# The Born Rule - Axiom or Result?

## INTRODUCTION

There is a wide range of attitudes regarding the Born rule and its status among the axioms of quantum theory.

Some accept it as an axiom, while others think that it should be (or has been) derived from the other axioms.

Still others believe that the truth lies somewhere in between - that a derivation is possible, but it requires one or more additional assumptions beyond the other axioms.

To better describe these positions, let us state the Born rule as it might appear in the context of the other axioms.

Here are the axioms as they are stated, more or less consistently, by textbooks in use over the last seventy years.

The following represents a sort of consensus among those listed, following none in particular.

I. *An isolated system is associated with a Hilbert space $\mathcal{H}$. Its pure states are described by unit vectors $|\psi\rangle$ in $\mathcal{H}$* (normalized to $\langle\psi|\psi\rangle = 1$) [2].

II. *The evolution of a isolated system is described by a unitary transformation: $|\psi(t)\rangle = U|\psi(0)\rangle$, where $U$ represents the solution of the time dependent Schrödinger equation, $i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$, and $H$ is the Hamiltonian of the system.*

If $H$ is time independent, then $U = \exp(-iHt/\hbar)$.

III. *Observable quantities $\Omega$ are represented by Hermitian (i.e., linear and self-adjoint) operators on $\mathcal{H}$.*

It will be useful to expand a state $|\psi\rangle$ in terms of eigenstates of $\Omega$, that is,

$$|\psi\rangle = \sum_i \alpha_i |\phi_i\rangle, \quad \text{where} \quad \Omega |\phi_i\rangle = \omega_i |\phi_i\rangle. \quad (1)$$

The eigenvalues $\omega_i$ are real because $\Omega$ is Hermitian.

IV. *A measurement of $\Omega$ yields a particular eigenvalue $(\omega_k)$ with probability*

$$p_k = |\langle \phi_k | \psi \rangle|^2. \quad (2)$$

This probability formula is called the Born rule - typically not a separate axiom, but part of this so-called "collapse" axiom.

V. *The state of the system immediately following the measurement is the eigenstate $|\phi_k\rangle$ corresponding to the measured outcome $\omega_k$. A subsequent measurement of $\Omega$ reproduces the same outcome with certainty.*

This fifth axiom is called "the post-measurement state update rule."

It is implicit in the von Neumann measurement formalism, although it is valid only if the measurement is ideal; that is, minimally disturbing and repeatable on the same system.

As a common counterexample, measurement sometimes destroys the system, as in a photon polarization measurement.

Axioms I - III are the "dynamical" axioms, which describe the continuous, deterministic evolution of the unobserved system (what von Neumann calls "type II" processes); while IV and V ("collapse and "state update") describe the abrupt and random change produced by measurement (which von Neumann calls "type I").

The word collapse, as used here, refers to the observed facts described by IV and V, and not to any interpretation- dependent concept of what might lie behind it.

The Born rule, while part of the collapse axiom, bridges the gap separating the two realms of behavior, referring only to input and output states, with no description of the processes connecting them.

The fifth axiom is sometimes absorbed into the fourth by saying that the outcome of a measurement is the output state, $|\phi k\rangle$, instead of the eigenvalue $\omega k$.

Sakurai puts this point most graphically: *"... the system is thrown into an eigenstate $|\phi k\rangle$ of $\Omega$"*.

One cannot dispute the facts represented by the axioms above, but interpretations differ regarding the status of the collapse axioms and the unanswered questions that lie behind them.

The Copenhagen Interpretation (and more modern descendants) accept collapse as an independent axiom, implying that its underpinnings are not a subject for study within quantum theory.

Other interpretations, including but not limited to many worlds, attempt to derive the collapse phenomenon, as well as the Born rule itself, from the dynamical axioms.

Decoherence theory, which is not tied to any particular interpretation, provides a formal background that describes the collapse phenomenon using axioms I - III.

However, it does not provide a derivation of the Born rule, because the Born rule is assumed implicitly in its formalism.

Standard quantum mechanics, as practiced by most physicists, accommodates a broad range of interpretations between the two extremes above, and it makes use of decoherence theory as a practical tool.

In this paper, by accepting the collapse phenomenon as a fact and focusing just on the Born rule itself, we shall present a derivation which is independent of interpretations, and which illustrates the apparent necessity of an additional assumption.

Stylistically, for the sake of transparency, we will be concise and not comprehensive.

We will tend to emphasize physics over mathematics, citing more detailed mathematical treatments as we progress.

Our aim is to provide an intuitive physical derivation of the quadratic dependence of the Born rule, and thus to clarify its status as part axiom and part derived.

## PHYSICAL ORIGIN OF THE BORN RULE

The Born rule is defined by Eq. 2.

The signature quadratic dependence is explicit, and considered by many to be synonymous with the Born rule itself.

But we draw attention to two other properties which are implicit: (i) the probability $p_k$ is independent of the other (unobserved) expansion coefficients ($\alpha_j$ with $j \neq k$), and perhaps less obviously (ii) $p_k$ is independent of the choice of the observable ($\Omega$), so long as $\Omega$ has the output state $|\phi_k\rangle$ as an eigenstate.

In what follows, we shall first derive the quadratic dependence by assuming (i) explicitly.

We shall then assume (ii) explicitly, and derive both (i) and the quadratic dependence from it.

So (ii) is the deeper, as well as the more physical assumption.

To set the stage, we begin by making neither assumption - we imagine that the probability $p_k$ could, in principle, depend on all the expansion coefficients in Eq. 1, with additional dependence on the observable $\Omega$ being measured; that is,

$$p_k = f_k(\{\alpha_i\}, \Omega), \qquad i = 1, ..., d. \qquad (3)$$

The expansion coefficients refer to the basis of eigenstates of $\Omega$, which is called the *measurement basis*.

Both of our proofs are based on the invariance of $p_k$ under unitary transformations of the complex vector $\{\alpha_i\}$ of expansion coefficients.

These transformations correspond to different representations of the same prepared initial state $|\psi\rangle$ (that is, to different measurement bases).

As a trivial preliminary example, note that the phases of the above basis states [i.e., $\gamma_k$ in $\alpha_k \equiv a_k \exp(\gamma_k)$] can be chosen arbitrarily, which means that $p_k$ can only depend on the magnitudes $a_k$:

$$p_k = f_k(\{a_i\}, \Omega), \qquad i = 1, ..., d, \qquad (4)$$

and the normalization condition for state $|\psi\rangle$ simplifies (slightly) to

$$\sum_{k=1}^{d} |\alpha_k|^2 = \sum_{k=1}^{d} a_k^2 = 1. \qquad (5)$$

But this simplification has a convenient geometrical interpretation: The unitary transformation on the vector of complex expansion coefficients,

$$(\alpha_1, .., \alpha_d)^T \rightarrow (\alpha'_1, ..., \alpha'_d)^T, \qquad (6)$$

reduces to an orthogonal transformation of the vector of moduli; that is,

$$\mathbf{a} = (a_1, ..., a_d)^T \rightarrow (a'_1, .., a'_d)^T = \mathbf{a}'. \qquad (7)$$

This is a rotation on the unit orthant (the $2^{-d}$ segment of the unit sphere in d real dimensions, in which all components are non-negative).

We now implement assumption (i) by assuming that $p_k$ depends on only the single amplitude ($a_k$) corresponding to the outcome state: $p_k = f(a_k)$.

We can then write the normalization sum for probabilities (as distinct from that of the state itself), as

$$\mathcal{N}(\{a_i\}) = \sum_{i=1}^{d} f(a_i). \qquad (8)$$

Now $\mathcal{N}$ must be stationary on the unit orthant.

Invoking this condition through a Lagrange multiplier $\lambda$, we find that

$$\frac{\partial}{\partial a_j} \left[ \mathcal{N}(\{a_i\}) - \lambda \left( \sum_{i=1}^{d} a_i^2 - 1 \right) \right] = \frac{\partial f}{\partial a_j} - 2\lambda a_j = 0. \qquad (9)$$

The solution is

$$f(x) = \lambda x^2 + \mu, \qquad (10)$$

and this, with the boundary conditions $f(0) = 0$ and $f(1) = 1$, reduces to the Born rule, $f(x) = x^2$.

This is intuitive - it shows that there is no function of a single variable, except for $x^2$, for which the probabilities can be normalized together with the state.

However, there is no a priori justification for the assumption (i) by itself.

It does not follow from the "dynamical axioms," and it begs the question of a physical origin.

Assumption (ii) provides a physical origin - namely, that $p_k$ is independent of the choice of observable to be measured, so long as it has the outcome state $\phi_k$ as one of its eigenstates.

To implement this assumption mathematically, we demand that $p_k$ be invariant under unitary transformations $U_k$ that preserve the single modulus $a_k$.

Such transformations produce arbitrary rotations of a at fixed $a_k$:

$$\left(a_k, \ \{a_i, \ (i \neq k)\}\right) \rightarrow \left(a_k, \ \{a_i', \ (i \neq k)\}\right), \qquad (11)$$

or more concisely, introducing the $(d-1)$-tuples $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{a}}'$ perpendicular to the k direction,

$$\left(a_k, \ \tilde{\mathbf{a}}\right) \rightarrow \left(a_k, \ \tilde{\mathbf{a}}'\right). \qquad (12)$$

Thus $U_k$ rotates $\tilde{\mathbf{a}}$ into $\tilde{\mathbf{a}}'$ on the orthant of radius $\rho = \sqrt{1 - a_k^2}$ in $d-1$ dimensions.

The stationarity of $p_k$ subject to this constraint is expressed by

$$\frac{\partial}{\partial a_j}\left[p_k - \lambda\left(\sum_{i \neq k} a_i^2 - \rho^2\right)\right] = \partial p_k/\partial a_j - 2\lambda a_j = 0, \quad (13)$$

for all $j \neq k$.

This has the solution

$$p_k = \lambda \sum_{j \neq k} a_j^2 + \mu = \lambda(1 - a_k^2) + \mu. \qquad (14)$$

Since this formula holds for any initial choice of $a_k$, it provides the functional form of $p_k$.

Applying the boundary conditions as above, we determine the parameters $\mu = -\lambda = 1$, resulting in $p_k = a^2_k$.

Thus we have derived the quadratic dependence on the single expansion coefficient corresponding to the output state.

We should comment that this proof, unlike some others, applies to the case $d = 2$, even though the sum over i (with $i \neq k$) contains only a single term.

The physical assumption employed above is what we have called "observable independence."

Essentially the same assumption made elsewhere has been called non-contextuality, a term which we have avoided up to now because it has multiple meanings.

Its meaning here refers to the context of alternate possible outcomes: $p_k$ does not depend on the particular identities of these outcomes.

But we prefer to emphasize the more physical underpinning of the Born rule by referring to the measured observable.

This suggests that the role of $\Omega$ is to generate a sort of prism which separates the component eigenstates (for example, into a set of non-overlapping paths) prior to detection.

One detects a particular path (k); the alternate paths are immaterial, as are the alternate eigenvalues of $\Omega$.

**An Example**

An instructive example is a J = 1 atom, with eigenstates of $J_z$ written as $|m\rangle$, with m = 1, 0, and $-1$.

But consider the alternative basis set consisting of $|\pm\rangle = (|1\rangle \pm |{-}1\rangle)/\sqrt{2}$ and again $|0\rangle$.

The latter are eigenstates of the operator $J_x{}^2 - J_y{}^2$.

Whatever the initial state $|\psi\rangle$, measurements of the two operators will yield the same probability of output for the state $|0\rangle$.

This is what we mean by observable independence.

**CONCLUSIONS**

We have presented a derivation of the quadratic dependence of the Born rule from the "dynamical axioms" (I - III), plus the single additional assumption that the probability is "non-contextual," or perhaps less ambiguously, "observable independent."

This derivation is one of many alternatives to Gleasons original proof.

It is aimed at maximizing the physical content and minimizing the formalism.

Strictly speaking, the assumption of observable independence is part of the Born rule itself, as this assumption is implicit in Eq. 2.

In this light, we have simply derived one aspect of the Born rule from another.

In principle, one could replace the Born rule as stated in axiom IV (through Eq 2) by the alternative statement that probability is observable-independent.

However, one equation (2) is worth a thousand words, and although in some sense redundant, its elegance argues for maintaining the statement as is - but to be understood as part axiom and part derived.