \tag{44}$$

Combining these results with (42), we obtain:

$$\int \frac{\partial \psi^*}{\partial t} \psi\,dx + \int \psi^* \frac{\partial \psi}{\partial t}\,dx = \frac{d}{dt} \left(\int \psi^* \psi\,dx \right) = 0. \tag{45}$$

Then, the "area under $\psi^*\psi$", which is just the total probability, must be independent of time.

That is, the probability is constant; it is conserved.

In what respects to the kernel, this means that, if f is the wave function at t_1, and ψ is the wave function at t_2, so that

$$\psi(2) = \int K(2,1) f(1)\,dx_1, \tag{46}$$

then the square integral of the wave functions at the different times must be the same:

$$\int \psi^*(2)\psi(2)\,dx_2 = \int f^*(1) f(1)\,dx_1. \tag{47}$$

Substituting (46) in (47), we obtain:

$$\iiint K^*(2;x_1',t_1)K(2;x_1,t_1)f^*(x_1')f(x_1)\,dx_1\,dx_1'\,dx_2$$

$$= \int f^*(x_1)f(x_1)\,dx_1. \tag{48}$$

Clearly, this equality will hold for any arbitrary f if and only if

$$\int K^*(2,x_1',t_1)K(2,x_1,t_1)\,dx_2 = \delta(x_1' - x_1). \tag{49}$$

Then, if we interpret ψ as a probability amplitude, the kernel must satisfy this equation.

Conclusion

In this work, we have presented the foundations of quantum mechanics through Feynman's path integral formalism, tracing his reasoning from the double-slit experiment and highlighting key insights into his thinking.

The article introduces the wave function and probability amplitudes, formulates Feynman's postulates, derives the Schrödinger equation from the path integral, and examines the conservation of probability, linking norm preservation to the structure of the propagator.

We emphasize how this formalism offers intuitive explanations for interference and other quantum phenomena and show its connection to the Hamiltonian approach.

In doing so, the article serves as a pedagogical bridge, helping learners appreciate the equivalence of the two formalisms, deepen their conceptual understanding with a unifying perspective, and prepare for more advanced topics in quantum theory.

Main reference:

Feynman, R. P. & Hibbs, A. R. Quantum Mechanics and Path Integrals (McGraw-Hill, New York, 1965).

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